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APPROXIMATION OF SINGULARITY SETS WITH ANALYTIC GRAPHS OVER THE BALL IN $\mathbf{C}^2$

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ABSTRACT. Let h be a smooth function on the ball in $\mathbf{C}^2$ whose gradient has length less than or equal to 1. We show that if h is uniformly near an analytic function on every complex affine one-dimensional slice then it must be near some function analytic on the whole ball. We use this to show the following: a singularity set over the ball which is near the graph of a function h with $|\nabla h| \leq 1$ must be near the graph of some analytic function over the ball.

Let B_2 be the open unit ball in $\mathbf{C}^2$, let $S = \partial B_2$ and let K be a compact subset of $\overline{B}_2 \times \mathbf{C}$. We say that K is a singularity set if $(B_2 \times \mathbf{C}) \backslash K$ is pseudoconvex. This implies that there exists an f analytic on $(B_2 \times \mathbf{C}) \backslash K$ which is singular at each point of the boundary of K in $B_2 \times \mathbf{C}$. If we let Δ denote the closed unit disk in $\mathbf{C}$ then we can make a similar definition of a singularity set in $(int\,\Delta) \times \mathbf{C}$. Singularity sets were studied as early as 1909 by Hartogs [3] and later by Oka [5] and Nishino [4]. One issue that has been studied is the question of whether such sets possess analytic structure, i.e., whether they contain analytic varieties. Wermer [9] and Słodkowski [8] showed that such an expectation is reasonable in general by proving a maximum modulus principle for singularity sets K projecting onto Δ; in particular, the following holds:

Proposition. *If $(z_0, w_0) \in K$ then for every polynomial Q,*

$$|Q(z_0, w_0)| \leq \sup_{(z,w) \in K \cap \{|z|=1\}} |Q(z, w)|.$$

(See (1) on p. 264 of [1].) In [1], Alexander and Wermer showed that a singularity set projecting onto the disk which is reasonably "thin" must be near an analytic graph. More precisely, they showed

Theorem of [1]**.** *Let $\lambda \to a(\lambda)$ be a continuous function defined for $|\lambda| \leq 1$ with $|a(\lambda)| \leq 1$ for all λ. Fix $r > 0$. Suppose that there exists a singularity set X projecting onto Δ such that X is contained in the tube $\{(\lambda, w) \,\big|\, |w - a(\lambda)| < r\}$. Then there exists an analytic function $\lambda \to f(\lambda)$ such that $|f(\lambda) - a(\lambda)| \leq 4r$ for each λ in the unit disk.*

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We shall make use of Alexander's and Wermer's technique in the proof of Theorem 2.

If $h : \mathbf{C}^2 \longrightarrow \mathbf{C}$ is a smooth function then $|\nabla h|$ shall denote the Euclidean length of the vector

$$\left(\frac{\partial h}{\partial x_1}, \frac{\partial h}{\partial x_2}, \frac{\partial h}{\partial y_1}, \frac{\partial h}{\partial y_2}\right),$$

where $(z_1, z_2) = (x_1 + x_2 i, y_1 + y_2 i)$. For K a singularity set over the ball we shall prove the following.

Theorem 2. *Let h be a C^1 function in a neighborhood of the closed ball in $\mathbf{C}^2$ with $|\nabla h| \leq 1$ on $\overline{B}_2$. Suppose K is a singularity set projecting onto $\overline{B}_2$ such that K is contained in the tube*

$$T = \{(z, w) \,\epsilon\, \overline{B}_2 \times \mathbf{C} \;\big|\; |w - h(z)| < \epsilon\}.$$

Then there exists an analytic polynomial F in $\mathbf{C}^2$ such that

$$|F(z) - h(z)| < 26\sqrt{\epsilon} \text{ on } \overline{B}_2.$$

Suppose that L is a complex affine subspace of $\mathbf{C}^2$ of complex dimension 1 which meets B_2 and let p be the point on L nearest to $(0, 0)$. Then the points on L which meet S form a circle in L with center p and $L \cap \overline{B}_2$ is a disk embedded complex affinely in $\mathbf{C}^2$. We shall write $\Delta = L \cap \overline{B}_2$ and refer to Δ as a complex affine slice of $\overline{B}_2$. Our technique shall be to use the fact that h is uniformly near an analytic function on every complex affine slice of $\overline{B}_2$ (from the results over the disk) and try to prove that h is uniformly near an analytic function on B_2 as a function of two variables. We let $C(\overline{B}_2)$ denote the space of continuous complex-valued functions on $\overline{B}_2$ with supremum norm and $A(B_2)$ the subspace of functions which are analytic on B_2.

Theorem 1. *Suppose h is C^1 in a neighborhood of the closed ball in $\mathbf{C}^2$, $|\nabla h| \leq 1$ on $\overline{B}_2$ and that given any complex affine slice Δ (a disk), there exists a polynomial g_Δ on Δ such that*

$$|h - g_\Delta| < \epsilon \text{ on } \Delta. \tag{1}$$

Then in $C(\overline{B}_2)$,

$$\operatorname{dist}(h, A(B_2)) < 13\sqrt{\epsilon}. \tag{2}$$

Proof. First we note that the theorem is trivial if $\epsilon \geq 1$, because then the fact that $|\nabla h| \leq 1$ means that h is uniformly within 1 of the function which is constantly $h(0)$, and $1 < 13\sqrt{1}$. Thus we assume that $\epsilon < 1$.

We shall show that $C[h]$, the Cauchy integral of h, satisfies

$$\operatorname{dist}(C[h]|_\Delta, h|_\Delta) < \sqrt{5}\,\epsilon \text{ in } L^2(\Delta) \tag{3}$$

where $\Delta =$ a diametrical complex linear slice of $\overline{B}_2$, $\nu_1 =$ normalized area measure on Δ and $L^2(\Delta) =$ the set of functions on Δ which are square integrable with respect to ν_1. We shall also define $A^2(\Delta) = \{f \text{ analytic in int } \Delta |\; f \in L^2(\Delta)\}$. Suppose first that $\Delta = \overline{B}_2 \cap \{(z_1, z_2) | z_2 = 0\}$. Let $\sigma =$ normalized volume measure on S. Using a technique used by Rudin in [6],

$$\begin{aligned} C[h](z_1,0) &= \int_{\partial B_2} \frac{h(\zeta_1,\zeta_2)}{(1-\langle z_1,\zeta_1\rangle)^2} d\sigma(\zeta_1,\zeta_2) \\ &= \int_{B_1} \frac{1}{(1-\langle z_1,\zeta_1\rangle)^2} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} h(\zeta_1,\zeta_2 e^{i\theta}) d\theta \right) d\nu_1(\zeta_1). \end{aligned}$$

(See [6], pp. 15, 39.) For fixed ζ_1 define $k(\zeta_1,\zeta_2)$ to be the harmonic extension of $h(\zeta_1,x)$, $|x| = \sqrt{1-\mid \zeta_1 \mid^2}$, to the region where $|x| \leq \sqrt{1-\mid \zeta_1 \mid^2}$. Then the above equation becomes

$$C[h](z_1,0) = \int_{B_1} \frac{k(\zeta_1,0)}{(1-\langle z_1,\zeta_1\rangle)^2} d\nu_1(\zeta_1). \tag{4}$$

We check that k is continuous when restricted to $\{z_2 = 0\}$. We have $k(z_1,0)= \frac{1}{2\pi}\int_{-\pi}^{\pi} h(z_1, e^{i\theta}\sqrt{1-|z_1|^2})d\theta$; for z_1' near z_1, $h(z_1', e^{i\theta}\sqrt{1-|z_1'|^2})$ is uniformly close to $h(z_1, e^{i\theta}\sqrt{1-|z_1|^2})$ so $k(z_1',0)$ is close to $k(z_1,0)$.

Claim. *For* $\zeta_1 \in \Delta$,

$$\mid h(\zeta_1,0) - k(\zeta_1,0) \mid < 2\epsilon. \tag{5}$$

Proof of Claim. From (1), we get that for fixed ζ_1, there exists a polynomial f_{ζ_1} in ζ_2 such that

$$\mid h(\zeta_1,\zeta_2) - f_{\zeta_1}(\zeta_2) \mid < \epsilon \text{ for } \mid \zeta_2 \mid \leq \sqrt{1-\mid \zeta_1 \mid^2}.$$

Thus

$$\begin{aligned} \mid h(\zeta_1,0) - k(\zeta_1,0) \mid &\leq \mid h(\zeta_1,0) - f_{\zeta_1}(0) \mid + \mid f_{\zeta_1}(0) - k(\zeta_1,0) \mid \\ &< \epsilon + \sup_{|\zeta_2|=\sqrt{1-|\zeta_1|^2}} \mid f_{\zeta_1}(\zeta_2) - k(\zeta_1,\zeta_2) \mid \\ &\quad \text{(since } f_{\zeta_1} \text{ and } k \text{ are harmonic in } \zeta_2\text{)} \\ &= \epsilon + \sup_{|\zeta_2|=\sqrt{1-|\zeta_1|^2}} \mid f_{\zeta_1}(\zeta_2) - h(\zeta_1,\zeta_2) \mid \\ &< 2\epsilon, \end{aligned}$$

as claimed.

Let $\Pi : L^2(\Delta) \longrightarrow A^2(\Delta)$ be orthogonal projection. From what we know about the Bergman kernel, (4) is the orthogonal projection of k to $A^2(\Delta)$, as a function of z_1. This means that $\Pi(k|_\Delta) = C[h]|_\Delta$. Then in $L^2(\Delta)$,

$$\operatorname{dist}(h|_\Delta, C[h]|_\Delta)^2 = \operatorname{dist}(h|_\Delta, \Pi(h|_\Delta))^2 + \operatorname{dist}(\Pi(h|_\Delta), C[h]|_\Delta)^2 \tag{6}$$

by the Pythagorean theorem.

Since $\operatorname{dist}(h|_\Delta, A(\Delta)) < \epsilon$ in $C(\Delta)$, $\operatorname{dist}(h|_\Delta, A^2(\Delta)) < \epsilon$ in $L^2(\Delta)$, so we have $\operatorname{dist}(h|_\Delta, \Pi(h|_\Delta)) < \epsilon$ in $L^2(\Delta)$. Thus (6) is now

$$\begin{aligned} &< \epsilon^2 + \operatorname{dist}(\Pi(h|_\Delta), \Pi(k|_\Delta))^2 \\ &\leq \epsilon^2 + \operatorname{dist}(h|_\Delta, k|_\Delta)^2 \\ &\leq \epsilon^2 + (2\epsilon)^2 \end{aligned}$$

from (5)

$$= 5\,\epsilon^2.$$

Thus (3) holds for $\Delta = \overline{B}_2 \cap \{(z_1, z_2)|z_2 = 0\}$.

Now if Δ is an arbitrary complex linear diametrical slice of $\overline{B}_2$, let U be a unitary transformation such that $U(\{z_2 = 0\} \cap \overline{B}_2) = \Delta$. Then $h \circ U$ satisfies the original hypotheses of the theorem, so

$$\operatorname{dist}(h \circ U|_{\{z_2=0\}}, C[h \circ U]|_{\{z_2=0\}}) < \sqrt{5}\epsilon \text{ in } L^2(B_2 \cap \{z_2 = 0\})$$

$$\Rightarrow \operatorname{dist}(h \circ U\ |_{\{z_2=0\}}, C[h] \circ U\ |_{\{z_2=0\}}) < \sqrt{5}\epsilon \text{ in } L^2(B_2 \cap \{z_2 = 0\})$$

(since the Cauchy integral operator commutes with unitary transformations). Since $U(\{z_2 = 0\} \cap \overline{B}_2) = \Delta$ and U induces a natural isometry from $L^2(\Delta)$ to $L^2(B_2 \cap \{z_2 = 0\})$, we have (3).

Now let $f = C[h]$. Fix any diametrical slice Δ. By (1) we can choose a polynomial $g_\Delta \,\epsilon\, A(\Delta)$ such that $|h - g_\Delta| < \epsilon$ on Δ. Then $\operatorname{dist}(h|_\Delta, g_\Delta) < \epsilon$ in $L^2(\Delta) \Rightarrow \operatorname{dist}(f|_\Delta, g_\Delta) \leq \operatorname{dist}(f, h|_\Delta) + \operatorname{dist}(h|_\Delta, g_\Delta)$ in $L^2(\Delta) < \sqrt{5}\epsilon + \epsilon$ from (3) $= (1 + \sqrt{5})\epsilon \equiv a\epsilon$ so that

$$\operatorname{dist}(f|_\Delta, g_\Delta) < a\epsilon \text{ in } A^2(\Delta) \tag{7}$$

Define

$$\begin{aligned} f^r(ze^{i\theta}) &= f(zre^{i\theta}), z \in \operatorname{int} \Delta, \\ g^r_\Delta(ze^{i\theta}) &= g_\Delta(zre^{i\theta}), z \in \Delta, \\ h^r(ze^{i\theta}) &= h(zre^{i\theta}), z \in \Delta. \end{aligned}$$

Claim. *For $r \leq 1 - \sqrt{\epsilon}$,*

$$\| f^r - g^r_\Delta \|_2 \leq a\epsilon^{\frac{3}{4}} \text{ in } H^2(\Delta \cap S). \tag{8}$$

Proof of claim. Suppose not. Note that $\| f^r - g^r_\Delta \|_2$ is increasing in r since f^r and g^r_Δ are analytic. Thus

$$\| f^r - g^r_\Delta \|_{H^2} \geq a\epsilon^{\frac{3}{4}} \text{ for } r > 1 - \sqrt{\epsilon}.$$

Using polar coordinates,

$$\begin{aligned} \int_\Delta |f - g_\Delta|^2 d\nu_1 &= 2\int_0^1 r dr \frac{1}{2\pi}\int_{-\pi}^{\pi} |f^r(e^{i\theta}) - g^r_\Delta(e^{i\theta})|^2 d\theta \\ &= 2\int_0^1 r \| f^r - g^r_\Delta \|^2_{H^2} dr \\ &> 2\int_{1-\sqrt{\epsilon}}^1 r \| f^r - g^r_\Delta \|^2_{H^2} dr \\ &\geq 2\int_{1-\sqrt{\epsilon}}^1 r a^2 \epsilon^{\frac{3}{2}} dr, \text{ from (8)} \\ &= 2a^2\epsilon^{\frac{3}{2}} \int_{1-\sqrt{\epsilon}}^1 r dr \geq a^2\epsilon^{\frac{3}{2}}(2\sqrt{\epsilon} - \epsilon) \\ &\geq a^2\epsilon^{\frac{3}{2}}(2\sqrt{\epsilon} - \sqrt{\epsilon}) = a^2\epsilon^{\frac{3}{2}}\sqrt{\epsilon} = a^2\epsilon^2 \end{aligned}$$

(recall $\epsilon < 1$) so $\| f - g_\Delta \|_2 > a\epsilon$ in $A^2(\Delta)$, a contradiction of (7). Thus the claim (8) holds. This means that for $r \leq 1 - \sqrt{\epsilon}$, $\| f^r - h^r \|_{L^2(\Delta \cap S)} \leq \| f^r - g^r_\Delta \|_{L^2(\Delta \cap S)}$

$+ \parallel g^r_\Delta - h^r \parallel_{L^2(\Delta\cap S)} < a\epsilon^{\frac{3}{4}} + \epsilon$ (from (1) and (8)) $\leq (2+\sqrt{5})\epsilon^{\frac{3}{4}}$ and we conclude

$$\parallel f^r - h^r \parallel_{L^2(\Delta\cap S)} \leq (2+\sqrt{5})\epsilon^{\frac{3}{4}} \text{ for } r \leq 1-\sqrt{\epsilon}. \tag{9}$$

Note that the conclusion holds for all diametrical slices Δ.

Now define $F : B_2 \longrightarrow \mathbf{C}$ by $F(z) = \int_R f(ze^{-it})\phi(t)dt$ where $\phi\,\epsilon\, C^\infty(\mathbf{R}), \phi \geq 0, spt\phi \subset [-\sqrt{\epsilon}, \sqrt{\epsilon}], \int \phi dx = 1$, and $\phi \leq \frac{1}{\sqrt{2\epsilon}}$. Also define $H(z) = \int_R h(ze^{-it})\phi(t)dt$. Since f and ϕ are smooth in B_2, f analytic, F is analytic in B_2 by differentiation under the integral sign. Then for $|z| \leq 1-\sqrt{\epsilon}$,

$$\begin{aligned}
|F(z) - H(z)| &\leq |\int_R (f(ze^{-it}) - h(ze^{-it}))\phi(t)dt| \\
&\leq \sqrt{\int_{-\sqrt{\epsilon}}^{\sqrt{\epsilon}} |f(ze^{-it}) - h(ze^{-it})|^2 dt}\sqrt{\int \phi(t)^2 dt} \\
&\leq (\sqrt{2\pi})(2+\sqrt{5})\epsilon^{\frac{3}{4}}\sqrt{2\sqrt{\epsilon}\sup_{t\in R}|\phi(t)|^2}, \text{ from (9)} \\
&\leq (\sqrt{2\pi})(2+\sqrt{5})\epsilon^{\frac{3}{4}}\sqrt{2\sqrt{\epsilon}\frac{1}{2\epsilon}} \\
&\leq (\sqrt{2\pi})(2+\sqrt{5})\sqrt{\epsilon}
\end{aligned} \tag{10}$$

and for $z \in B_2$,

$$\begin{aligned}
|H(z) - h(z)| &= |\int_R (h(ze^{-it}) - h(z))\phi(t)dt| \\
&\leq \sup_{t\in spt\phi} |h(ze^{-it}) - h(z)| \parallel \phi \parallel_1 \\
&\leq \sqrt{\epsilon} \text{ (since } |\nabla h| \leq 1).
\end{aligned} \tag{11}$$

Thus for $|z| \leq 1-\sqrt{\epsilon}$,

$$\begin{aligned}
|F(z) - h(z)| &\leq |F(z) - H(z)| + |H(z) - h(z)| \\
&\leq (\sqrt{2\pi})(2+\sqrt{5})\sqrt{\epsilon} + \sqrt{\epsilon}, \text{ from (10) and (11)} \\
&< 12\sqrt{\epsilon}.
\end{aligned} \tag{12}$$

Lastly define $J : \overline{B}_2 \longrightarrow \mathbf{C}$ by $J(z) = F(z(1-\sqrt{\epsilon}))$. Then $J \in A(B_2)$. For $z \in \overline{B}_2$,

$$\begin{aligned}
|J(z) - h(z)| &= |F(z(1-\sqrt{\epsilon})) - h(z)| \\
&\leq |F(z(1-\sqrt{\epsilon})) - h(z(1-\sqrt{\epsilon}))| + |h(z(1-\sqrt{\epsilon})) - h(z)| \\
&< 12\sqrt{\epsilon} + \sqrt{\epsilon}, \text{ from (12) and since } |\nabla h| \leq 1 \\
&\leq 13\sqrt{\epsilon}.
\end{aligned}$$

Thus $\text{dist}(h, A(B_2)) < 13\sqrt{\epsilon}$ in $C(\overline{B}_2)$, which is (2). □

We now prove Theorem 2, which we state again for the reader's convenience.

Theorem 2. *Let h be a C^1 function in a neighborhood of the closed ball in $\mathbf{C}^2$ with $|\nabla h| \leq 1$ on $\overline{B}_2$. Suppose K is a singularity set projecting onto $\overline{B}_2$ such that K is contained in the tube*

$$T = \{(z,w)\,\epsilon\, \overline{B}_2 \times \mathbf{C}\ \big| |w - h(z)| < \epsilon\}. \tag{13}$$

Then there exists an analytic polynomial F in $\mathbf{C}^2$ such that

$$|F(z) - h(z)| < 26\sqrt{\epsilon} \text{ on } \overline{B}_2. \tag{14}$$

Proof. Our main goal is to show that on every complex affine slice Δ of $\overline{B}_2$, there exists a polynomial g_Δ such that

$$|h - g_\Delta| < 4\epsilon. \tag{15}$$

We can then apply Theorem 1 with (1) replaced by (15) and conclude (14) since $13\sqrt{4} = 26$. Let us now fix such a Δ.

To prove (15), we can follow a method used by Alexander and Wermer in [2] to show that $\text{dist}(h|_{\Delta\cap S}, A(\Delta)) < 2\epsilon$. Let $P : \overline{B}_2 \times \mathbf{C} \longrightarrow \overline{B}_2$ be projection and let K' be the set $\{(z,w) \,\big|\, |w - h(z)| \le \epsilon, z \in \overline{B}_2\}$. We claim that: some element of the polynomial convex hull of $K' \cap P^{-1}(\partial\Delta)$ lies over $int\,\Delta$. To see this, we first note that $K \cap P^{-1}(int\,\Delta)$ is a singularity set in $P^{-1}(\Delta)$ since the intersection of a pseudoconvex set with an affine subspace is pseudoconvex in that affine subspace. Then by the Proposition, the elements of $K \cap P^{-1}(int\,\Delta)$ (which is nonempty by assumption) are in the polynomial hull of $K \cap P^{-1}(\partial\Delta)$. Since $K' \supset K$, the claim holds. Then by Theorem 1 of [2], there exists $\phi \,\epsilon\, H^\infty(\Delta)$ such that $(z, \phi(z)) \in K'$ for a.e. $z \in \Delta \cap S$. This means that $\| h - \phi \|_\infty \le \epsilon < 2\epsilon$. Then we conclude that since h is continuous, $\text{dist}(h|_{\Delta\cap S}, A(\Delta)) < 2\epsilon$. Choose a polynomial g_Δ on Δ such that

$$|h - g_\Delta| < 2\epsilon \text{ on } \Delta \cap S. \tag{16}$$

Now consider the polynomial $w - g_\Delta$ on the set $K \cap (\Delta \times \mathbf{C})$. Let $(z_0, w_0) \in K \cap (\Delta \times \mathbf{C})$. Since (z_0, w_0) is in the polynomial hull of $K \cap P^{-1}(\partial\Delta)$,

$$|w_0 - g_\Delta(z_0, w_0)| \le \sup_{(z,w)\in K, z\in\Delta\cap S} |w - g_\Delta(z, w)|. \tag{17}$$

Then

$$\begin{aligned}
\sup_{z\in\Delta} |h(z) - g_\Delta(z)| &\le \sup_{(z,w)\in(K\cap(\Delta\times C))} |h(z) - w| + \sup_{(z,w)\in(K\cap(\Delta\times C))} |w - g_\Delta(z)| \\
&\le \epsilon + \sup_{(z,w)\in K, z\in\Delta\cap S} |w - g_\Delta(z)|, \text{ from (13) and (17)} \\
&\le \epsilon + \sup_{(z,w)\in K, z\in\Delta\cap S} |w - h(z)| + \sup_{(z,w)\in K, z\in\Delta\cap S} |h(z) - g_\Delta(z)| \\
&< \epsilon + \epsilon + 2\epsilon, \text{ from (13) and (16)} \\
&\le 4\epsilon.
\end{aligned}$$

This shows that (15) holds, so the theorem is proven. □

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