HARMONIC POLYNOMIALS AND THE DIVISIBILITY PROBLEM

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Abstract. An easy way to construct a first harmonic polynomial component of any polynomial is given.

If we tried to divide any polynomial \( P(x) \) \( (x \in \mathbb{R}^n) \) by the polynomial \( L(x) \) using the ordinary Euclidean algorithm then for \( n > 1 \) we would meet the problem: What should be a residue? We would not find a reasonable answer if we were keeping in mind that the degree of a residue must be less than the degree of \( L(x) \). Nevertheless we may define a “division” of the polynomial \( P(x) \) by the polynomial \( L(x) \) by the equality \( P(x) = Q(x)L(x) + H(x) \), where the residue \( H(x) \) is determined not as a polynomial of degree less than degree of \( L(x) \) but as a polynomial solution of the equation \( L(D)H(x) = 0 \). Here operator \( L(D) \) is obtained from the polynomial \( L(x) \) by replacing each variable \( x_i \) on the differential operator \( \partial/\partial x_i \). In this case for each polynomial \( P(x) \) there exist the only polynomials \( Q(x) \) and \( H(x) \) such that the equality \( P(x) = Q(x)L(x) + H(x) \) holds under the condition \( L(D)H(x) = 0 \) [1].

If \( L(x) = |x|^2 = x_1^2 + \cdots + x_n^2 \) this fact was proved in [2]. The proof’s method of the above statement does not permit us to construct the polynomial \( H(x) \) by the polynomial \( P(x) \). In the general case it is not a simple problem. Let us consider an easy way to find the polynomial \( H(x) \) for the special form of the polynomial \( L(x) \), i.e., if \( L(x) = |x|^2 \) and \( n > 2 \).

Let \( L(D) \) be the Laplace operator, i.e., \( L(D) = \Delta \).

Lemma. Let \( H_m(x) \) be a homogeneous harmonic polynomial of \( m \)-th degree, \( H_m^*(x) \) be the Kelvin transformation of \( H_m(x) \) \( (H_m^*(x) = |x|^{2-n}H_m(x/|x|^2)) \) and \( (k,2)_m = k(k+2)\cdots(k+2m-2) \). If \( n > 2 \) and \( x \neq 0 \) then the following formula holds:

\[
H_m^*(x) = \frac{(-1)^m}{(n-2,2)_m}H_m(D)|x|^{2-n}.
\]

Proof. We shall employ the induction on \( m \). Set \( m = 1 \). Then for \( k = 1, \ldots, n \) we can easily write the equality

\[
\frac{\partial}{\partial x_k} \frac{|x|^{2-n}}{2-n} = \frac{1}{|x|^{n-2}} \frac{x_k}{|x|^2} = x_k^*,
\]

and therefore for \( m = 1 \) Eq.(1) is true.
Suppose that for \( m < k \) the lemma is true and prove it for \( m = k \). Let us use the Euler formula for homogeneous functions. We get

\[
H_k(x) = \sum_{i=1}^{n} \frac{x_i}{k} H_k^{(i)}(x),
\]

where \( H_k^{(i)}(x) = \partial/\partial x_i H_k(x) \). Obviously the polynomials \( H_k^{(i)}(x) \) are harmonic polynomials of degree \( k - 1 \). Making use of Eq.(2), by the induction hypotheses we can write

\[
H_k(D)|x|^{2-n} = \sum_{i=1}^{n} \frac{1}{k} \frac{\partial}{\partial x_i} H_k^{(i)}(D)|x|^{2-n}
\]

\[
= \sum_{i=1}^{n} \frac{(-1)^{k-1}}{k} (n-2,2)_{k-1} \frac{\partial}{\partial x_i} \left( H_k^{(i)} \right)^{*}(x).
\]

Keeping in mind that

\[
\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( H_k^{(i)} \right)^{*}(x) = -(2k + n - 4) \sum_{i=1}^{n} x_i H_k^{(i)}(x),
\]

we get

\[
H_k(D)|x|^{2-n} = (-1)^k (n-2,2)_{k-1}(2k + n - 4) \left( \sum_{i=1}^{n} \frac{x_i}{k} H_k^{(i)} \right)^{*}(x).
\]

Again, making use of Eq.(2) and observing that \((n-2,2)_{k-1}(2k + n - 4) = (n-2,2)_{k}\) we get Eq.(1), and the proof is complete.

Let \( P(x) \) be an arbitrary polynomial. Represent it in the form \( P(x) = \sum_{m} P_m(x) \).

**Theorem.** Suppose that \( P(x) = Q(|x|^2) + H(x) \) and \( H(x) \) is a harmonic polynomial. Then \( H(x) \) can be found from the equality

\[
H(x) = \sum_{m} (-1)^m \frac{|x|^{2m+n-2}}{(n-2,2)_m} P_m(D)|x|^{2-n}.
\]

**Proof.** Since \( P(x) = Q(|x|^2) + H(x) \), then for \( x \neq 0 \) we get

\[
P_m(D)|x|^{2-n} = H_m(D)|x|^{2-n}.
\]

If we take advantage of the lemma then we easily get the desired result.

**References**

[1] V. V. Karachik, O polinomialnyh reshenijah sistem linejnyh differenzialnyh uravnenij. Voprosi Vychislitelnoy i prikladnoy matematiki. v.82, Tashkent, 1987, s.41-48. MR 91f:34008