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ON THE EIGENVALUES OF THE p-LAPLACIAN WITH VARYING p

YIN XI HUANG

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Abstract. We study the nonlinear eigenvalue problem

$$-\text{div}\ (|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where $p \in (1, \infty)$, Ω is a bounded smooth domain in $\boldsymbol{R}^N$. We prove that the first and the second variational eigenvalues of (1) are continuous functions of p. Moreover, we obtain the asymptotic behavior of the first eigenvalue as $p \to 1$ and $p \to \infty$.

0. Introduction

In this paper we study the nonlinear eigenvalue problem

$$-\Delta_p u = \lambda |u|^{p-2}u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{0.1}$$

where $p \in (1, \infty)$, Ω is a bounded smooth domain in $\boldsymbol{R}^N$, and $\Delta_p = \text{div}\ (|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian. We are mainly interested in the continuity of the eigenvalue λ as a function of the parameter p and its asymptotic behavior as $p \to 1$ and $p \to \infty$.

The p-Laplacian has attracted growing interest, and figures in a variety of applications in applied fields. For example, it appears in non-Newtonian fluids ($p > 2$ for dilatant fluids and $1 < p < 2$ for pseudoplastic fluids), in reaction-diffusion problems, in flow through porous media, in nonlinear elasticity, in petroleum extraction ([D]), and in torsional creep problems ([K1]).

Our study is mainly motivated by the "jumping nonlinearity" problem

$$-\Delta_p u = f(x, u) \tag{0.2}$$

with $f(x, u)$ satisfying conditions of the following type:

$$\lambda_k(p) \le \liminf_{t \to \rho_1} \frac{f(x,t)}{|t|^{p-2}t} \le \limsup_{t \to \rho_2} \frac{f(x,t)}{|t|^{p-2}t} \le \lambda_{k+l}(p), \tag{0.3}$$

where $\rho_1, \rho_2 = 0^+$ or $\pm\infty$, and $\lambda_k(p)$ is the k-th eigenvalue of (0.1). In the case $p = 2$, sophisticated techniques have been developed and deep results have been obtained by many mathematicians in the past decades. See [LM] for more details and references. However, for $p \neq 2$, those well established and widely employed tools are no longer applicable due to the nonlinear and nonsymmetric nature of the

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p-Laplacian. One naturally speculates that, via certain continuation arguments, one may be able to derive some properties of the p-Laplacian from the special case $p = 2$. In doing so certain homotopy type results will be crucial. As a matter of fact, a similar procedure (in essence) was used in the O.D.E. ($N = 1$) case to study the existence of solutions to a problem of the type (0.2) by Del Pino, Elgueta and Manasevich ([DEM]), where the continuity of $\lambda_k(p)$ as a function of p is obvious. Meanwhile, Lindqvist carried out a series of studies in this direction: the dependence on p for the solutions of $-\Delta_p u = f(x)$ in [L1], and the continuity of $\lambda_1(p)$ and $u_1(p)$ (the first eigenfunction associated with $\lambda_1(p)$) in [L4]. Recently Kawohl [K1] discussed the behavior of solutions to the problem $-\Delta_p u = 1$ as $p \to 1$ and $p \to \infty$, which appears in torsional creep problems.

In this paper we obtain the continuity of $\lambda_k(p)$ for $k = 1, 2$, thus extend the results in [L4], and derive partial results regarding the asymptotic behavior of $\lambda_1(p)$ as $p \to 1$ and $p \to \infty$. We note that even though our continuity result is expected to be true by most experts and thus is not surprising at all, its proof is not that trivial. Moreover, our proof reveals an interesting fact, that is, the continuity of the eigenvalues on p follows from the linear independence of the eigenfunctions (see Remark 2.2). The latter is not yet proved except for the O.D.E. case (i.e. $N = 1$) when $p \neq 2$.

We organize this paper as follows. In Section 1 we introduce notations needed in the sequel. In Section 2 we prove the continuity of $\lambda_k(p)$ by constructing appropriate sets of genus k for $k = 1, 2$. Finally we discuss in Section 3 the asymptotic behavior of $\lambda_1(p)$.

1. Notation

Consider the eigenvalue problem

$$-\Delta_p u = \lambda |u|^{p-2} u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where $p \in (1, \infty)$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, and Ω is a bounded smooth domain in $\boldsymbol{R}^N$. $u \in W_0^{1,p}(\Omega)$ is a solution of (1.1) if for some $\lambda \in \boldsymbol{R}$ and any $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi = \lambda \int_\Omega |u|^{p-2} u \varphi.$$

In this paper, all integrals will be on Ω unless otherwise stated.

We introduce notation we need later in this paper. Let

$$U_p = \left\{ u \in W_0^{1,p}(\Omega) : \|u\|_p = 1 \right\},$$

where $\|\cdot\|_p$ is the norm in the L^p space, and

$$\Gamma_k(p) = \left\{ A \subset U_p \ : \ A \text{ is symmetric, compact, } \gamma(A) = k \right\},$$

where $\gamma(A)$ is the genus of A, i.e. the smallest integer k such that there exists an odd continuous map from A to $\boldsymbol{R}^k \setminus \{0\}$. See e.g. [R] and [Sz] for details of the genus. By the standard Ljusternik-Schnirelmann theory, see e.g. [AA] and [Sz], and the known regularity theory, cf. [Eg] and [T], we have the following

Proposition 1.1. *There exist $u(k,p) \in A(k,p) \in \Gamma_k(p)$ and $\lambda_k(p)$ such that*

$$\lambda_k(p) = \int |\nabla u(k,p)|^p = \sup_{u \in A(k,p)} \int |\nabla u|^p = \inf_{A \in \Gamma_k(p)} \sup_{u \in A} \int |\nabla u|^p, \tag{1.2}$$

and

$$-\Delta_p u(k,p) = \lambda_k(p)|u(k,p)|^{p-2}u(k,p). \tag{1.3}$$

Moreover, for some $\alpha(k,p) > 0$ *and* $\beta(k,p) > 0$,

$$|\nabla u(k,p)(x) - \nabla u(k,p)(y)| \le \beta(k,p)|x-y|^{\alpha(k,p)}, \tag{1.4}$$

for $x, y \in \bar{\Omega}$.

Without loss of generality, we assume $|\Omega| = 1$. In fact, if $|\Omega| = a$, we define

$$\Omega' = \{y \ : a^{1/N}y \in \Omega\}.$$

Then $|\Omega'| = 1$. Now, for $u \in U_p$ satisfying

$$-\Delta_p u = \tilde{\lambda}|u|^{p-2}u,$$

define $v(y) = a^{1/p}u(a^{1/N}y) = a^{1/p}u(x)$, with $y \in \Omega'$, $x = a^{1/N}y \in \Omega$. Then $\int_{\Omega'} |v(y)|^p = 1$, and $\nabla_y v(y) = a^{1/N+1/p}\nabla_x u(x)$. Therefore we have $-\Delta_p v = (a^{p/N}\tilde{\lambda})|v|^{p-2}v$, i.e. the new eigenvalue problem on the domain Ω' is modified by a factor $a^{p/N}$, which is continuous in p.

2. Continuity of $\lambda_k(p)$, $k = 1, 2$

We start with some preparation. Define the following sets:

$$\begin{aligned} A_k(p) = \Big\{ \sum_{i=1}^k \alpha_i \bar{u}(i,p) \ : \sum_{i=1}^k \alpha_i = 1, \alpha_i \ge 0, \\ \bar{u}(i,p) \in \{u(i,p), -u(i,p)\}, i = 1, \cdots, k \Big\}, \end{aligned} \tag{2.1}$$

and for $q \in (1, \infty)$,

$$A_k^q(p) = \Big\{ u/\|u\|_q \in W_0^{1,q}(\Omega) \ : \ u \in A_k(p) \Big\}. \tag{2.2}$$

First we have

Lemma 2.1. *If* $0 \notin A_k(p)$, *then* $A_k^q(p)$ *is well defined and* $\gamma(A_k^q(p)) = k$.

Proof. The regularity theory guarantees that for $u \in A_k(p)$, $u \in W_0^{1,q}(\Omega)$. Observe that if $0 \notin A_k(p)$, then $A_k^q(p)$ is well defined and is symmetric and compact. We define a map $f_k : A_k^q(p) \to \boldsymbol{R}^k \setminus \{0\}$ as follows. Let $e_i^k, i = 1, \cdots, k$, be the standard unit basis of $\boldsymbol{R}^k$, define

$$f_k(\pm u(i,p)/\|u(i,p)\|_q) = \pm e_i^k,$$

and for $u = \sum_{i=1}^k \alpha_i \bar{u}(i,p)/\|\sum_{i=1}^k \alpha_i \bar{u}(i,p)\|_q$,

$$f_k(u) = \sum_{i=1}^k \alpha_i f_k(\bar{u}(i,p)/\|\bar{u}(i,p)\|_q).$$

f_k so defined is an odd homeomorphism between $A_k^q(p)$ and the unit sphere of $\boldsymbol{R}^k$. It follows from Proposition 2.3 (a) of [Sz] that $\gamma(A_k^q(p)) = k$. □

Corollary 2.1. *If the eigenfunctions* $u(i,p)$ *are linearly independent, then* $\gamma(A_k^q(p)) = k$ *for* $k = 1, 2, \cdots$.

In particular, since $u(1,p)$ is of one sign and $u(2,p)$ changes sign, obviously $0 \notin A_1(p) \cup A_2(p)$. Thus we have

Corollary 2.2. *$\gamma(A_k^q(p)) = k$ for $k = 1, 2$.*

From now on we consider the case $k = 1, 2$.

Lemma 2.2. *$\lambda_k(p)$ is a bounded function of p on any bounded interval $(1, s]$ for $k = 1, 2$.*

Proof. By Corollary 2.2, $\gamma(A_k^p(s)) = k$. Observe that

$$\lambda_k(p) \le \sup_{v \in A_k^p(s)} \int |\nabla v|^p \le \sup_{u \in A_k(s)} \frac{\int |\nabla u|^p}{\|u\|_p^p} \le \sup_{u \in A_k(s)} \frac{(\int |\nabla u|^s)^{p/s}}{\|u\|_p^p}.$$

Here we have used the assumption that $|\Omega| = 1$. Note that $\inf_{u \in A_k(s)} \int |u|^p = c(p,s) > 0$. Indeed, if $\inf_{u \in A_k(s)} \int |u|^p = c(p,s) = 0$, since $\int |\cdot|^p$ is a continuous function of u, and $A_k(s)$ is compact in $W_0^{1,s}(\Omega)$ and hence is compact in $W_0^{1,p}(\Omega)$, there exists a $u_0 \in A_k(s)$ with $\int |u_0|^p = 0$. This implies that $u_0 = 0$, a contradiction. Consequently we have

$$0 < \lambda_k(p) \le \frac{\sup_{u \in A_k(s)} (\int |\nabla u|^s)^{p/s}}{\inf_{u \in A_k(s)} \int |u|^p} \le [\lambda_k(s)]^{p/s} / c(p,s) < \infty.$$

The lemma is proved. □

Lemma 2.3. *For $p, q \in (1, s]$, $\lim_{p \to q} \inf_{u \in A_k(p)} \int |u|^p = 1$.*

Proof. Let $u(p) \in A_k(p)$ be such that

$$u(p) \to \lim_{p \to q} \inf_{u \in A_k(p)} \int |u|^p.$$

For $\varepsilon > 0$ small, we have $p > q - \varepsilon$. Noting that $\int |\nabla u|^{q-\varepsilon} \le (\int |\nabla u|^p)^{(q-\varepsilon)/p}$, Lemma 2.2 and the fact that $u(p)$ is a convex combination of $u(1,p)$ and $u(2,p)$ imply that $u(p)$ is bounded in $W_0^{1,q-\varepsilon}(\Omega)$. Consequently for a subsequence $\{p_n\}$ and some $u_0 \in W_0^{1,q-\varepsilon}(\Omega)$, $u(p_n) \to u_0$ weakly in $W_0^{1,q-\varepsilon}(\Omega)$ and strongly in $L^{q-\varepsilon}(\Omega)$. It then follows that

$$\int |u_0|^{q-\varepsilon} = \lim_{p_n \to q} \int |u(p_n)|^{q-\varepsilon} \le \lim_{p_n \to q} \Big(\int |u(p_n)|^{p_n} \Big)^{(q-\varepsilon)/p_n} = 1.$$

Letting $\varepsilon \to 0$ we obtain $\int |u_0|^q \le 1$.

On the other hand, Lemma 4.2 of [L2] implies that

$$\int |u_0|^{p_n} \ge \int |u(p_n)|^{p_n} + p_n \int |u(p_n)|^{p_n - 2} u(p_n)(u_0 - u(p_n)).$$

The second integral on the right-hand side approaches 0 as $p_n \to q$. Thus $\int |u_0|^q \ge 1$. The lemma is proved. □

Now we are ready to prove our main result.

Theorem 2.1. *$\lambda_k(p)$ is continuous in p for $k = 1, 2$.*

Proof. By Lemma 2.2, for any sequence $p_n \to q$, there exists a subsequence, still denoted by $\{p_n\}$, such that $\lambda_k(p_n) \to \tilde{\lambda}$. Our objective is to prove $\tilde{\lambda} = \lambda_k(q)$.

As in the proof of Theorem 4.1 of [L4], a standard diagonal argument produces a subsequence of $\{p_n\}$, still denoted by $\{p_n\}$, such that for some $u_0 \in C^1(\Omega)$,

$u(k,p_n) \to u_0$ and $\nabla u(k,p_n) \to \nabla u_0$ locally uniformly in Ω. The boundedness of $\{\lambda_k(p_n)\}$ further implies that $u(k,p_n) \to u_0$ in $W_0^{1,q-\varepsilon}(\Omega)$ weakly for $\varepsilon > 0$ small enough (cf. [L2], [L4]). Observe that

$$\int |\nabla u(k,p_n)|^{p_n-2}\nabla u(k,p_n)\nabla\varphi = \lambda_k(p_n)\int |u(k,p_n)|^{p_n-2}u(k,p_n)\varphi \tag{2.3}$$

for $\varphi \in C_0^\infty(\Omega)$. Letting $p_n \to q$ in (2.3), and noting that we have uniform convergence of $u(k,p_n)$ to u_0 on the support of φ, we obtain

$$\int |\nabla u_0|^{q-2}\nabla u_0\nabla\varphi = \tilde\lambda\int |u_0|^{q-2}u_0\varphi, \tag{2.4}$$

whence we have $-\Delta_q u_0 = \tilde\lambda|u_0|^{q-2}u_0$ and $u_0 \in C^{1+\alpha}(\bar\Omega)$. Moreover, the proof of Lemma 2.3 implies that $\|u_0\|_q = 1$.

Observe that

$$\begin{aligned}\lambda_k(q) &\le \sup_{v\in A_k^q(p_n)}\int |\nabla v|^q = \sup_{u\in A_k(p_n)}\frac{\int|\nabla u|^q}{\|u\|_q^q}\\ &\le \frac{\sup_{u\in A_k(p_n)}\int|\nabla u|^q}{\inf_{u\in A_k(p_n)}\|u\|_q^q}.\end{aligned} \tag{2.5}$$

By (1.4) and the convergence of $u(k,p_n)$ to u_0, we know that for p_n close enough to q and $k = 1,2$, $u(k,p_n) \in C^{1+\alpha}(\bar\Omega)$ for some $\alpha > 0$. Thus for $u \in A_k(p_n)$, $u \in C^{1+\alpha}(\bar\Omega)$. Consequently for $\varepsilon > 0$ such that $p_n + \varepsilon > q$, we have

$$\int |\nabla u|^q \le \Big(\int |\nabla u|^{p_n+\varepsilon}\Big)^{q/(p_n+\varepsilon)} \le \Big(\|\nabla u\|_\infty^\varepsilon \cdot \int |\nabla u|^{p_n}\Big)^{q/(p_n+\varepsilon)} \tag{2.6}$$

for $u \in A_k(p_n)$, and

$$\Big(\int |\nabla u|^{p_n}\Big)^{1/p_n} \le \sum_{i=1}^k \alpha_i\Big(\int |\nabla\bar u(i,p)|^{p_n}\Big)^{1/p_n} \le \lambda_k(p). \tag{2.7}$$

Combining (2.5), (2.6) and (2.7), letting $p_n \to q$, then $\varepsilon \to 0$, and using Lemma 2.3, we conclude that

$$\lambda_k(q) \le \tilde\lambda.$$

On the other hand, we have

$$\lambda_k(p_n) \le \sup_{v\in A_k^{p_n}(q)}\int |\nabla v|^{p_n}.$$

Hence the similar procedure yields

$$\lambda_k(q) \ge \tilde\lambda.$$

Thus $\lambda_k(p)$ is continuous. This concludes the proof. □

Remark 2.1. The continuity of $\lambda_1(p)$ is proved by Lindqvist in [L4]. For the O.D.E. case, the explicit formula of $\lambda_k(p)$ is given in [GV]; see also [DEM] and [HM].

Remark 2.2. Our proof breaks down for the case $k \ge 3$ because it is not clear for this case whether $0 \in A_k(p)$ or not. However, our proof shows that, if all the (variational) eigenfunctions are linearly independent, then $\lambda_k(p)$ is continuous with respect to p, since then $0 \notin A_k(p)$ for all $k > 0$. We note that the independence of the eigenfunctions is a standing open question for the p-Laplacian for $p \ne 2$.

3. Asymptotic behavior of $\lambda_1(p)$

We discuss the asymptotic behavior of $\lambda_1(p)$ as $p \to 1$ and $p \to \infty$.

Theorem 3.1. *Assume $B(x_0, \tau) \subset \Omega$. Then*

$$\lambda_1(p) \leq \frac{(N+p)\cdots(p+1)}{N!\tau^p}. \tag{3.1}$$

Moreover, $\lim_{p\to\infty} \lambda_1(p) = 0$ *if* $\tau > 1$.

Proof. Take $\varphi = \tau - |x - x_0|$ if $|x - x_0| \leq \tau$, and $\varphi = 0$ if $|x - x_0| > \tau$. Then

$$\lambda_1(p) \leq \frac{\int |\nabla\varphi|^p}{\int |\varphi|^p}.$$

Calculation shows that

$$\int |\nabla\varphi|^p = \frac{\tau^N}{N}\frac{2\pi^{N/2}}{\Gamma(N/2)},$$

and

$$\int |\varphi|^p = \frac{2\pi^{N/2}}{\Gamma(N/2)}\tau^{p+N}\int_0^1 (1-s)^p s^{N-1} ds,$$

where $\Gamma(\cdot)$ is the Γ-function. It is known that

$$\int_0^1 (1-s)^p s^{N-1} ds = \frac{\Gamma(N)\Gamma(p+1)}{\Gamma(N+p+1)} = \frac{(N-1)!}{(N+p)\cdots(p+1)},$$

hence we obtain

$$\lambda_1(p) \leq \frac{(N+p)\cdots(p+1)}{N!\tau^p}.$$

The theorem then follows. □

We will see later that the condition $\tau > 1$ is optimal.

To obtain an estimate for $\lambda_1(p)$ as $p \to 1$, we first establish the following Barta type inequality.

Lemma 3.1. *For any positve $v \in C_0^2(\Omega)$, let $\beta = \inf_{x\in\Omega}(-\Delta_p v/v^{p-1})$. Then $\lambda_1(p) \geq \beta$. Moreover $\lambda_1(p) > \beta$ if $v \neq cu(1,p)$, where c is a constant.*

Proof. We write $u_1 = u(1,p)$ for short. Observe that

$$-\Delta_p u_1 = \lambda_1(p) u_1^{p-1},$$

and

$$-\Delta_p v \geq \beta v^{p-1}.$$

By using the monotonicity of $-\Delta_p\cdot$ and $(\cdot)^{p-1}$, and integrating by parts, we obtain

$$\begin{aligned} 0 &\leq \int (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla v|^{p-2}\nabla v)(\nabla u_1 - \nabla v) \\ &\leq (\lambda_1(p) - \beta)\int (u_1^{p-1} - v^{p-1})(u_1 - v). \end{aligned}$$

This implies the desired inequality. The proof is completed. □

Theorem 3.2. *Assume that* $|\Omega| = \delta$. *Then for* $1 < p < 2$,

$$\lambda_1(p) \geq \frac{(N+p-2)}{(2(p-1)/p)^{p-1}}\Big(\frac{p}{2-p}\Big)^{1-p/2}\tau^{-p-1}, \tag{3.2}$$

where $\tau = (2\pi^{N/2})^{-1/N}[N\Gamma(N/2)\delta]^{1/N}$. *Moreover,* $\lim_{p\to 1}\lambda_1(p) \geq (N-1)\tau^{-2}$.

Proof. Let Ω^* be the symmetrization of Ω, i.e. Ω^* is radially symmetric and $|\Omega^*| = \delta$. Then $\Omega^* = B(0,\tau)$. Let $\lambda_1^*(p) > 0$ be the first eigenvalue of the p-Laplacian on Ω^*; then $\lambda_1^*(p) \leq \lambda_1(p)$ (cf. Corollary 2.33 of [K2]). Next we estimate $\lambda_1^*(p)$.

Take $v = \tau^2 - |x|^2$ in Ω^*. Then $-\Delta_p v = 2^{p-1}(N+p-2)|x|^{p-2}$. Let $\beta = \inf(-\Delta_p v/v^{p-1})$. One easily checkes that, for $1 < p < 2$, β is attained at $|x| = ((2-p)/p)^{1/2}\tau$ and

$$\beta = \frac{(N+p-2)}{(2(p-1)/p)^{p-1}}\Big(\frac{p}{2-p}\Big)^{1-p/2}\tau^{-p-1}.$$

The theorem then follows from the above estimate and Lemma 3.1. □

We now turn our attention to the one dimensional case and obtain more accurate asymptotic behavior. It is known (see, e.g., Theorem 1 of [GV], and [DEM]) that the k-th eigenvalue of the problem

$$-(|u'|^{p-2}u')' = \lambda(p)|u|^{p-2}u, \quad u(0) = u(L) = 0 \tag{3.3}$$

is given by

$$\begin{aligned}\lambda_k(p) = k^p\lambda_1(p) &= k^pL^{-p}(p-1)\Big[2\int_0^1 (1-t^p)^{-1/p}dt\Big]^p \\ &= k^p(p-1)\Big[\frac{2}{L}\cdot\frac{\pi}{p\sin(\pi/p)}\Big]^p.\end{aligned} \tag{3.4}$$

Theorem 3.3. (i) $\lim_{p\to\infty}\lambda_1(p) = \infty$ *if* $L \leq 2$. (ii) $\lim_{p\to\infty}\lambda_1(p) = 0$ *if* $L > 2$. (iii) $\lim_{p\to 1^+}\lambda_1(p) = 2/L$.

Proof. For $L \neq 2$, (i) and (ii) follow from the fact that $\lim_{p\to\infty}\pi(p\sin(\pi/p))^{-1} = 1$.

For $L = 2$, we observe that, since $\ln(\pi/p) - \ln(\sin(\pi/p)) > 0$,

$$\ln\Big[(p-1)^{1/p}\frac{\pi}{p\sin(\pi/p)}\Big]^p = p\Big[\ln\frac{\pi/p}{\sin(\pi/p)} + \frac{1}{p}\ln(p-1)\Big] \to +\infty$$

as $p \to \infty$. So $\lambda_1(p) \to \infty$.

To prove (iii), we discuss the limit $\lim_{p\to 1^+}(\sin(\pi/p))^{-p}(p-1)$. Let

$$y = (\sin(\pi/p))^p;$$

then

$$\begin{aligned}\ln y &= p\ln\sin(\pi/p) \\ &= p\ln\Big[\frac{\sin(\pi/p)}{(p-1)\pi}\Big] + p\ln[(p-1)\pi] = o(1) + p\ln[(p-1)\pi].\end{aligned}$$

Therefore $y = e^{o(1)}((p-1)\pi)^p$. Observe that

$$\begin{aligned}\lim_{p\to 1^+}\frac{p-1}{(\sin(\pi/p))^p} &= \lim_{p\to 1^+}\frac{(p-1)}{e^{o(1)}(p-1)^p\pi^p} \\ &= \lim_{p\to 1^+}\frac{1}{e^{o(1)}\pi^p}\frac{1}{(p-1)^{p-1}} = \frac{1}{\pi}.\end{aligned}$$

(iii) then follows from (3.4). The theorem is proved. □

Remark 3.1. We see from (i) that the condition $\tau > 1$ in Theorem 3.1 cannot be improved in general. Using (3.4), we can also obtain asymptotic behavior for other $\lambda_k(p)$.

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Department of Mathematical Sciences, University of Memphis, Memphis, Tennessee 38152

E-mail address: `huangy@mathsci.msci.memphis.edu`