LEMPERT MAPPINGS AND SYMPLECTIC FORMS

ZOLTAN BALOGH AND CHRISTOPH LEUENBERGER

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Abstract. We use Lempert’s version of Riemann mapping to construct non-equivalent symplectic forms on an ellipsoid in \( \mathbb{C}^n \).

Any bounded pseudoconvex domain \( D \subseteq \mathbb{C}^n \) admits a strictly plurisubharmonic exhausting function \( u \). If \( u \) is smooth (or at least \( C^2 \)) then \( \omega = dd^c u = 2i \partial \bar{\partial} u \) is a symplectic form on \( D \). The following theorem was proved in [2].

**Theorem A** (Eliashberg, Gromov). Let \( u_1, u_2 \) be two smooth strictly plurisubharmonic exhausting functions on a bounded strictly pseudoconvex domain \( D \):

\[
D = \{ u_1 < \infty \} = \{ u_2 < \infty \}.
\]

Then the forms \( \omega_1 = dd^c u_1 \) and \( \omega_2 = dd^c u_2 \) are symplectomorphic, i.e. there exists a diffeomorphism \( \psi : D \to D \) such that \( \psi^* \omega_2 = \omega_1 \).

As \( u_i(z) \to \infty \) for \( z \to \partial D \) it can easily be seen that the volume of \( D \) with respect to the volume forms \( \Omega_i = \omega_1 \wedge \cdots \wedge \omega_i \) (\( i = 1, 2 \)) is infinite. We will construct an example which shows that Theorem A no longer holds true for the finite volume case, i.e. when we give the domain not by exhausting but by defining functions.

Let \( D \) be a bounded strictly pseudoconvex domain represented by a strictly plurisubharmonic smooth defining function \( u_1 \), i.e. \( D = \{ u_1(z) < 0 \} \subseteq \mathbb{C}^n \) with \( du_1|_{\partial D} \neq 0 \). \( D \) is a symplectic manifold with the 2-form \( \omega_1 = dd^c u_1 \). Observe that the volume of \( D \) with respect to the volume form \( \Omega_1 = \omega_1 \wedge \cdots \wedge \omega_1 \) is finite.

If \( u_2 \) is another defining function for \( D \) that is equal to \( u_1 \) in a neighbourhood of the boundary \( \partial D \) then \( \int_D \Omega_1 = \int_D \Omega_2 \) as follows from Stokes’ theorem. Moreover, using Moser’s deformation argument (see e.g. [1], p.20) we can easily construct a symplectomorphism between \( (D, \omega_1) \) and \( (D, \omega_2) \). If \( u_1 \) and \( u_2 \) are not equal in a neighbourhood of \( \partial D \) the volume condition alone is not sufficient to guarantee the existence of a symplectomorphism as will be shown in the following example.

The basis of our construction is Lempert’s theorem stating that for any strictly convex domain \( D \subseteq \mathbb{C}^n \) there is a homeomorphism of the unit ball \( \rho : B \to D \) which has many similar properties to the usual Riemann mapping in \( \mathbb{C} \). The property we need here is that the function \( u : D \to R, u(z) = \frac{1}{2}(|\rho^{-1}(z)|^2 - 1) \) is a plurisubharmonic defining function of \( D \) and moreover the relation \( dd^c u = \rho^{-1}_* \omega_1 \) holds where \( \omega_1 = \sum dy_j \wedge dx_i = \frac{1}{2}dd^c(|z|^2 - 1) \) is the standard symplectic form in \( \mathbb{C}^n \) (see e.g. [5], Lemma 2.6).

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We are going to use furthermore the fundamental theorem of Gromov which is according to A. Weinstein the geometric expression of the uncertainty principle (see e.g. [4], p.358 for explanation). Gromov’s theorem states that if there exists a symplectic embedding 
\[(B(r), \omega_1) \hookrightarrow (Z_1(R), \omega_1)\]
of the ball in \(\mathbb{R}^{2n}\) of radius \(r\) into the cylinder of radius \(R\)
\[Z_1(R) = \{(x_1, y_1, \ldots, x_n, y_n) \in \mathbb{R}^{2n} : x_1^2 + y_1^2 < R^2\}\]
then necessarily \(r \leq R\).

Let \(A : \mathbb{C}^2 \rightarrow \mathbb{C}^2, A(x_1, y_1, x_2, y_2) = (ax_1, ay_1, b_1 x_2, b_2 y_2)\) be a real linear map and \(B = \{|z| < 1\}\) be the unit ball in \(\mathbb{C}^2\). Consider the real ellipsoid
\[E = A^{-1}(B) = \{z \in \mathbb{C}^2 : |A(z)|^2 - 1 < 0\}\]
\[= \{x \in \mathbb{R}^4 : a^2 x_1^2 + a^2 y_1^2 + b_1^2 x_2^2 + b_2^2 y_2^2 - 1 < 0\}.\]

We are going to construct two plurisubharmonic defining functions \(u_1\) and \(u_2\) for \(E\) and consider the symplectic forms \(\omega_i = dd^c u_i, i = 1, 2\). These will have the properties:

- \(u_i\) are smooth and \(\int_E \omega_1 \wedge \omega_1 = \int_E \omega_2 \wedge \omega_2;\)
- there exists no diffeomorphism \(\psi : E \rightarrow E, \psi^* \omega_2 = \omega_1.\)

First we choose the positive numbers \(a, b_1, b_2\) carefully to fulfill the relations
1. \(2a^2 = b_1^2 + b_2^2,\)
2. \(a^2 b_1 b_2 = 1,\)
3. \(a^2 > 1\) and \(b_1 b_2 < 1.\)

(For example, choose \(a = 2, b_1 = \frac{1}{2\sqrt{2}}(\sqrt{15} + \sqrt{17}), b_2 = \frac{1}{4b_1}.\))

The function \(u_1(z) = \frac{1}{2a^2}(|A(z)|^2 - 1)\) is a plurisubharmonic defining function for \(E\) and by relation 1 we have \(\omega_1 = dd^c u_1 = \sum_{i=1}^2 dy_i \wedge dx_i.\)

On the other hand we consider the Lempert mapping \(\rho : B \rightarrow E, \rho(0) = 0,\) which exists as \(E\) is strictly convex, and put \(u_2(z) = \frac{1}{4}(|\rho^{-1}(z)|^2 - 1).\) Then \(u_2\) is another plurisubharmonic defining function for \(E\) and \(\omega_2 = \rho^* \omega_1\) where \(\omega_1\) is now the standard symplectic structure on the unit ball \(B\). The forms \(\omega_1\) and \(\omega_2\) give the same volume for \(E\). Indeed by relation 2:

\[
\text{vol}_{\omega_1}(E) = \int_E \omega_1 \wedge \omega_1 = \int_B (A^{-1})^* (\omega_1 \wedge \omega_1) = \frac{1}{a^2 b_1 b_2} \int_B \omega_1 \wedge \omega_1 = \int_B \omega_1 \wedge \omega_1, 
\]

\[\text{vol}_{\omega_2}(E) = \int_E \omega_2 \wedge \omega_2 = \int_{\rho^{-1}(E)} \rho^* (\omega_2 \wedge \omega_2) = \int_B \omega_1 \wedge \omega_1.\]

We claim that there is no symplectomorphism \(\psi : (E, \omega_2) \rightarrow (E, \omega_1).\) For if such \(\psi\) existed we could consider the map \(\phi = \psi \circ \rho : B \rightarrow E\) which has the property \(\phi^* \omega_1 = (\rho^* \circ \psi^*) \omega_1 = \rho^* \omega_2 = \omega_1.\)

On the other hand \(E\) is contained in the cylinder
\[Z_1\left(\frac{1}{a}\right) = \{(x \in \mathbb{R}^4 : x_1^2 + y_1^2 < \frac{1}{a^2}\} \subseteq \mathbb{C}^2\]
and we have obtained an embedding
\[\phi : B \hookrightarrow Z_1\left(\frac{1}{a}\right), \quad \phi^* \omega_1 = \omega_1.\]
But as $a > 1$ by property 3 this is a contradiction to Gromov’s theorem.

The deficiency of this construction is that the 2-form $\omega_2 = \rho^{-1} \omega_1$ might have discontinuous coefficients at $0 \in E$ as Lempert mappings $\rho$ are generally not smooth at the origin. But the situation can be rescued by pushing the singularity near the boundary with a suitable symplectic mapping.

To be more precise, denote by $B(r)$ the open ball of radius $r$: $B(r) = \{ |z| < r \} \subseteq \mathbb{C}^n$. As $\rho : B \rightarrow E$ is a homeomorphism and $\rho(0) = 0$ we can choose a small neighbourhood $V$ of the origin in $E$ such that $\rho^{-1}(V) \subseteq B(\epsilon)$ for a given $\epsilon$. Applying standard convolution arguments we can smoothen $u_2$ to a $C^\infty$ plurisubharmonic function $\tilde{u}_2$ coinciding with $u_2$ outside of $V$.

By Stokes’ theorem the volume of $E$ with respect to $\tilde{\omega}_2 = dd^c \tilde{u}_2$ will still be the same. Now choose a compactly supported smooth Hamiltonian function $H$ on $B = B(1)$ with

$$H = \begin{cases} x_1 & \text{on } B(1 - 2\epsilon), \\ 0 & \text{on } B \setminus B(1 - \epsilon). \end{cases}$$

Let $h$ be the time-$(1 - 3\epsilon)$-map of the Hamiltonian vectorfield $X_H$ defined by

$$\iota_{X_H} \omega_1 = dH.$$ 

As $X_H = \frac{\partial}{\partial \eta_1}$ on $B(1 - 2\epsilon)$ the symplectomorphism $h$ maps the ball $B(\epsilon)$ around the origin into $B \setminus B(1 - 4\epsilon)$.

Let us now assume the existence of a diffeomorphism $\psi : E \rightarrow E$ with $\psi^* \omega_1 = \tilde{\omega}_2$. As before we put $\phi = \psi \circ \rho : B(1) \rightarrow E$ and now the mapping $\phi \circ h^{-1}|_{B(1 - 4\epsilon)}$ again yields a symplectic embedding

$$(B(1 - 4\epsilon), \omega_1) \hookrightarrow (Z_{1}(\frac{1}{a}), \omega_1).$$

Choosing $\epsilon$ small enough we arrive again at a contradiction to Gromov’s theorem.

Thus we obtain

**Theorem 1.** There exist two smooth defining functions $u_1$ and $u_2$ of a strictly pseudoconvex domain $E$ such that their 2-forms $\omega_1$, $\omega_2$ give rise to the same volume but they are not symplectically equivalent.$\square$

Let us finish this note with the following simple observation:

**Proposition 1.** Let $D \subseteq \mathbb{C}^n$ be a complete circular domain with defining function $u$. If $dd^c u = \omega_1$ then $D$ is a ball.

**Proof.** Since $\omega_1 = \sum dy_i \wedge dx_i = dd^c u_0$ for $u_0 = \frac{1}{4} \sum |z_i|^2$ the function $h = u_0 - u$ is pluriharmonic in $D$ and satisfies the following Dirichlet-type problem:

$$\begin{cases} dd^c h = 0, \\ h|_{\partial D} = u_0|_{\partial D}. \end{cases}$$ (2)

Let $L_p$ be a complex line through the origin and a point $p \in \partial D$. Then $L_p \cap D = p \cdot \Delta = \{ p \zeta : \zeta \in \Delta \}$ as $D$ is complete circular. But $h|_{p \cdot \Delta}$ is harmonic and $h|_{\partial (p \cdot \Delta)} = \frac{1}{2} |p|^2$. Hence $h \equiv \frac{1}{2} |p|^2$; in particular $h(0) = u(0) = \frac{1}{4} |p|^2$. This proves that $|p|^2$ is independent of the choice of $p$. $\square$
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REFERENCES


MATHMATICS INSTITUTE, UNIVERSITY OF BERN, SIDLERSTRASSE 5, 3012 BERN, SWITZERLAND

E-mail address: zoltan@math-stat.unibe.ch
E-mail address: leuenb@math-stat.unibe.ch