

ON THE CLASS OF NORM LIMITS OF NILPOTENTS

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. It is known that every operator on a Hilbert space \mathcal{H} whose invariant subspace lattice is possibly $\{(0), \mathcal{H}\}$ is a norm-limit of a sequence of nilpotent operators. In this note we study properties of such approximating sequences.

1. INTRODUCTION

Let \mathcal{H} denote a separable, infinite dimensional, complex Hilbert space and $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on \mathcal{H} . If $T \in \mathcal{L}(\mathcal{H})$ we write, as usual, $\sigma(T)$ for the spectrum of T and $\sigma_e(T)$ for the essential (i.e., Calkin) spectrum of T . Recall that an operator T in $\mathcal{L}(\mathcal{H})$ is called *quasitriangular* [3] if there exists an increasing sequence $\{P_n\}_{n=1}^\infty$ of finite-rank projections converging strongly to $I_{\mathcal{H}}$ such that

$$\|P_n T P_n - T P_n\| \rightarrow 0,$$

and T is called *biquasitriangular* (notation: $T \in \mathcal{BQT}(\mathcal{H})$) if both T and T^* are quasitriangular. In this note we study a certain subset $\mathcal{C}(\mathcal{H})$ of $\mathcal{L}(\mathcal{H})$, defined as follows.

Definition. An operator T in $\mathcal{L}(\mathcal{H})$ belongs to the class $\mathcal{C}(\mathcal{H})$ if $T \in \mathcal{BQT}(\mathcal{H})$, both $\sigma(T)$ and $\sigma_e(T)$ are connected subsets of the complex plane \mathbf{C} , and $0 \in \sigma_e(T)$.

A first reason that the class $\mathcal{C}(\mathcal{H})$ is interesting is that it has the following beautiful characterization, due to Apostol, Foiaş, and Voiculescu [2].

Theorem 1.1. *An operator T in $\mathcal{L}(\mathcal{H})$ belongs to $\mathcal{C}(\mathcal{H})$ if and only if there is a sequence $\{N_k\}_{k \geq 1}$ of nilpotent operators in $\mathcal{L}(\mathcal{H})$ such that $\|N_k - T\| \rightarrow 0$.*

A second reason that the class $\mathcal{C}(\mathcal{H})$ is interesting is that the invariant subspace problem for operators in $\mathcal{L}(\mathcal{H})$ is equivalent to the invariant subspace problem for operators in $\mathcal{C}(\mathcal{H})$; cf. [4, Chapter 6]:

Theorem 1.2. *Every operator in $\mathcal{L}(\mathcal{H}) \setminus \mathcal{C}(\mathcal{H})$ has a nontrivial hyperinvariant subspace (n.h.s.).*

For completeness, we briefly sketch the proof of this result. First, an operator T in $\mathcal{L}(\mathcal{H})$ belongs to $\mathcal{BQT}(\mathcal{H})$ if and only if for any complex number λ such that $T - \lambda I_{\mathcal{H}}$ is a semi-Fredholm operator, the (Fredholm) index of $T - \lambda I_{\mathcal{H}}$ is

Received by the editors January 30, 1996 and, in revised form, June 26, 1996.

1991 *Mathematics Subject Classification.* Primary 47A15, 47A65.

Key words and phrases. Sequences of nilpotents, invariant subspaces.

equal to zero [1]. It follows trivially that every operator in $\mathcal{L}(\mathcal{H}) \setminus \mathcal{BQT}(\mathcal{H})$ has a n.h.s. For, every such operator or its adjoint has point spectrum, and thus a n.h.s. Furthermore, if $\sigma(T) \neq \sigma_e(T)$, then T is not a scalar and either T or T^* has point spectrum (and therefore a n.h.s.), so it suffices to consider those T in $\mathcal{BQT}(\mathcal{H})$ such that $\sigma(T) = \sigma_e(T)$. Moreover, if $\sigma(T)$ is not connected, then, by integrating around some connected component of $\sigma(T)$, one easily produces a nontrivial idempotent that commutes with the commutant of T , and thus a n.h.s. for T . Finally, since translation of an operator by a scalar preserves the existence of a n.h.s., one may suppose that $0 \in \sigma(T)$, which completes the sketch.

If we write $\mathcal{N} = \mathcal{N}(\mathcal{H})$ for the class of all nilpotent operators in $\mathcal{L}(\mathcal{H})$, and \mathcal{S}^- for the norm-closure of a subset \mathcal{S} of $\mathcal{L}(\mathcal{H})$, then Theorem 1.2 can be paraphrased by saying that if $T \in \mathcal{L}(\mathcal{H})$ and T is not known to have a n.h.s., then $T \in \mathcal{N}(\mathcal{H})^- = \mathcal{C}(\mathcal{H})$. Thus, for such a T one knows that there exists a sequence $\{N_k\}_{k \geq 1}$ of nilpotent operators from $\mathcal{L}(\mathcal{H})$ such that $\|N_k - T\| \rightarrow 0$.

The purpose of this note is to study the properties of such an approximating sequence $\{N_k\}_{k \geq 1}$ and, in particular, to derive as much structure as possible for this sequence, with the hope that nice properties of the sequence might, in the future, lead to invariant subspaces for the limit operator T .

2. APPROXIMATING SEQUENCES

Let us fix an arbitrary operator T in $\mathcal{N}(\mathcal{H})^-$ and a sequence $\{N_k\}_{k \geq 1}$ of nilpotent operators in $\mathcal{L}(\mathcal{H})$ such that

$$\|N_k - T\| \rightarrow 0.$$

Let $m_k \geq 1$ be the index of nilpotence of each N_k . For the purpose of showing that T has a n.h.s., we may suppose that $\lim_k(m_k) = +\infty$. (For otherwise there exist a natural number p and a subsequence $\{m_{k_n}\}_{n \geq 1}$ of $\{m_k\}$ such that $m_{k_n} \leq p$ for each $n \in \mathbb{N}$. Consequently, since $\|T^p - N_{k_n}^p\| \rightarrow 0$, T is nilpotent of index at most p and has a n.h.s. for trivial reasons.) Moreover, there is obviously no loss of generality in supposing that the sequence $\{m_k\}$ is strictly increasing.

Lemma 2.1. *Let N be an operator in $\mathcal{N}(\mathcal{H})$ having index of nilpotence $m > 1$. Then there exists a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m$ such that $\dim \mathcal{H}_i = \aleph_0$, $i = 1, \dots, m$, and such that with respect to this decomposition the matrix of N has the strictly lower triangular form*

$$(1) \quad N = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ N_{2,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ N_{m,1} & N_{m,2} & \cdots & 0 \end{pmatrix},$$

(where, of course, for $1 \leq i, j \leq m$, $N_{i,j} \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i)$).

Proof. The proof is by induction on the index m of nilpotence of N . We begin by noting that for an arbitrary $m > 1$, we may write $\mathcal{H} = \mathcal{H}'_1 \oplus \cdots \oplus \mathcal{H}'_m$, where $\mathcal{H}'_i = \text{Ker} \{(N^*)^i\} \ominus \text{Ker} \{(N^*)^{i-1}\} = (\text{Ran } N^{i-1})^- \ominus (\text{Ran } N^i)^-$, $i = 1, \dots, m$.

Elementary considerations show that with respect to this decomposition of \mathcal{H} , the matrix of N will have the form

$$N = \begin{pmatrix} 0 & 0 & \dots & 0 \\ N'_{2,1} & 0 & \dots & 0 \\ N'_{3,1} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ N'_{m,1} & N'_{m,2} & \dots & 0 \end{pmatrix}$$

where, of course, for $1 \leq i, j \leq m$, $N'_{i,j}$ maps \mathcal{H}'_j into \mathcal{H}'_i . (Easy examples show, however, that the dimension of some \mathcal{H}'_i may be finite.) For $m = 2$, with respect to the decomposition $\mathcal{H} = \mathcal{H}'_1 \oplus \mathcal{H}'_2$ the matrix of N has the form

$$(2) \quad N = \begin{pmatrix} 0 & 0 \\ N'_{2,1} & 0 \end{pmatrix},$$

where $\mathcal{H}'_1 = \mathcal{H} \ominus (\text{Ran } N)^- = \text{Ker } N^*$ (which is infinite dimensional, being the kernel of a nilpotent operator), and $\mathcal{H}'_2 = (\text{Ran } N)^- \subset \text{Ker } N$. If $\dim \mathcal{H}'_2 = \aleph_0$, then setting $\mathcal{H}_1 = \mathcal{H}'_1$ and $\mathcal{H}_2 = \mathcal{H}'_2$ completes the proof in this case. On the other hand, if N has finite rank, then we may write $\text{Ker } N = \mathcal{K}_1 \oplus \mathcal{K}_2$ where $\dim \mathcal{K}_1 = \dim \mathcal{K}_2 = \aleph_0$ and $\text{Ran } N \subset \mathcal{K}_2$. Now set $\mathcal{H}_1 = \mathcal{H} \ominus \mathcal{K}_2$ and $\mathcal{H}_2 = \mathcal{K}_2$. Then $\mathcal{H}_1 \subset \mathcal{H}'_1$ and clearly \mathcal{H}_1 and \mathcal{H}_2 are infinite dimensional subspaces such that with respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, N has a matrix of the desired form.

Suppose now that the index of nilpotence m of N is greater than 2 and that the lemma has been proved for all nilpotent operators with index of nilpotence less than m . With respect to the decomposition $\mathcal{H} = \text{Ker } N^* \oplus \text{Ran } N^-$, the matrix of N has the form

$$N = \begin{pmatrix} 0 & 0 \\ N' & N'' \end{pmatrix}.$$

It is easy to see that N'' is nilpotent of index $m - 1$. Once again, one knows that $\dim \text{Ker } N^* = \aleph_0$, and if N is not of finite rank, we may apply the induction hypothesis to the operator N'' , and conclude that there exists a decomposition $\text{Ran } N^- = \mathcal{H}_2 \oplus \dots \oplus \mathcal{H}_m$ such that each \mathcal{H}_i , $2 \leq i \leq m$, is infinite dimensional and the matrix of N'' with respect to this decomposition has the form (1). By setting $\mathcal{H}_1 = \text{Ker } N^*$, we obtain the desired decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_m$. On the other hand, if N has finite rank, then the same idea used in the case $m = 2$ above, together with another application of the induction hypothesis, provides the desired decomposition.

Lemma 2.2. *Let $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ where \mathcal{K} has dimension \aleph_0 . Then for any $\varepsilon > 0$ there are decompositions $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$ and an operator $T' \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ such that $\mathcal{H}_1, \mathcal{H}_2, \mathcal{K}_1, \mathcal{K}_2$ are infinite dimensional, $\|T - T'\| < \varepsilon$, and the matrix of T' with respect to the above decompositions has the form*

$$T' = \begin{pmatrix} * & T'_{1,2} \\ * & * \end{pmatrix},$$

where $T'_{1,2}$ is an invertible operator in $\mathcal{L}(\mathcal{H}_2, \mathcal{K}_1)$.

Proof. If T is a compact operator, we choose arbitrary decompositions $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$ where the \mathcal{H}_i and \mathcal{K}_i are infinite dimensional, $i = 1, 2$. With respect to these decompositions, the matrix of T has the form

$$T = \begin{pmatrix} * & T_{1,2} \\ * & * \end{pmatrix},$$

where $T_{1,2}$ is compact. Therefore, for any $\varepsilon > 0$ there exists λ satisfying $0 < \lambda < \varepsilon$ such that $T'_{1,2} = T_{1,2} + \lambda$ is invertible, and hence one may take T' to have the same matrix as T except that $T_{1,2}$ is replaced by $T'_{1,2}$.

If T is not compact, we consider the polar decomposition $T = U|T|$, and, of course, $|T|$ is not compact either. Let $E(\cdot)$ be the spectral measure of $|T|$. Then there exists $\theta > 0$ such that the range of $E[\theta, +\infty)$ is an infinite dimensional subspace of \mathcal{H} , and the restriction of T to the range \mathcal{M} of $E[\theta, +\infty)$ is an invertible operator in $\mathcal{L}(\text{Ran } E[\theta, +\infty), \text{Ran } T|_{\mathcal{M}})$. Now we decompose $\mathcal{M} = \mathcal{L}_1 \oplus \mathcal{L}_2$ with both \mathcal{L}_1 and \mathcal{L}_2 infinite dimensional, and define $\mathcal{H}_1 = \mathcal{L}_2^\perp$, $\mathcal{H}_2 = \mathcal{L}_2$, $\mathcal{K}_2 = T\mathcal{H}_2$, and $\mathcal{K}_1 = \mathcal{K}_2^\perp$. It is clear that $T_{21} = T|_{\mathcal{H}_2}$ is invertible, and the lemma is proved.

Lemma 2.3. *Let N be an operator in $\mathcal{N}(\mathcal{H})$ with index of nilpotence $m > 1$, and let ε be any positive number. Then there exist $N' \in \mathcal{N}(\mathcal{H})$ with index of nilpotence $p \geq 2m$ and a decomposition $\mathcal{H} = \mathcal{K}_1 \oplus \dots \oplus \mathcal{K}_p$ such that $\|N - N'\| < \varepsilon$, $\dim \mathcal{K}_i = \aleph_0$, $i = 1, \dots, p$, and the matrix of N' with respect to this decomposition has the strictly lower triangular form*

$$N' = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ N_{2,1} & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ N_{p-1,1} & N_{p-1,2} & \ddots & 0 & 0 \\ N_{p,1} & N_{p,2} & \dots & N_{p,p-1} & 0 \end{pmatrix},$$

where $N_{2,1} = 0$ and $N_{i+1,i}$ is either invertible or zero for each $i = 2, \dots, p-1$.

Proof. We prove the lemma by induction on the index of nilpotence m of N . First we assume that $m = 2$. By Lemma 2.1, there exists a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ such that $\dim \mathcal{H}_1 = \dim \mathcal{H}_2 = \aleph_0$ and the matrix of N with respect to this decomposition has the form

$$N = \begin{pmatrix} 0 & 0 \\ N'_{2,1} & 0 \end{pmatrix}.$$

Applying Lemma 2.2 to the operator $N'_{2,1} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$, we get some infinite dimensional subspaces $\mathcal{K}_1, \dots, \mathcal{K}_4$ such that $\mathcal{H}_1 = \mathcal{K}_1 \oplus \mathcal{K}_2$, $\mathcal{H}_2 = \mathcal{K}_3 \oplus \mathcal{K}_4$, and an operator $N''_{2,1}$ with $\|N'_{2,1} - N''_{2,1}\| < \varepsilon$ such that with respect to these decompositions, the matrix of $N''_{2,1}$ has the form

$$N''_{2,1} = \begin{pmatrix} * & N_{2,1} \\ * & * \end{pmatrix}$$

where $N_{2,1}$ is invertible. With respect to the decomposition $\mathcal{H} = \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \mathcal{K}_3 \oplus \mathcal{K}_4$ the matrix of N has the form

$$(3) \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & N_{2,1} & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix},$$

so the proof is complete in case $m = 2$. Now suppose that $m > 2$ and that the lemma has been proved for all nilpotent operators of index less than m . We apply Lemma 2.1 to N to obtain a decomposition $\mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_m$ and a matrix for N with respect to this decomposition of the form (1). The operator $N|_{\mathcal{H}_2 \oplus \dots \oplus \mathcal{H}_m} \in \mathcal{L}(\mathcal{H}_2 \oplus \dots \oplus \mathcal{H}_m)$ is easily seen to be nilpotent of index $m - 1$. Applying the induction hypothesis, we get a decomposition $\mathcal{H}_2 \oplus \dots \oplus \mathcal{H}_m = \mathcal{K}'_4 \oplus \mathcal{K}_5 \oplus \dots \oplus \mathcal{K}_p$, where the direct sum on the right has q summands (for some $q \geq 2m - 2$) all of which are infinite dimensional, such that the matrix of $N|_{\mathcal{H}_2 \oplus \dots \oplus \mathcal{H}_m}$ has the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ * & N_{6,5} & 0 & 0 & \dots & 0 \\ * & * & N_{7,6} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & * & \dots & 0 \end{pmatrix},$$

where $N_{6,5}$ is invertible and each subdiagonal entry $N_{6+i,5+i}$ is either 0 or is invertible. With respect to the decomposition $\mathcal{H}_1 \oplus \mathcal{K}'_4 \oplus \mathcal{K}_5 \oplus \dots \oplus \mathcal{K}_p$, the matrix of N has the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ A_{2,1} & 0 & 0 & 0 & \dots & 0 \\ * & 0 & 0 & 0 & \dots & 0 \\ * & * & N_{6,5} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & * & \dots & 0 \end{pmatrix},$$

where $A_{2,1}$ is the compression of $N_{2,1}$ to the subspace \mathcal{K}'_4 . Next we apply a similar argument like that in the proof of the case $m = 2$ to the operator

$$N''' = \begin{pmatrix} 0 & 0 \\ A_{2,1} & 0 \end{pmatrix},$$

and we get some infinite dimensional subspaces $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4$ such that $\mathcal{H}_1 = \mathcal{K}_1 \oplus \mathcal{K}_2, \mathcal{K}'_4 = \mathcal{K}_3 \oplus \mathcal{K}_4$, and the matrix of N''' with respect to the decomposition $\mathcal{H}_1 \oplus \mathcal{K}'_4 = \mathcal{K}_1 \oplus \dots \oplus \mathcal{K}_4$ has the form (3). With respect to the decomposition $\mathcal{H} = \mathcal{K}_1 \oplus \dots \oplus \mathcal{K}_p$ with $p = q - 1 + 4 \geq 2m - 3 + 4 = 2m + 1 \geq 2m$, the matrix of N has the desired form. Thus the lemma is proved.

Corollary 2.4. *Let N be an operator in $\mathcal{N}(\mathcal{H})$ with index of nilpotence $m > 1$, and let ε be any positive number. Then there exist N' in $\mathcal{N}(\mathcal{H})$ with index of nilpotence $p \geq 2m$, and a decomposition $\mathcal{H} = \mathcal{K}_1 \oplus \dots \oplus \mathcal{K}_p$ such that $\|N - N'\| < \varepsilon$, $\dim \mathcal{K}_i = \aleph_0, i = 1, \dots, p$, and the matrix of N' with respect to this decomposition*

has the strictly lower triangular form

$$(4) \quad N' = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ N_{2,1} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ N_{p-1,1} & N_{p-1,2} & \dots & 0 & 0 \\ N_{p,1} & N_{p,2} & \dots & N_{p,p-1} & 0 \end{pmatrix},$$

where $N_{i+1,i}$, is invertible for $i = 1, \dots, p-1$.

Proof. Since in the previous lemma the subdiagonal operators $N_{i+1,i}$ are either invertible or zero, we can replace those operators $N_{i+1,i}$ which are zero by $(\varepsilon/2)U_{i+1,i}$, where $U_{i+1,i}$ is a unitary operator from \mathcal{H}_i onto \mathcal{H}_{i+1} and thereby obtain the desired approximation.

Proposition 2.5. *Let N be a nilpotent operator such that the matrix representation of N with respect to a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_m$, $m > 1$, has each subdiagonal entry $N_{i+1,i}$, $i = 1, \dots, m-1$, invertible. Then N is unitarily equivalent to a nilpotent operator N' such that each subdiagonal entry $N'_{i+1,i}$, $i = 1, \dots, m-1$, is a positive operator.*

Proof. We prove the proposition by induction with respect to the index of nilpotence m . If $m = 2$, then $N = \begin{pmatrix} 0 & 0 \\ N_{2,1} & 0 \end{pmatrix}$, with $N_{2,1}$ invertible. Then the polar decomposition $N_{2,1} = U_{2,1}P_{2,1}$ has $U_{2,1}$ unitary operator. Consider $U = \begin{pmatrix} I & 0 \\ 0 & U_{2,1} \end{pmatrix}$. Then $U^*NU = \begin{pmatrix} 0 & 0 \\ P_{2,1} & 0 \end{pmatrix}$. Suppose that the proposition is true for any index k , $1 < k \leq m$, and we prove that it remains true for $m+1$. Let $N = \begin{pmatrix} 0 & 0 \\ N_1 & N_2 \end{pmatrix}$ be the matrix form of N with respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1^\perp$. The operator N_2 is nilpotent of index m , and thus by the induction hypothesis there exists a unitary operator U' on $\mathcal{H}_2 \oplus \dots \oplus \mathcal{H}_{m+1}$ such that

$$U'^*N_2U' = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ P_{3,2} & 0 & 0 & \dots & 0 & 0 \\ * & P_{4,3} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ * & * & * & \ddots & 0 & 0 \\ * & * & * & \dots & P_{m+1,m} & 0 \end{pmatrix},$$

with $P_{i+1,i}$, $i = 2, \dots, m$, positive operators. Consider $U'' = I \oplus U_{2,1} \oplus I \oplus \dots \oplus I$, where $N_{2,1} = U_{2,1}P_{2,1}$ is the polar decomposition of the invertible operator $N_{2,1}$. Then

$$U''^*(I \oplus U')^*N(I \oplus U')U''$$

has the desired matrix form, so the proof is complete.

The following is our main result on the structure of approximating sequences of nilpotents.

Theorem 2.6. *Let T be an operator in $\mathcal{N}(\mathcal{H})^- \setminus \mathcal{N}(\mathcal{H})$. Then there exist a sequence $\{N_k\}_{k \geq 1}$ of nilpotent operators, a strictly increasing sequence $\{m_k\}$ of positive integers, and a sequence $\{\mathcal{H} = \mathcal{K}_1^{(k)} \oplus \dots \oplus \mathcal{K}_{m_k}^{(k)}\}$ of decompositions of \mathcal{H} such that*

- (a) m_k is the index of nilpotence of N_k ,
- (b) $\|N_k - T\| \rightarrow 0$,
- (c) $\dim \mathcal{K}_i^{(k)} = \aleph_0$ for each positive integer k , and each $i = 1, \dots, m_k$,
- (d) the elements $(N_k)_{i+1,i}$, $1 \leq i \leq m_k - 1$, of the matrix N_k corresponding to the decomposition $\mathcal{H} = \mathcal{K}_1^{(k)} \oplus \dots \oplus \mathcal{K}_{m_k}^{(k)}$ are invertible positive operators,
- (e) $\bigcup_{k=1}^\infty \text{Ran} N_k \neq \mathcal{H}$.

Proof. The first four conclusions of this corollary come directly from the previous lemmas. To prove (e), note that it follows easily from (d) that for each positive integer k , the range of N_k is the subspace $(0) \oplus \mathcal{K}_2^{(k)} \oplus \dots \oplus \mathcal{K}_{m_k}^{(k)}$ which is a proper subspace of \mathcal{H} (necessarily nowhere dense in \mathcal{H}). By the Baire category theorem, $\bigcup_{k=1}^\infty \text{Ran} N_k \neq \mathcal{H}$.

In the next proposition we show that the nilpotent operators N_k appearing above are all similar to operators acting on a direct sum of finite number of copies of \mathcal{H} and having matrices of the form

$$(5) \quad J = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ I & 0 & 0 & \dots & 0 \\ 0 & I & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I & 0 \end{pmatrix}.$$

where each entry on the first subdiagonal of J is the identity operator. Such a matrix is called a Jordan block [5]. More precisely, we have the following.

Proposition 2.7. *Let $N \in \mathcal{L}(\mathcal{H} = \mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_m)$ be a nilpotent operator of index $m > 1$ of the form (4) where each $N_{i+1,i}$ is invertible. Then there exist an invertible operator S from $\mathcal{H}^{(m)}$ (the direct sum of m copies of \mathcal{H}) to \mathcal{H} and a Jordan block operator J in $\mathcal{L}(\mathcal{H}^{(m)})$ as in (5), such that $S^{-1}NS = J$.*

Proof. The proof is by induction on the index of nilpotence m . If $m = 2$, then

$$N = \begin{pmatrix} 0 & 0 \\ N_{2,1} & 0 \end{pmatrix},$$

where $N_{2,1}$ is invertible in $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. Let \mathcal{H}_1 and \mathcal{H}_2 be identified with \mathcal{H} via Hilbert space isomorphisms. Then N is unitarily equivalent to an operator $N' \in \mathcal{L}(\mathcal{H}^{(2)})$ whose matrix has the form $\begin{pmatrix} 0 & 0 \\ M & 0 \end{pmatrix}$ with M invertible. Then

$$\begin{pmatrix} M & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & 0 \\ M & 0 \end{pmatrix} \begin{pmatrix} M^{-1} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}.$$

Next suppose that $m \geq 3$ and that the conclusion of Lemma 2.6 is true for any nilpotent operator with index less than m . Let N' be a nilpotent operator of index m with matricial form (4) and satisfying the hypothesis of the lemma. As above, we can identify $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_m$ with \mathcal{H} . Then N is unitarily equivalent to $N' \in$

$\mathcal{L}(\mathcal{H}^{(m)})$ whose matrix has the form

$$N' = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ N'_{2,1} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ N'_{m,1} & N'_{m,2} & \dots & N'_{m,m-1} & 0 \end{pmatrix}$$

with $N'_{2,1}, N'_{3,2}, \dots, N'_{m,m-1}$ invertible. By applying the induction hypothesis to the $(m-1) \times (m-1)$ matrix that is the upper left-hand corner of N' , say N'' , we get an invertible operator S' in $\mathcal{L}(\mathcal{H}^{(m-1)})$ such that $S'^{-1}N''S' = J'$, where J' is the $(m-1) \times (m-1)$ Jordan block. Considering $S_1 = \begin{pmatrix} S' & 0 \\ 0 & I \end{pmatrix}$ an invertible operator in $\mathcal{L}(\mathcal{H}^{(m)})$, then

$$S_1^{-1}N'S_1 = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ I & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ M_1 & M_2 & \dots & M_{m-1} & I \end{pmatrix}$$

with M_{m-1} invertible. (We should point here that M_{m-1} is invertible because of the lower triangular matrix form of S' .) We put now

$$S_2 = \begin{pmatrix} I & 0 & \dots & 0 & 0 \\ 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I & 0 \\ 0 & M_1 & \dots & M_{m-2} & I \end{pmatrix}.$$

Then

$$S_2^{-1}(S_1^{-1}N'S_1)S_2 = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ I & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I & 0 \\ 0 & 0 & \dots & M_{m-1} & 0 \end{pmatrix}.$$

Finally, we take $S_3 = I \oplus \dots \oplus I \oplus M_{m-1} \in \mathcal{L}(\mathcal{H}^{(m)})$. After a computation we get $S_3^{-1}[S_2^{-1}(S_1^{-1}N'S_1)S_2]S_3 = J$, where J is the $m \times m$ Jordan block and $S_1S_2S_3$ has lower triangular matrix form. Thus the proof is done.

We write $\mathcal{H} = \mathcal{K}_1 \dot{+} \dots \dot{+} \mathcal{K}_t$ to mean that \mathcal{H} is the (not necessarily orthogonal) direct sum of the subspaces \mathcal{K}_i .

Corollary 2.8. *Let T be an operator in $\mathcal{N}(\mathcal{H}) \setminus \mathcal{N}(\mathcal{H})$. Then there exist a sequence $\{N_k\}_{k \geq 1}$ of nilpotent operators, a strictly increasing sequence $\{m_k\}$ of positive integers, and a sequence $\{\mathcal{H} = \mathcal{K}_1^{(k)} \dot{+} \dots \dot{+} \mathcal{K}_{m_k}^{(k)}\}$ of direct sum decompositions of \mathcal{H} with (a), (b), (c), (e) as in Theorem 2.6 and also (d') each matrix N_k is a Jordan block.*

This note raises some problems that would seem to be interesting:

1) Can one use the properties of an approximating sequence of nilpotents to establish the existence of an invariant subspace for the limit operator?

1') Suppose T is the limit of a sequence of Jordan block matrices. Does T have a n.i.s?

2) Can additional properties of an approximating sequence of nilpotents be deduced when a) T is quasinilpotent? b) $T = N + K \in \mathcal{N}(\mathcal{H})^-$, where N is normal and K is compact?

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