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NON-SEPARABLE SURFACES IN CUBED MANIFOLDS

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Abstract. We show that there are 3-manifolds with cubings of non-positive curvature such that their fundamental groups are not subgroup separable (LERF). We also give explicit examples of non-separable surfaces in certain cubed manifolds.

1. Introduction

In recent years the theory of topological 3-manifolds has seen much progress. Motivated by Thurston's Geometrization Conjecture, much attention has been given to compact 3-manifolds which admit a metric of non-positive curvature. One of the ways to study these manifolds in particular is the concept of *polyhedral metrics of non-positive curvature*, introduced in part by Gromov [Gm] and further developed by Aitchison and Rubinstein as they specifically focused their attention on the cubing of 3-manifolds in [AR]. In this first section, we quickly review some basic facts about cubed manifolds.

For a definition of cubing, see [AR], [Mo], [Ma], etc. Throughout this paper, we assume that the term *cubing* refers to a cubing of non-positive curvature and all manifolds are closed unless otherwise specified. Intuitively, a cubing of a 3-manifold M is a partition of M into Euclidean cubes $\mathbb{E}^3$ identified along their faces such that the dihedral angle around each edge of the cubes is at least 2π and each vertex of the cubes satisfies the following *Vertex Condition*: for each vertex v, $lk(v)$, a triangulated sphere, has the properties that every 1-cycle of $lk(v)$ has at least 3 edges and that every 1-cycle consisting of exactly 3 edges bounds a triangle contained in exactly one cube. The number of cubes around an edge e, or more precisely the dihedral degree of the edge e divided by $\frac{\pi}{2}$, is called the *degree* of e and denoted $deg(e)$. As each cube is declared Euclidean, the negative curvature is concentrated on the 1-skeleton of the cubing.

Every cubed 3-manifold M has the property that the universal cover $\widetilde{M}$ of M is homeomorphic to $\mathbb{R}^3$ (see [AR] and [BGS]), and thus M is irreducible, aspherical, and $\pi_1(M)$ is infinite. Some examples of cubed 3-manifolds are given in [AR] and [Ma].

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While all closed orientable surfaces except S^2 admit many squarings, there are classes of 3-manifolds for which it is unknown whether any cubing (of non-positive curvature) exists. Perhaps the most important class of such manifolds is the class of hyperbolic manifolds. There are also some manifolds which are known *not* to admit any cubing. Some sufficient conditions for admitting a cubing are stated in [AR], and some interesting properties of the fundamental groups of cubed manifolds are investigated in [Sk], [Re], and [Mo]. One of the nicest properties of cubed manifolds is the existence of canonical incompressible surfaces immersed in them. They satisfy the so-called 4-plane and 1-line properties [HS] which guarantee that cubed manifolds are topologically rigid, i.e., determined by their fundamental groups [AR].

We say that an incompressible immersed surface S in a 3-manifold M is *separable* if there is a finite-degree covering space of M in which S lifts to an embedding. It is still unknown whether every canonical surface in a cubed manifold is separable. Canonical surfaces are a natural starting point since D. Long [Lo] has shown that surfaces that are totally geodesic in *hyperbolic manifolds* are in fact separable. In this paper we will be concerned with the separability of immersed surfaces in cubed manifolds. In particular, we will show that not all incompressible surfaces in cubed manifolds are separable.

2. Cubed manifolds with non-LERF fundamental groups

It is true that every π_1-injective closed curve in a surface S lifts to a simple closed curve in a finite cover of S [Sc]. This fact comes from the group-theoretical concept of LERF-ness.

Definition 2.1. A group G is called LERF (or subgroup separable) if for every finitely generated subgroup H of G and an arbitrary element $g \in G \backslash H$, there is a finite-index subgroup G' of G such that $H \subset G'$ and $g \notin G'$.

It is shown in [Sc] that the LERF-ness of $\pi_1(M)$ guarantees the separability of all incompressible surfaces immersed in M. Hence, if all compact 3-manifold groups were LERF, then every such surface would be separable. In 1989, however, Burns, Karrass, and Solitar [BKS] gave an example of a compact 3-manifold (say Γ) whose fundamental group

$$K = \langle y, \alpha, \beta | (y^{-1})\alpha y = \alpha\beta,\ (y^{-1})\beta y = \beta \rangle$$

is not LERF. This Γ turns out to be a graph manifold and there are immersed surfaces in Γ that are not separable [Ma1]. This fact was proved by a separability criterion due to Rubinstein and Wang [RW].

In order to construct our example of a *cubed* manifold whose fundamental group is not LERF, we use the following lemma found in [Ma1], pointing out that K is an HNN extension.

Lemma 2.2. *Let*

$$G = \langle t, y_0, y_1 | [y_0, y_1] = 1, (t^{-1})y_0 t = y_1 \rangle.$$

Then, $G \cong K = \langle y, \alpha, \beta | (y^{-1})\alpha y = \alpha\beta, (y^{-1})\beta y = \beta \rangle$.

With this new presentation we can now give a few more examples of 3-manifolds with non-LERF fundamental groups.

Theorem 2.3. *Suppose M is any Seifert fiber space with boundary having at least two components (which are tori). Take two of these components and glue them by a homeomorphism of the torus which does not preserve the Seifert fibers, and call the resulting graph manifold M'. Then, $\pi_1(M')$ is not LERF.*

Proof. We will prove this by showing that $\pi_1(M')$ contains a subgroup isomorphic to K. Let T_1 and T_2 be the two distinct boundary components of M glued together in a non-trivial way so that a vertical fiber S_1 (a simple closed curve) on T_1 is glued to S_2 (also a simple closed curve) on T_2. By construction, S_2 is not a vertical fiber. There is a vertical annulus A in M joining S_1 to a fiber on T_2. After the gluing, T_1 and T_2 are identified as, say T, and we have a 2-complex $T \cup A$ which is π_1-injective in M'. $\pi_1(T \cup A) = (\mathbb{Z} \oplus \mathbb{Z})*_{\mathbb{Z}}$, where the $*_{\mathbb{Z}}$ indicates the HNN extension determined by the annulus A, which conjugates two distinct elements of $\pi_1(T)$, say c and d. In other words,

$$\pi_1(T \cup A) = (\mathbb{Z} \oplus \mathbb{Z})*_{\mathbb{Z}} = \langle a, b, t | [a, b] = 1, (t^{-1})ct = d\rangle.$$

This group would be K if $c = a$, $d = b$. Now, take the subgroup K' of $\pi_1(T \cup A)$ generated by c, d, and t. Then,

$$K' = \langle c, d, t | [c, d] = 1, (t^{-1})ct = d\rangle,$$

which is isomorphic to K. Since $\pi_1(M')$ contains the non-LERF subgroup K', it is not LERF. $\square$

Remark. Long and Niblo [LN] recently gave some more non-trivial examples of 3-manifold groups containing K. One such example is

$$G = (F_2 \times \mathbb{Z}) *_{\mathbb{Z}} (F_3 \times \mathbb{Z}).$$

This implies in particular that an amalgamated free product of two LERF groups over $\mathbb{Z}$ is not necessarily LERF.

We are now ready to show the following theorem.

Theorem 2.4. *There exist 3-manifolds M with cubings of non-positive curvature such that $\pi_1(M)$ is not LERF.*

Proof. We construct such cubed manifolds and show that the curvature conditions still hold. Take any closed orientable surface S (of genus g) with a squaring of non-positive curvature (see Figure 1 for a specific example with $g = 2$). Remove two disjoint squares D_1 and D_2 of S to obtain S_g, the twice-punctured surface of genus $g \geq 1$. Since we can always subdivide a given squaring, we assume that two disjoint squares can actually be removed this way. Let us assign the length 1 to each edge of the cubing. So each boundary component of S_g has length 4.

Now take $S_g \times S^1$, where the S^1 factor has length 4 (i.e., 4 cubes high), and call it M'. M' is a Seifert fiber space with two isometric torus boundary components, T_1 and T_2, since each of them is a 4×4 squared torus. M' is the same manifold obtained when one begins with S (genus-g closed surface) $\times S^1$ and then removes the interior of two disjoint solid tori. Notice that before the removal, $S \times S^1$ is cubed as a closed manifold, and in particular it satisfies the Vertex Condition. This fact will be important later. Now, identify these tori with a $\frac{\pi}{2}$ twist, that is, with the gluing homeomorphism $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The result is a closed graph manifold M, and by Theorem 2.3, $\pi_1(M)$ contains the non-LERF group isomorphic to K since M

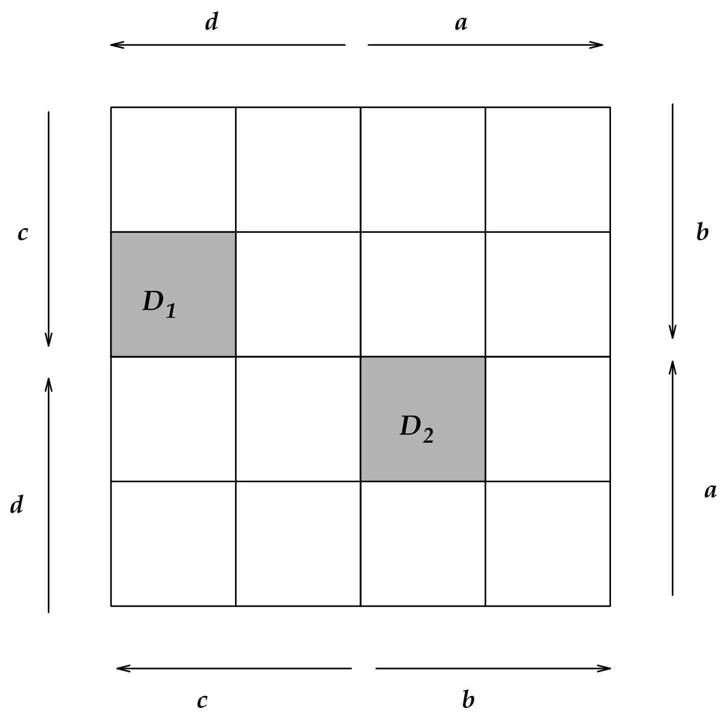

FIGURE 1. Squared genus-2 surface

is obtained by gluing two boundary torus components of a Seifert fiber space in a non-fiber-matching way.

It remains to show that M inherits a cubing of non-positive curvature. We check the degrees of edges and the Vertex Condition along the boundary tori, where the gluing took place. First, consider the degrees of the new edges of M, where the tori are glued together. Each new edge e is obtained from two originally distinct edges, e_0 and e_1, of $S_g \times S^1$, one of which (say e_0) is parallel to the S^1 factor, and the other (e_1) is perpendicular to it. By construction, $deg(e_1) = 2$ as we cut out a vertical solid torus. Also, since S is of non-positive curvature and only one square touching e_0 was removed from S to construct S_g and the torus, $deg(e_0) \geq 3$. Then,

$$deg(e) = deg(e_0) + deg(e_1) \geq 5.$$

Hence, each edge of M has dihedral angle at least 2π; in fact, each edge along the glued tori has strictly negative curvature.

Finally we check the Vertex Condition at those vertices where the gluing took place. Suppose a new vertex v of M was obtained from the two originally distinct vertices v_0 and v_1. In this proof, "$lk(v_i)$" and "$deg(v_i)$" refer to the link and degree of v_i in $S_g \times S^1$, before the gluing takes place. So $lk(v_i)$ is almost a triangulated 2-sphere, except that two adjacent 2-simplices were removed when we took away two adjacent cubes sharing v_i to make the boundary torus (see Figure 2 for $lk(v_i)$ assuming v_i was Euclidean in $S_g \times S^1$). Suppose $deg(v_i) = n_i$, $i = 0, 1$, in $S_g \times S^1$. Diagrammatically then, $lk(v_i)$ in $S_g \times S^1$ looks like a triangulated disc consisting of n_i triangles (Figure 3). As stated before, the closed manifold $S \times S^1$ (before the removal of the two solid tori) satisfies the Vertex Condition, so each of these

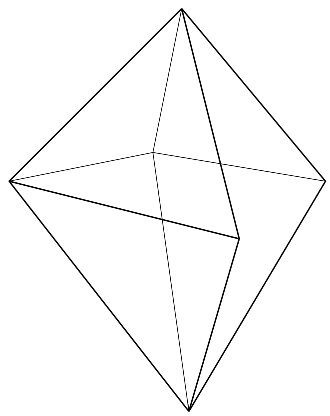

FIGURE 2. $lk(v_i)$

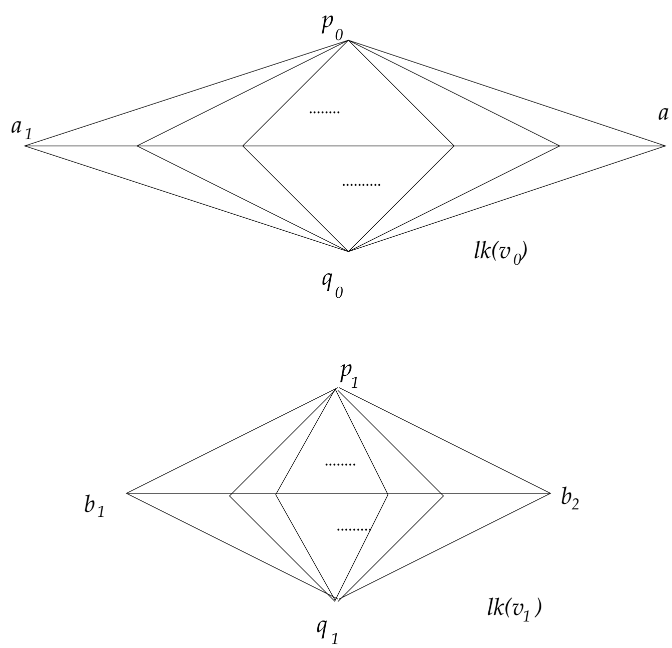

FIGURE 3. Diagrams of $lk(v_i)$ for general v_i

triangles (2-simplices) of $lk(v_i)$ is contained in exactly one cube. Now, $lk(v)$ can be represented as the union of these two triangulated discs $lk(v_0)$ and $lk(v_1)$ glued along the boundary circles. This is the triangulated sphere obtained by gluing $\partial(lk(v_0))$ and $\partial(lk(v_1))$ via the edge-gluing isometry sending

$$p_0 \longmapsto b_1, \quad a_1 \longmapsto q_1, \quad q_0 \longmapsto b_2, \quad a_2 \longmapsto p_1$$

in Figure 3.

In other words, the 4 external edges are identified with a $\frac{\pi}{2}$ twist. Therefore, each 2-simplex in $lk(v)$ is a triangle in $lk(v_0)$ or in $lk(v_1)$. Clearly, every 1-cycle of $lk(v)$ must have at least 3 edges. Now, suppose δ is a 1-cycle of exactly 3 edges in $lk(v)$. Again, it is obvious that δ bounds one of the triangles in $lk(v_0)$ or $lk(v_1)$, contained in exactly one cube as stated earlier. Hence, the Vertex Condition holds, and we conclude that M inherits a cubing of non-positive curvature. □

Remark. As illustrated in this lemma, it is often useful to construct cubed manifolds by taking the product of S^1 with some squared surface S. [Ma] gives several distinct squarings for every closed orientable surface and shows that $2g-1$ is the sharp lower bound for the number of squares required for the genus g surface.

The fact that $\pi_1(M)$ is not LERF does not necessarily imply the existence of a non-separable surface in M. However, the next theorem shows that there are indeed such surfaces immersed in cubed manifolds. The proof depends heavily on the criterion of Rubinstein and Wang (see [RW] and [Ma2]).

Theorem 2.5. *There exist cubed* 3*-manifolds of non-positive curvature admitting non-separable immersed surfaces.*

Proof. We will explicitly construct (infinitely many) surfaces which are non-separable by Rubinstein and Wang's criterion [RW]. Let F be any twice-punctured compact orientable surface of genus $g \geq 1$. Let $X' = F \times S^1$, and let T_1 and T_2 be the framed boundary tori, which are the components of $\partial F \times S^1$. Now, identify T_1 and T_2 by the $\frac{\pi}{2}$-twist, i.e., by the gluing map $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, as before, and call the resulting 3-manifold X. Such a manifold admits a cubing of non-positive curvature coming from the squaring of F by Theorem 2.4, and we know $\pi_1(X)$ is not LERF.

Now we construct an immersed surface S in X, using a lemma in [RW] repeatedly. Let S_1 be a compact orientable surface with 4 boundary components $c_{1j}\,(1 \leq j \leq 4)$ such that c_{11} and c_{12} map to $T_1 \subset X'$ with the coordinates $(1,-1)$ and $(1,-2)$, c_{13} and c_{14} map to $T_2 \subset X'$ with the coordinates $(1,1)$ and $(1,2)$, respectively (see Figure 4). Here, the first coordinate indicates how many times the boundary curve wraps around the meridian (horizontal), and the second coordinate, around in the fiber (vertical) direction. Lemma 2.2 of [RW] guarantees that such S_1 exists and is a connected orientable surface; note that the second coordinates add up to 0 and that the union of these closed curves is homologous to 0 in $F \times S^1$. One can actually construct S_1 (and the other S_i) by starting from the appropriate covering space of F (depending on the first coordinates) immersed at one level of $X' = F \times S^1$ and then modifying the immersion by appropriate Dehn twists along some properly embedded arcs (depending on the second coordinates). Similarly, construct S_2, S_3, and S_4, each of which is a compact orientable surface with 4 boundary components (2 projecting to each component of $\partial X'$), immersed in X' as shown in Figure 4. (The genus of each S_i depends on F; in this figure F is taken to be the twice-punctured torus.) The coordinates c_{ij} are assigned in such a way that they indeed bound surfaces and they respect the gluing of X. After the gluing we obtain $S = \cup_{i=1}^{4} S_i$, an orientable connected closed surface immersed in the cubed manifold X. Being a horizontal surface, it is incompressible.

It is clear that S is not embedded in X. Now we use Rubinstein and Wang's criterion [RW] (also see [Ma2]) to determine the separability of S. Let p be the oriented simple closed curve in $S_1 \cup S_2$ shown in Figure 4. Denote the first coordinate

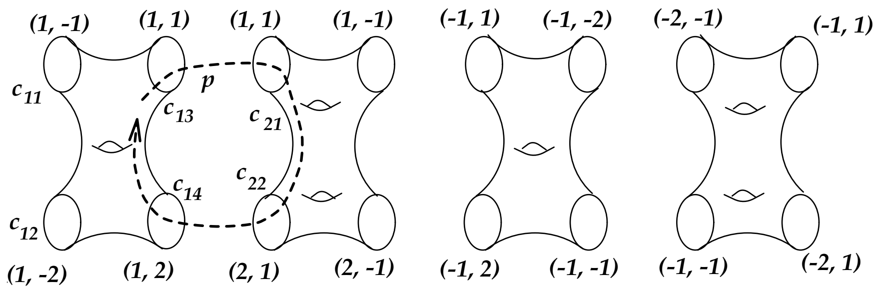

FIGURE 4. Non-separable surface in a cubed manifold

of the circle c_{ij} by a_{ij}. Then,

$$s_p = \frac{a_{13}}{a_{21}} \cdot \frac{a_{22}}{a_{14}} = \frac{1}{1} \cdot \frac{2}{1} = 2 \neq 1.$$

For the definition of this "fiber-intersection ratio" s_p, see Section 2 of [RW] or Definition 2.1 of [Ma2]. The criterion states that S is separable if and only if $s_p = 1$ for every simple closed curve p on S; hence, this surface S cannot lift to an embedding in any finite cover of X. □

Note that this construction allows F to be of any positive genus, giving infinitely many cubed manifolds with non-separable immersed surfaces.

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