TRIEBEL-LIZORKIN SPACES ASSOCIATED WITH LAGUERRE AND HERMITE EXPANSIONS

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Abstract. It is proved that Triebel-Lizorkin spaces for some Laguerre and Hermite expansions are well-defined.

1. Introduction

Let $D$ be a self-adjoint positive operator acting on $L^2(\mathbb{R})$, and let $dE$ be its spectral resolution, that is,

$$Df = \int_0^\infty \lambda dE(\lambda)f.$$ (1.1)

For $\alpha \in \mathbb{R}$, $0 < p, q < \infty$, and a $C^\infty$ function $\varphi$ satisfying

$$\text{supp } \varphi \subset [1/2, 2], \quad |\varphi(\lambda)| > c > 0 \quad \text{for } \lambda \in [3/4, 7/4],$$ (1.2)

we define the Triebel-Lizorkin norm associated with $D$ (and with $\varphi$) by

$$\|f\|_{D^\alpha_{pq}(\varphi)} = \left\| \sum_{\mu \in \mathbb{Z}} (2^{\mu \alpha}|Q_\mu f|)^q \right\|^{1/q}_{L^p(X)},$$ (1.3)

where

$$Q_\mu f = \varphi(2^{-\mu}D)f = \int_0^\infty \varphi(2^{-\mu}\lambda)dE(\lambda)f.$$ (1.4)

Note that if $D = \Delta = -\sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$ is the Laplacian on $\mathbb{R}^d$, then the norm $\|f\|_{\Delta^\alpha_{pq}(\varphi)}$ is equivalent to the classical Triebel-Lizorkin norm $\|f\|_{F^\alpha_{pq}}$.

Triebel-Lizorkin spaces associated with the one-dimensional Hermite operator

$$\mathcal{H} = -\frac{\partial^2}{\partial x^2} + x^2$$

were studied by J. Epperson in [E1] and [E2]. It was proved there, using Mehler’s formula, that the definition of the corresponding space $F^\alpha_{\mathcal{H}^sigma}$ is independent of the particular choice of the function $\varphi$.

The present paper continues these studies. We consider Triebel-Lizorkin spaces associated with some Laguerre expansions and multidimensional Hermite expansions. We use some ideas from [E1] combined with Heisenberg group methods (cf. [HJ]). Symbolic calculus for sublaplacians on Heisenberg groups (cf. Theorem 2.3) plays an essential role in our paper.

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2. Symbolic calculus on Heisenberg groups

Let \( \mathbb{H}_d \) be the \( 2d + 1 \) dimensional Heisenberg group, that is, \( \mathbb{H}_d = \mathbb{C}^d \times \mathbb{R} \) with the multiplication \( hh' = (z,t)(z',t') = (z+z', t+t' + \frac{1}{2} \Im(z\bar{z}')) \). Let \( X_j, Y_j \) be the elements of the Lie algebra of \( \mathbb{H}_d \) which we identify with the left-invariant vector fields
\[
X_j = \frac{\partial}{\partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} + \frac{1}{2} x_j \frac{\partial}{\partial t}.
\]
(2.1)

The corresponding right-invariant vector fields are:
\[
\tilde{X}_j = \frac{\partial}{\partial x_j} + \frac{1}{2} y_j \frac{\partial}{\partial t}, \quad \tilde{Y}_j = \frac{\partial}{\partial y_j} - \frac{1}{2} x_j \frac{\partial}{\partial t}.
\]
(2.2)

The sublaplacian \( L \) on \( \mathbb{H} \) defined by
\[
L = -\sum_{j=1}^{d} X_j^2 + Y_j^2
\]
is a positive, homogeneous of degree 2, left-invariant subelliptic differential operator. Let \( dE \) be the spectral resolution for \( L \), that is, \( Lf = \int_0^\infty \lambda dE(\lambda) f \). If \( m \) is a bounded function on \((0, \infty)\), then the operator
\[
m(L)f = \int_0^\infty m(\lambda)dE(\lambda)f
\]
is left-invariant and bounded on \( L^2(\mathbb{H}_d) \).

The following theorem due to Hulanicki (cf. \[H\]) is the basic tool in our paper.

**Theorem 2.3.** If \( m \in S(\mathbb{R}) \), then
\[
m(L)f = f * M,
\]
(2.4)

with \( M \) in the Schwartz space \( S(\mathbb{H}_d) \) of functions on \( \mathbb{H}_d \).

Moreover, if for \( s > 0 \) we set \( m^s(\lambda) = m(s\lambda) \), then
\[
m^s(L)f = f * M_s,
\]
(2.5)

where
\[
M_s(h) = M_s(z,t) = s^{-Q/2} M\left(\frac{z}{\sqrt{s}}, \frac{t}{s}\right).
\]
(2.6)

Here \( Q = 2d + 2 \) is the homogeneous dimension of \( \mathbb{H}_d \).

3. Laguerre functions

Let
\[
L_k^m(w) = (2\pi)^{-1/2} \left( \frac{k!}{(k+m)!} \right)^{1/2} w^{m/2} L_k^m(\sqrt{w}) e^{-w/2}, \quad w > 0,
\]
(3.1)

be the Laguerre function of type \( m \), \( m = 0, 1, 2, ... \), where
\[
L_k^m(w) = \sum_{j=0}^{k} \binom{k+m}{k-j} \frac{(-w)^j}{j!}
\]
(3.2)

is the corresponding Laguerre polynomial of type \( m \), \( m = 0, 1, 2, ... \).

Let \( \mathbb{H}/\Gamma \) denote the reduced Heisenberg group, where \( \Gamma = \{(0,2\pi n) : n \in \mathbb{Z}\} \) is a normal discrete central subgroup of \( \mathbb{H} = \mathbb{H}_1 \). For \( p > 0 \) and nonnegative integer
we consider the space $L^p_m(\mathbb{H}/\Gamma)$ which consists of $L^p$ functions $f$ which have the form
\begin{equation}
    f(z,t) = e^{it}e^{-im\theta}f_0(r), \quad z = re^{i\theta}.
\end{equation}

It is well known (cf. [T]) that if a $C^2$ function $f$ on $\mathbb{H}/\Gamma$ has the form (3.3), then $Lf$ is of the same form, where $L = -X^2 - Y^2$ is the sublaplacian on $\mathbb{H}/\Gamma$. Moreover, the functions
\begin{equation}
    \phi_k^m(z,u) = e^{iu}e^{-im\theta}L_k^m(|z|^2/2)
\end{equation}
form an orthonormal basis of $L^2_m(\mathbb{H}/\Gamma)$, and
\begin{equation}
    L\phi_k^m = d_k\phi_k^m, \quad \text{where } d_k = 2k + 1.
\end{equation}

Note that the map $W : L^p(\mathbb{R}^+) \to L^p_m(\mathbb{H}/\Gamma)$ given by
\begin{equation}
    f(z,u) = (Wg)(z,u) = e^{iu}e^{-im\theta}g(|z|^2/2), \quad \text{where } z = e^{i\theta}|z|,
\end{equation}
is an isometry from $L^p(\mathbb{R}^+)$ onto $L^p_m(\mathbb{H}/\Gamma)$. If, moreover, $f$ and $g$ are related by (3.6), then
\begin{equation}
    \langle g, L_m^m \rangle = \langle f, \phi_k^m \rangle.
\end{equation}
Consequently, if $g = \sum_k \langle g, L_k^m \rangle L_k^m$, then
\begin{equation}
    Wg = \sum_k (Wg, \phi_k^m)\phi_k^m.
\end{equation}

4. Triebel-Lizorkin spaces for Laguerre expansions

Let $\varphi$ be a $C^\infty$ function satisfying (1.2). We define the linear operators $Q_\mu$ on $L^2(\mathbb{R}^+)$ by
\begin{equation}
    Q_\mu L_k^m = \varphi(2^{-\mu}d_k)L_k^m.
\end{equation}
For $g = \sum_k a_k L_k^m$, $0 < p, q < \infty$, and $\alpha \in \mathbb{R}$, the Triebel-Lizorkin norm $\|g\|_{L^\alpha_q(\varphi)}$ is defined by
\begin{equation}
    \|g\|_{L^\alpha_q(\varphi)} = \left\| \left[ \sum_{\mu \in \mathbb{Z}} (2^{\mu\alpha}|Q_\mu g|)^q \right]^{1/q} \right\|_{L^p(\mathbb{R}^+)).
\end{equation}

On the space $L^2_m(\mathbb{H}/\Gamma)$ we define the corresponding operators $\hat{Q}_\mu$ by setting
\begin{equation}
    \hat{Q}_\mu \phi_k^m = \varphi(2^{-\mu}d_k)\phi_k^m.
\end{equation}
 Obviously for $f$ and $g$ related by (3.6),
\begin{equation}
    \|g\|_{L^\alpha_q(\varphi)} = \left\| \left[ \sum_{\mu \in \mathbb{Z}} (2^{\mu\alpha}|\hat{Q}_\mu f|)^q \right]^{1/q} \right\|_{L^p(\mathbb{H}/\Gamma)}.
\end{equation}

Our goal in this section is the following

**Theorem A.** Let $\alpha \in \mathbb{R}$, $0 < p < \infty$, and $0 < q < \infty$. If $\varphi^{(1)}$ and $\varphi^{(2)}$ are two $C^\infty$ functions satisfying (1.2), then there exists a constant $C$ such that
\begin{equation}
    C^{-1}\|g\|_{L^\alpha_q(\varphi^{(1)})} \leq \|g\|_{L^\alpha_q(\varphi^{(2)})} \leq C\|g\|_{L^\alpha_q(\varphi^{(1)})}.
\end{equation}
On the reduced Heisenberg group $\mathbb{H}/\Gamma$ let $d((z, t), (z', t'))$ be a distance function given by

$$d((z, t), (z', t')) = \inf_{n \in \mathbb{Z}} \{(z, t)^{-1}(z', t')(0, 2\pi n)\},$$

where $|(z, t)| = |z| + |t|^{1/2}$ is a homogeneous norm on $\mathbb{H}$.

For $a > 0$ and $f$ of the form (3.3) we define an analogue of the Peetre maximal operator:

$$\tilde{A}_\mu f(z, t) = \sup_{(z', t') \in \mathbb{H}/\Gamma} \frac{|\tilde{Q}_\mu f(z', t')|}{(1 + 2\mu^2 d((z, t), (z', t'))^a)}.$$  

Note that if

$$\tilde{A}_\mu f(z) = \sup_{z' \in \mathbb{R}^2} \frac{|\tilde{Q}_\mu f(z', 0)|}{(1 + 2\mu^2 |z - z'|^a)},$$

then

$$\tilde{A}_\mu f(z) = \tilde{A}_\mu f(z, t).$$

Let

$$\tilde{B}_\mu f(z, t) = \sup_{(z', t') \in \mathbb{H}/\Gamma} \frac{|
abla \tilde{Q}_\mu f(z', t')|}{(1 + 2\mu^2 d((z, t), (z', t'))^a)},$$

where $|
abla \tilde{Q}_\mu f(z', t')| = |X \tilde{Q}_\mu f(z', t')| + |Y \tilde{Q}_\mu f(z', t')|.$

**Lemma 4.10.** For every $a > 0$ there is a constant $C > 0$ such that

$$\tilde{B}_\mu f(z, t) \leq C 2^{\mu/2} \tilde{A}_\mu f(z, t).$$

**Proof.** Let $\psi$ be a $C^\infty$ function satisfying (1.2) such that

$$\sum_{\mu \in \mathbb{Z}} \psi(2^{-\mu} \lambda) \varphi(2^{-\mu} \lambda) = 1 \quad \text{for} \quad \lambda > 0.$$  

For the function $\zeta(\lambda) = \sum_{j=-1}^{\infty} \varphi(2^j \lambda) \psi(2^j \lambda)$ we denote by $M_{2^{-\mu}}(z, t)$ the convolution kernel on $\mathbb{H}$ that corresponds to the operator $\zeta(2^{-\mu} L)$, where $L$ is the sublaplacian on $\mathbb{H}$. By Theorem 2.3

$$|X \tilde{Q}_\mu f(z, t)| = |X \int_{\mathbb{H}/\Gamma} \sum_{n \in \mathbb{Z}} \tilde{Q}_\mu f(z', t') M_{2^{-\mu}}((z', t')^{-1}(z, t)(0, 2\pi n)) dz' dt'|$$

$$= \left| 2^{5\mu/2} \int_{\mathbb{R}^2} \int_{\mathbb{R}} (\pi M)(2^{\mu/2}(z - z'), 2^{\mu}(t - t' - \frac{1}{2}\Delta(z' \bar{z}) + 2\pi n)) \times \tilde{Q}_\mu f(z', t') dz' dt' \right|$$

$$\leq 2^{5\mu/2} \int_{\mathbb{R}^2} \int_{\mathbb{R}} |(\pi M)(2^{\mu/2}(z - z'), 2^{\mu}(-t'))| \tilde{Q}_\mu f(z', 0)| dz' dt'$$

$$\leq C h 2^{3\mu/2} \int_{\mathbb{R}^2} (1 + 2^{\mu/2}|z - z'| - b|\tilde{Q}_\mu f(z', 0)|)| dz'$$

$$\leq C h 2^{\mu/2} \tilde{A}_\mu f(z'')(1 + 2^{\mu/2}|z'' - z|)^a.$$  

Similarly,

$$|Y \tilde{Q}_\mu f(z, t)| \leq C h 2^{\mu/2} \tilde{A}_\mu f(z'')(1 + 2^{\mu/2}|z'' - z|)^a.$$  

Now, applying (4.8), we get (4.11).
Lemma 4.13. \( \tilde{A}_\mu f(z,t) \leq C[M(\tilde{Q}_\mu f)|^r](z)^{1/r} \), where \( M \) is the classical Hardy-Littlewood maximal operator on \( \mathbb{R}^2 \) and \( r = 2/a \).

Proof. We conclude from the main value theorem for stratified groups (cf. [FS], Theorem 1.41) that there is a constant \( C \) such that for \( h_3 \in B_0(2^{-\mu/2}\delta) \)
\[
|\tilde{Q}_\mu f(h_1 h_2)| \leq C|\tilde{Q}_\mu f(h_1 h_2 h_3)| + C2^{-\mu/2}\delta \sup_{a \in B_0(C2^{-\frac{\mu}{2}\delta})} |\nabla \tilde{Q}_\mu f(h_1 h_2 h_4)|.
\]
This gives
\[
|\tilde{Q}_\mu f(h_1 h_2)| \leq C \left( \left| B_0(2^{-\mu/2}\delta) \right|^{-1} \int_{B_0(2^{-\mu/2}\delta)} \left| \tilde{Q}_\mu f(h_1 h_2 h_3) \right|^r dh_3 \right)^{1/r} + C2^{-\mu/2}\delta \sup_{h_4 \in B_0(C2^{-\frac{\mu}{2}\delta})} \left( \left| \nabla \tilde{Q}_\mu f(h_1 h_2 h_4) \right| (1 + 2\mu/2d(0, h_2 h_4))^a \right)
\]
\[
\leq C \left( \left| B_0(2^{-\mu/2}\delta) \right|^{-1} \int_0^{\min(2\delta, 2^{-\frac{\mu}{2}\delta})} \int_{|z_3| < 2^{-\frac{\mu}{2}\delta}} \left| \tilde{Q}_\mu f(h_1 h_2 h_3) \right|^r dh_3 \right)^{1/r} + C2^{-\mu/2}\delta \sup_{h_4 \in B_0(C2^{-\frac{\mu}{2}\delta})} \left( 1 + 2\mu/2d(0, h_2 h_4) \right)^a
\]
\[
\leq C \left( 2^{-\mu/2}\delta \right)^{-2} \int_{|z_3| < 2^{-\frac{\mu}{2}\delta}} \left| \tilde{Q}_\mu f(z_1 + z_2 + z_3) \right|^r dz_3 \quad \text{and} \quad C2^{-\mu/2}\delta (1 + \delta + 2\mu/2d(0, h_2))^a \tilde{B}_\mu f(h_1).
\]
Using Lemma 4.10, we get
\[
|\tilde{Q}_\mu f(h_1 h_2)| \leq C \left( \frac{2^{-\mu/2}\delta + |z_2|^2}{2^{-\mu/2}\delta} \right)^{1/r} \left( \frac{1}{2^{-\mu/2}\delta + |z_2|^2} \int_{|z_3| < 2^{-\frac{\mu}{2}\delta} + |z_2|} \left| \tilde{Q}_\mu f(z_1 + z_3) \right|^r dz_3 \right)^{1/r} + C\delta (1 + \delta + 2\mu/2d(0, h_2))^a \tilde{A}_\mu f(h_1).
\]
Finally there is a constant \( C \) such that for any \( \delta \in (0, 1) \)
\[
|\tilde{Q}_\mu f(h_1 h_2)| \leq C\delta^{-2/r} \left( 1 + 2\mu/2d(0, h_2) \right)^{2/r} (M(\tilde{Q}_\mu f(z_1)|^r) \right)^{1/r} + C\delta (1 + 2\mu/2d(0, h_2))^a \tilde{A}_\mu f(h_1),
\]
which completes the proof of the lemma. \( \square \)

Proof of Theorem A. Let \( 0 < r < \min\{p, q\} \) and \( a = 2/r \). For \( \varphi^{(2)} \) let \( \psi^{(2)} \) be a \( C^\infty \) function satisfying (1.2) such that (4.12) holds. If \( \tilde{R}_\nu^{(2)} \) is the linear operator determined by \( \tilde{R}_\nu^{(2)} \phi_k^{(m)} = \psi^{(2)}(2^{-\nu}d_k)\phi_k^{(m)} \), then
\[
(4.14) \quad \tilde{Q}_\mu^{(1)} = \sum_{\nu=\mu}^{\mu+1} \tilde{Q}_\mu^{(1)} \tilde{R}_\nu^{(2)} \tilde{Q}_\nu^{(2)}.
\]
By Theorem 2.3 the kernels $K_{\nu,\mu}((z,t), (z',t'))$ of the operators $\tilde{Q}_{\mu}^{(1)} \tilde{R}_{\nu}^{(2)}$, $|\nu-\mu| \leq 1$, are bounded by $C_b 2^{b2/2} (1 + 2^{\nu/2} |z' - z|)^{-b}$, thus

$$|\tilde{Q}_{\mu}^{(1)} f(z,t)| \leq C_b \sum_{\nu = \mu}^{\mu+1} \int_{\mathbb{R}^d} 2^{b2/2} (1 + 2^{\nu/2} |z' - z|)^{-b} \tilde{A}_{\nu}^{(2)} f(z) \, dz' \leq C_b \sum_{\nu = \mu}^{\mu+1} \tilde{A}_{\nu}^{(2)} f(z).$$

From Lemma 4.13 we conclude

$$|\tilde{Q}_{\mu}^{(1)} f(z,t)| \leq C \sum_{\nu = \mu}^{\mu+1} [\mathcal{M}(|\tilde{Q}_{\nu}^{(2)} f|^r)(z)]^{1/r}.$$

Using the Fefferman-Stein vector-valued maximal inequality, we get

$$\|f\|_{H_{\nu}^{\alpha,q}(\varphi^{(1)})} \leq C \left( \sum_{\nu = -\infty}^{\infty} (2^{\mu\alpha} [\mathcal{M}(|\tilde{Q}_{\nu}^{(2)} f|^r)(z)]^{1/r})^q \right)^{1/q} \|f\|_{L^p} \leq C \|f\|_{H_{\nu}^{\alpha,q}(\varphi^{(2)})}. \quad \square$$

5. TRIEBEL-LIZORKIN SPACES ASSOCIATED WITH THE HERMITE OPERATOR

Let

$$H = -\Delta + |x|^2$$

be the Hermite operator on $\mathbb{R}^d$. Our main goal in the present section is to prove the following theorem which states that the definition of the Triebel-Lizorkin space $H_{\nu}^{\alpha,q}(\varphi)$ does not depend on the particular choice of $\varphi$ (cf. (1.3)).

**Theorem B.** Let $\alpha \in \mathbb{R}$, $0 < p < \infty$, and $0 < q < \infty$. If $\varphi^{(1)}$ and $\varphi^{(2)}$ are two $C^\infty$ functions satisfying (1.2), then there exists a constant $C$ such that

$$C^{-1} \|f\|_{H_{\nu}^{\alpha,q}(\varphi^{(1)})} \leq \|f\|_{H_{\nu}^{\alpha,q}(\varphi^{(2)})} \leq C \|f\|_{H_{\nu}^{\alpha,q}(\varphi^{(1)})}.$$

Let $h_m$ be normalized eigenfunctions of $H$ with corresponding eigenvalues $a_m$, that is, $H h_m = a_m h_m$.

For $m \in \mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ is the Schwartz class of functions on $\mathbb{R}$, and $\mu \in \mathbb{Z}$ define

$$(5.1) \quad Q_{\mu} = m(2^{-\mu} H).$$

Then

$$(5.2) \quad Q_{\mu} h_m = m(2^{-\mu} a_m) h_m.$$ 

Obviously, for $f = \sum_m (f, h_m) h_m$, we have

$$Q_{\mu} f(x) = \sum_m m(2^{-\mu} a_m) (f, h_m) h_m(x) = \int_{\mathbb{R}^d} f(y) K_{\mu}(x,y) \, dy,$$

where

$$K_{\mu}(x,y) = \sum_m m(2^{-\mu} a_m) h_m(x) h_m(y).$$
Let $\pi$ be the Schrödinger representation of $\mathbb{H}_d$ defined by
\begin{equation}
\pi((z,t))f(u) = e^{i(x \cdot u + \frac{1}{2} x \cdot y + t)}f(y + u).
\end{equation}
Then $\pi_{Y_j} = \frac{\partial}{\partial x_j}$, $\pi_{X_j} = ix_j$, and consequently $\pi_L = -\Delta + |x|^2$ is the Hermite operator on $\mathbb{R}^d$. For $m \in \mathcal{S}(\mathbb{R})$, let $M(x,y,t)$ be the convolution kernel for the operator $m(L)$ on $\mathbb{H}_d$ (cf. Theorem 2.3). Then
\begin{equation}
m(H)f = \pi M f = \int_{\mathbb{H}_d} M(x,y,t)\pi(x,y,t)fdxdydt.
\end{equation}
Applying (5.4) and Theorem 2.3, we get
\begin{equation}
K_{\mu}(x,y) = \int_{\mathbb{R}^d} \int_{-\infty}^{\infty} 2^{d\mu/2}M(w,2^{\mu/2}(y-x),t)e^{i\frac{1}{2}2^{\mu/2}w \cdot (x+y)+2^{-\mu}t}dtdw.
\end{equation}

**Lemma 5.6.** For every $b > 0$ there is a constant $C_b$ such that
\begin{equation}
|K_{\mu}(x,y)| \leq C_b 2^{d\mu/2}(1 + 2^{\mu/2}|x-y|)^{-b}.
\end{equation}

**Proof.** The estimate (5.7) is a consequence of (5.5) and Theorem 2.3.

In order to obtain (5.8) we use the fact that $\frac{\partial}{\partial x_j}K_{\mu}(x,y)$ is the kernel which corresponds to $\pi_{Y_j}\pi_{M_{2-\mu}}$. As $\pi_{Y_j}\pi_{M_{2-\mu}} = -\pi_{Y_j}\pi_{M_{2-\mu}}$, we have
\begin{equation}
\frac{\partial}{\partial x_j}K_{\mu}(x,y) = -\int_{\mathbb{R}^d} \int_{-\infty}^{\infty} 2^{\mu/2}(\tilde{Y}_j M)_{2-\mu}(w,y-x,t)e^{i\frac{1}{2}w \cdot (x+y)+t}dtdw.
\end{equation}
This gives
\begin{equation}
|\frac{\partial}{\partial x_j}K_{\mu}(x,y)| \leq 2^{\mu/2}2^{d\mu/2} \int_{\mathbb{R}^d} \int_{-\infty}^{\infty} |\tilde{Y}_j M(w,2^{\mu/2}(y-x),t)|dtdw.
\end{equation}
The function $\tilde{Y}_j M$ is in the Schwartz class on $\mathbb{H}_d$, therefore (5.8) follows from (5.9).

Using Lemma 5.6, Theorem B can be proved in the same way as Theorem 1.1 in [E1].

**Remark.** Standard arguments show that the norms of the Triebel-Lizorkin spaces for Laguerre and Hermite expansions for parameters $\alpha = 0$, $1 < p < \infty$, and $q = 2$ are equivalent to the $L^p$ norms.

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