PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 12, December 1997, Pages 3541–3545
S 0002-9939(97)03977-4

A COMMUTATIVITY THEOREM FOR SEMIBOUNDED OPERATORS IN HILBERT SPACE

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(Communicated by Palle E. T. Jorgensen)

Dedicated to Allen Devinatz

ABSTRACT. Let A and B be semibounded (bounded from below) operators in a Hilbert space $\mathfrak{H}$ and $\mathfrak{D}$ a dense linear manifold contained in the domains of AB, BA, A^2, and B^2, and such that $ABx = BAx$ for all x in $\mathfrak{D}$. It is shown that if the restriction of $(A+B)^2$ to $\mathfrak{D}$ is essentially self-adjoint, then A and B are essentially self-adjoint and $\bar{A}$ and $\bar{B}$ commute, i.e. their spectral projections permute.

1. Introduction

Let A and B be symmetric operators in a Hilbert space $\mathfrak{H}$ and $\mathfrak{D}$ a dense linear manifold contained in the domains of AB, BA, A^2, and B^2, and such that $ABx = BAx$ for all x in $\mathfrak{D}$. E. Nelson [3] proved that if the restriction of $A^2 + B^2$ to $\mathfrak{D}$ is essentially self-adjoint, then A and B are essentially self-adjoint and $\bar{A}$ and $\bar{B}$ commute. For a simple proof see [5]. B. Fuglede [1] extended the result of Nelson by showing that the condition $A^2 + B^2$ is essentially self-adjoint may be replaced by $p(A, B)$ is essentially self-adjoint, where $p(x, y)$ is a polynomial with real coefficients of degree ≤ 2 whose terms of degree 2 form a definite quadratic form (such a polynomial is called *elliptic*).

Nelson also showed in [3] that the condition $A^2 + B^2$ is essentially self-adjoint cannot be replaced by $A + B$ is essentially self-adjoint. In fact he did show: *There exist symmetric operators A and B with common invariant domain $\mathfrak{D}$ such that $AB = BA$ on $\mathfrak{D}$, $\overline{aA + bB}$ is self-adjoint for all real numbers a and b, and $\bar{A}$ and $\bar{B}$ do not commute.*

Based on another example of Nelson [4, p. 273] B. Fuglede [1] proved the following result: There exist symmetric operators A and B with common invariant domain $\mathfrak{D}$ in a separable Hilbert space such that

(1) $AB\varphi = BA\varphi$ for all φ in $\mathfrak{D}$;
(2) $\bar{A}$ and $\bar{B}$ are self-adjoint;
(3) $\bar{A}$ and $\bar{B}$ do not permute;
(4) if $p(x, y)$ is a real polynomial of degree ≤ 2, then $p(A, B)$ is essentially self-adjoint if and only if p is non-elliptic.

Thus in particular $(A+B)^2$ is essentially self-adjoint and $\bar{A}$ and $\bar{B}$ do not commute.

Received by the editors April 30, 1996.
1991 *Mathematics Subject Classification.* Primary 47B25.

2. Main result

Theorem. *Let A and B be semibounded (bounded from below) operators in a Hilbert space $\mathfrak{H}$ and $\mathfrak{D}$ a dense linear manifold contained in the domains of AB, BA, A^2, and B^2, and such that $ABx = ABx$ for all x in $\mathfrak{D}$. If the restriction of $(A+B)^2$ to $\mathfrak{D}$ is essentially self-adjoint, then A and B are essentially self-adjoint and $\bar{A}$ and $\bar{B}$ commute.*

Proof. We may assume without loss of generality that $A \geq I$ and $B \geq I$. To see this we only need to show that if $S = A + B$, $R = A + B + cI$, $c \in R$, and

$$\Delta = S^2|_{\mathfrak{D}} \text{ the restriction of } S^2 \text{ to } \mathfrak{D},$$

$$\Delta' = R^2|_{\mathfrak{D}} \text{ the restriction of } R^2 \text{ to } \mathfrak{D},$$

then Δ essentially self-adjoint implies that $\bar{\Delta}'$ is self-adjoint.

We first show that $\bar{\Delta}$ self-adjoint implies that $\bar{S}$ is self-adjoint and $\bar{\Delta} = \bar{S}^2$.

$$\Delta \subset \bar{S}^2 \subset S^*\bar{S}, \bar{S}S^*$$

and therefore

$$\bar{\Delta} \subset S^*\bar{S}, \bar{S}S^*.$$

Since $S^*\bar{S}, \bar{S}S^*$ and $\bar{\Delta}$ are self-adjoint it follows that $\bar{\Delta} = S^*\bar{S} = \bar{S}S^*$. Thus $\bar{S}$ is normal and symmetric and hence self-adjoint and

$$\bar{\Delta} = \bar{S}^2.$$

Next

$$\Delta' \subset R^2 \subset \bar{R}^2 = \bar{S}^2 + 2c\bar{S} + c^2 I.$$

If $x \in \mathfrak{D}(\bar{R}^2) = \mathfrak{D}(\bar{S}^2) = \mathfrak{D}(\bar{\Delta})$, there exists a sequence $x_n \in \mathfrak{D}$ such that

$$x_n \to x$$

and

$$\Delta x_n = S^2 x_n \to \bar{\Delta}x = \bar{S}^2 x.$$

Hence

$$\|S(x_n - x_m)\|^2 = (S^2(x_n - x_m)|x_n - x_m) \to 0 \quad \text{as } n, m, \to \infty.$$

Thus

$$Sx_n \to \bar{S}x$$

and

$$\Delta' x_n = R^2 x_n = S^2 x_n + 2cSx_n + c^2 x_n \to \bar{S}^2 x + 2c\bar{S}x + c^2 x = \bar{R}^2 x.$$

Hence

$$x \in \mathfrak{D}(\bar{\Delta}') \quad (\text{and } \bar{\Delta}' x = \bar{R}^2 x).$$

Thus $\bar{R}^2 \subset \bar{\Delta}'$ and $\bar{\Delta}' = \bar{R}^2$ is self-adjoint.

Let $\mathcal{L}[x,y] = (\Delta x|Ay) = (S^2 x|Ay), x, y \in \mathfrak{D}$. $\mathcal{L}$ is a symmetric form since $AB = BA$ on $\mathfrak{D}$ and $\mathcal{L}[x] = \mathcal{L}[x,x] = (S^2x|Ax) = (A^2x|Ax) + 2(BAx|Ax) + (B^2x|Ax) = (A^2x|Ax) + 2(BAx|Ax) + (ABx|Bx) \geq \|Ax\|^2 + 2\|Ax\|^2 + \|Bx\|^2 \geq \|Ax\|^2 \geq \|x\|^2$ for $x \in \mathfrak{D}$. Thus $\mathcal{L}$ is a symmetric form bounded from below.

The symmetric form $\mathcal{L}$ is closable: Suppose

$$x_n \underset{\mathcal{L}}{\to} 0;$$

that is,

$$x_n \to 0 \quad \text{and} \quad \mathcal{L}[x_n - x_m] \to 0, \qquad x_n, x_m \in \mathfrak{D} \text{ as } n, m \to \infty$$
$$\mathcal{L}[x_n] = \mathcal{L}[x_n, x_n - x_m] + \mathcal{L}[x_n, x_m].$$

Therefore

$$\mathcal{L}[x_n] \leq \mathcal{L}[x_n]^{\frac{1}{2}} \mathcal{L}[x_n - x_m]^{\frac{1}{2}} + |\mathcal{L}[x_n, x_m]|.$$

Since $\mathcal{L}[x_n - x_m] \to 0$ there exists for a given $\varepsilon > 0$ an integer N such that

$$\mathcal{L}[x_n - x_m] \leq \varepsilon^2 \quad \text{if } n, m \geq N.$$

Hence

$$\mathcal{L}[x_n] \leq \mathcal{L}[x_n]^{\frac{1}{2}} \varepsilon + |\mathcal{L}[x_n, x_m]| \quad \text{for } n, m \geq N. \tag{1}$$

Now $|\mathcal{L}[x_n, x_m]| = |(S^2 x_n | A x_m)| \to 0$ as $m \to \infty$ since $A x_m \to 0$ as $m \to \infty$ because

$$\mathcal{L}[x_n - x_m] = (S^2(x_n - x_m) | A(x_n - x_m)) \geq \|A(x_n - x_m)\|^2 \to 0 \quad \text{as } n, m \to \infty.$$

Thus $A x_n \to y$ as $\to \infty$. Therefore $\bar{A} 0 = y$ and $y = 0$. Letting $m \to \infty$ in (1) it follows that

$$\mathcal{L}[x_n] \leq \mathcal{L}[x_n]^{\frac{1}{2}} \varepsilon \quad \text{for } n \geq N,$$
$$\mathcal{L}[x_n] \leq \varepsilon^2 \quad \text{for } n \geq N.$$

Thus

$$\mathcal{L}[x_n] \to 0 \quad \text{as } n \to \infty.$$

Let $\tilde{\mathcal{L}}$ be the closure of the symmetric form $\mathcal{L}$ and T the self-adjoint operator associated with $\tilde{\mathcal{L}}$ (cf. [2, p. 322]); then by the 'Second representation theorem' (cf. loc.cit. p. 331)

$$\mathfrak{D}(T^{\frac{1}{2}}) = \mathfrak{D}(\tilde{\mathcal{L}}) \quad \text{and} \quad \tilde{\mathcal{L}}[x, y] = (T^{\frac{1}{2}} x | T^{\frac{1}{2}} y), \qquad x, y \in \mathfrak{D}(\tilde{\mathcal{L}}).$$

Furthermore, $\mathfrak{D}$ is a core of $T^{\frac{1}{2}}$ since it is a core of $\tilde{\mathcal{L}}$.

Now, if $x \in \mathfrak{D}$, then

$$\tilde{\mathcal{L}}[x] = (\Delta x | A x) = \|T^{\frac{1}{2}} x\|^2 \geq \|A x\|^2,$$

and therefore

$$\|\Delta x\| \, \|A x\| \geq \|T^{\frac{1}{2}} x\|^2 \geq \|A x\|^2;$$

hence

$$\|\Delta x\| \geq \|T^{\frac{1}{2}} x\| \geq \|A x\| \quad \text{for all } x \in \mathfrak{D}.$$

It follows that

$$\mathfrak{D}(\bar{S}^2) = \mathfrak{D}(\bar{\Delta}) \subset \mathfrak{D}(T^{\frac{1}{2}}) \subset \mathfrak{D}(\bar{A})$$

and

$$\|\bar{S}^2 x\| \geq \|T^{\frac{1}{2}} x\| \quad \text{for all } x \in \mathfrak{D}(\bar{S}^2),$$

$$\|T^{\frac{1}{2}} x\| \geq \|\bar{A} x\| \quad \text{for all } x \in \mathfrak{D}(T^{\frac{1}{2}}).$$

From

$$\mathcal{L}[x,y] = (S^2x|Ay), \qquad x,y \in \mathfrak{D},$$

it now follows that

$$(\bar{S}^2x|\bar{A}y) = (T^{\frac{1}{2}}x|T^{\frac{1}{2}}y) \quad \text{for } x \in \mathfrak{D}(\bar{S}^2) \text{ and } y \in \mathfrak{D}(T^{\frac{1}{2}}).$$

Let $y \in \mathfrak{D}(T)$; then

$$(\bar{S}^2x|\bar{A}y) = (x|Ty) \quad \text{for all } x \in \mathfrak{D}(\bar{S}^2).$$

Therefore, since $\bar{S}^2$ is self-adjoint, $\bar{A}y \in \mathfrak{D}(\bar{S}^2)$ and

$$\bar{S}^2\bar{A}y = Ty.$$

Thus

$$T \subset \bar{S}^2\bar{A},$$

whence

$$\bar{S}^{-2}T \subset \bar{A}.$$

Since $\bar{S}^{-2}$ is a bounded everywhere defined operator in $\mathfrak{H}$,

$$(\bar{S}^{-2}T)^* = T\bar{S}^{-2} \supset A^* \supset \bar{A} \supset \bar{S}^{-2}T.$$

Thus

$$\bar{S}^{-2}T \subset T\bar{S}^{-2},$$

i.e. $\bar{S}^{-2}$ permutes with T. Hence

$$\overline{\bar{S}^{-2}T} = \overline{T\bar{S}^{-2}} = T\bar{S}^{-2}$$

and $\bar{A} = T\bar{S}^{-2}$.

Thus $\bar{A}$ is self-adjoint and $\bar{S}^{-2}$ permutes with $\bar{A}$. Similarly $\bar{B}$ is self-adjoint and $\bar{S}^{-2}$ permutes with $\bar{B}$. $\bar{A}\bar{S}^{-2}$ and $\bar{B}\bar{S}^{-2}$ are everywhere defined since $\mathfrak{D}(\bar{S}^2) \subset \mathfrak{D}(\bar{A}), \mathfrak{D}(\bar{B})$. It remains to show that $\bar{A}$ and $\bar{B}$ permute. Let $(AB)_0$ be the restriction of AB to $\mathfrak{D}$; then $(AB)_0 \subset AB$ and therefore $(AB)_0^* \supset (AB)^* \supset B^*A^* = \bar{B}\bar{A}$ and

$$[\bar{S}^{-4}(AB)_0]^* = (AB)_0^*\bar{S}^{-4} \supset \bar{B}\bar{A}\bar{S}^{-4} \supset \bar{B}\bar{S}^{-2}\bar{A}\bar{S}^{-2}.$$

It follows, since the domain of $(\bar{B}\bar{S}^{-2})(\bar{A}\bar{S}^{-2})$ is $\mathfrak{H}$, that $[\bar{S}^{-4}(AB)_0]^* = \bar{B}\bar{A}\bar{S}^{-4}$ and therefore

$$\bar{B}\bar{A}\bar{S}^{-4} = \bar{A}\bar{B}\bar{S}^{-4},$$

whence

$$\bar{S}^4\bar{A}^{-1}\bar{B}^{-1} = \bar{S}^4\bar{B}^{-1}\bar{A}^{-1}.$$

Therefore, since $\bar{A}^{-1}$ and $\bar{B}^{-1}$ permute with $\bar{S}^2$,

$$\bar{A}^{-1}\bar{B}^{-1}\bar{S}^4 \subset \bar{S}^4\bar{A}^{-1}\bar{B}^{-1} = \bar{S}^4\bar{B}^{-1}\bar{A}^{-1},$$

and hence

$$\bar{A}^{-1}\bar{B}^{-1} \subset \bar{S}^4\bar{B}^{-1}\bar{A}^{-1}\bar{S}^{-4}.$$

Therefore

$$\bar{A}^{-1}\bar{B}^{-1} = \bar{S}^4\bar{B}^{-1}\bar{A}^{-1}\bar{S}^{-4} \supset \bar{B}^{-1}\bar{S}^4\bar{A}^{-1}\bar{S}^{-4} \supset \bar{B}^{-1}\bar{A}^{-1},$$

and hence

$$\bar{A}^{-1}\bar{B}^{-1} = \bar{B}^{-1}\bar{A}^{-1},$$

which implies that $\bar{A}$ and $\bar{B}$ permute. □

Remark. $\overline{A+B} = \bar{A}+\bar{B}$ for $S = A+B \subset \bar{A}+\bar{B}$ and $\bar{A}+\bar{B}$ is self-adjoint because $\bar{A}$ and $\bar{B}$ are permuting self-adjoint operators which are bounded from below. It follows that $\bar{S} \subset \bar{A}+\bar{B}$ and, since $\bar{S}$ is self-adjoint, $\bar{S} = \bar{A}+\bar{B}$.

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