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ON THE FREDHOLM ALTERNATIVE FOR THE p-LAPLACIAN

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ABSTRACT. Consider

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda|u|^{p-2}u + f(x), \; x \in (0,1), \\ u(0) = \beta u'(0), \quad u'(1) = 0, \end{cases}$$

where $p > 1$ and $\beta \in \mathbb{R} \cup \{\infty\}$ and let λ_1 be the principal eigenvalue of the problem with $f(x) \equiv 0$. For $\lambda = \lambda_1$, we discuss for which values of p and β the Fredholm alternative holds.

1. INTRODUCTION

We consider the solvability of the problem

$$-(|u'|^{p-2}u')' = \lambda|u|^{p-2}u + f(x), \; x \in I, \tag{1.1}$$

$I = (0,1)$, in relation to its homogeneous counterpart

$$-(|u'|^{p-2}u')' = \lambda|u|^{p-2}u, \; x \in I, \tag{1.2}$$

subject to the general boundary condition

$$u(0) = \beta u'(0), \quad u'(1) = 0, \tag{1.3}$$

where $p > 1$ and β is a real number. Note that for $\beta = 0$ we get the mixed boundary condition and for $\beta = \infty$ we have the Neumann boundary condition. It is possible to extend $u(x)$ by letting $u(x) = u(2-x)$ for $x > 1$. (For example, $u(x)$ satisfies the Dirichlet boundary condition on $[0,2]$ if $\beta = 0$.) In this way our results can be extended to more general boundary conditions than $u'(1) = 0$, but for simplicity we consider only (1.3).

For linear operators the following Fredholm Alternative holds (cf. Proposition 19.16 of [Z]).

Theorem A. *Let $T \neq 0$ be a linear, symmetric compact operator on a Hilbert space H.* (i) *If λ is not an eigenvalue of T, then the equation $\lambda u - Tu = f$ has a unique solution for any $f \in H$.* (ii) *If λ is an eigenvalue of T, then the equation $\lambda u - Tu = f$ has a solution if and only if $(f, u) = 0$ for all eigenvectors u of T corresponding to λ.*

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Applying Theorem A to (1.1) with $p = 2$, $H = L^2(I)$, we get

Corollary A. *The equation*

$$-u'' - \lambda_1 u = f(x) \text{ in } I$$

subject to (1.3) has a solution for $f \in L^2(I)$ *if and only if* $\int_I fu_1 \, dx = 0$*, where* λ_1 *is the principal eigenvalue and* u_1 *is the principal eigenfunction, respectively.*

Fučik *et al.* [FNSS] studied the Fredholm alternative for nonlinear operators. They extended Theorem A (i) to the so-called (K, L, a)-homeomorphism (of which the p-Laplacian is a prototype) between two Banach spaces X and Y. As a consequence of their results (cf. Theorem 3.2 of Chapter II of [FNSS]), we get

Theorem B. *If* λ *is not an eigenvalue of (1.2)-(1.3), then (1.1)-(1.3) has a solution for any* $f \in L^q(I)$*, where* $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p < \infty$.

It is then logical to consider the extension of Theorem A (ii) to the p-Laplacian. For the Neumann boundary condition, where the principal eigenvalue $\lambda_1 = 0$ and we may take $u_1 = 1$, we have

Theorem C. *Problem (1.1)-(1.3) with* $\lambda = 0$ *and* $\beta = \infty$ *has a solution for* $f \in L^1(I)$ *if and only if* $\int_I f = 0$.

Note that Theorem C also holds for the PDE case. For simplicity we only present a proof of the ODE version in Section 3. (A similar result for weak solutions on certain domains in $\mathbb{R}^N$ has been given recently (for $p \geq 2$) by Li and Zhen [LZ].)

One might suspect, then, that the Fredholm alternative is another in the growing list of properties of the usual ($p = 2$) Laplacian that permits extension to the case $p \neq 2$. Actually, we shall show the opposite, namely, that the counterpart of Theorem A (ii) for the p-Laplacian fails when $p \neq 2$, except for the Neumann boundary conditions. Letting (λ_1, u_1) be the principal eigenpair of (1.2)-(1.3) (to be given precisely in Section 2), we have

Theorem D. *Let* $p \neq 2$*. If* β *is finite then there exists* $f \in L^q(I)$ *so that (1.1)-(1.3) with* $\lambda = \lambda_1$ *has a solution and* $\int_I fu_1 \neq 0$.

2. The homogeneous problem

From now on we assume $p \neq 2$. We will use $[u]^{p-1}$ to denote $(\text{sgn} u)|u|^{p-1}$. Note that $[u]^{p-1} = |u|^{p-2}u$. We start with the standard eigenvalue problem

$$\begin{aligned} &-(|u'|^{p-2}u')' = \lambda |u|^{p-2}u, \; x \in (0,2), \\ &u(0) = u(2) = 0. \end{aligned} \tag{2.1}$$

We need the following result whose proof can be found in [D] and [HM].

Lemma 1. *Problem (2.1) has a positive eigenfunction* φ_0 *associated with a positive eigenvalue* λ_0 *such that* $\varphi'(0) = 1$, $\varphi'(x) > 0$ *for* $x \in I$, $\varphi_0'(1) = 0$, $\varphi'(x) < 0$ *for* $x \in (1,2)$ *and* φ_0 *is symmetric with respect to* $x = 1$.

Our first result is

Theorem 1. (i) *For any* $\beta \in (-\infty, +\infty)$*, an eigenpair* (λ, u) *of problem (1.2)-(1.3) exists with* $\lambda > 0$, $u \in C^1$ *and* $u' > 0$ *on* $[0,1)$. (ii) *If* $\beta \geq 0$ *then* $u > 0$ *on* $(0,1]$. (iii) *If* $\beta < 0$ *then* $u(0) < 0 < u(1)$*. Moreover if* $u(z) = 0$ *then* $z < \frac{1}{2}$ *and* $u(x) = -u(2z - x)$ *for* $0 \leq x \leq z$. (iv) *If* $\beta \neq 0$ *then* $u(0) \neq \lambda[\beta]^{p-1} \int_I u$.

Proof. (i), (ii) We will show that a rescaling and translating of φ_0 will give us the desired eigenpair. Assume first $\beta > 0$. Let $u(x) = \varphi_0(k(x+s))$ with $k > 0$ and $s > 0$. Then $u(x)$ satisfies

$$-([u']^{p-1})' = k^p \lambda_0 [u]^{p-1}, \; x \in I,$$
$$u(0) = \varphi_0(ks), \quad u'(1) = k\varphi_0'(k(1+s)).$$

In order that (1.3) is satisfied, we must have $k(1+s) = 1$, i.e., $ks = 1-k$, and

$$\beta = \frac{u(0)}{u'(0)} = \frac{\varphi_0(1-k)}{k\varphi_0'(1-k)}.$$

Let $t = 1-k$. We then study the solvability of

$$h(t) := \frac{\varphi_0(t)}{(1-t)\varphi_0'(t)} = \beta$$

for $t \in I$. First we observe that

$$\lim_{t\to 0^+} h(t) = 0, \quad \lim_{t\to 1^-} h(t) = +\infty. \tag{2.2}$$

We calculate that

$$h'(t) = \frac{1}{1-t} + \frac{\varphi_0(t)}{(1-t)^2\varphi_0'(t)} - \frac{\varphi_0(t)\varphi_0''(t)}{(1-t)(\varphi_0'(t))^2}. \tag{2.3}$$

On the other hand, integrating (2.1) with $u = \varphi_0$ over $(0,t)$ for $t < 1$ we obtain

$$(\varphi_0'(t))^{p-1} = 1 - \lambda_0 \int_0^t \varphi_0^{p-1},$$

i.e.,

$$\varphi_0'(t) = \Big(1 - \lambda_0 \int_0^t \varphi_0^{p-1}\Big)^{\frac{1}{p-1}}.$$

Differentiating this equation we get

$$\varphi_0''(t) = \frac{1}{p-1}\Big(1 - \lambda_0 \int_0^t \varphi_0^{p-1}\Big)^{\frac{2-p}{p-1}} (-\lambda_0(\varphi_0(t))^{p-1}) < 0$$

for $t \in I$. We then conclude from (2.3) that $h(t)$ is strictly increasing. This together with (2.2) implies that $h(t) = \beta$ has a unique solution $t \in I$ for any $\beta > 0$. Hence problem (1.2)-(1.3) has a unique eigenpair for $\beta > 0$. For $\beta = 0$ we set $u = \varphi_0$.

(iii) Extending φ_0 to $x < 0$ by

$$\varphi_0(x) = -\varphi_0(-x) \tag{2.4}$$

one can deal with the case $\beta < 0$. The point z is the image of 0 after translating and shifting, and symmetry on $[0, 2z]$ follows from (2.4).

(iv) Without loss of generality we assume $|u(0)| = 1$.

Suppose first that $\beta > 0$, so $u'(0) = 1/\beta > 0$. Since $-([u']^{p-1})\Big|_0^1 = \lambda \int_I u^{p-1}$ we obtain

$$\frac{1}{\beta^{p-1}} = \lambda \int_I u^{p-1} > \lambda \int_I u$$

if $p > 2$, with the inequality reversed if $p < 2$. In both cases, then, $u(0) = 1 \neq \lambda[\beta]^{p-1} \int_I u$.

Now suppose that $\beta < 0$. Then we have $u(0) = -1$ and $u'(0) = -1/\beta > 0$. Again, since $-([u']^{p-1})\Big|_0^1 = \lambda \int_I [u]^{p-1}$ we obtain

$$-\frac{1}{[\beta]^{p-1}} = \lambda \int_I [u]^{p-1}.$$

But by the symmetry in (iii), we have, if $p < 2$,

$$\int_I [u]^{p-1} = \int_{2z}^1 [u]^{p-1} < \int_{2z}^1 u = \int_I u$$

since $u > 1 = u(2z)$ on $(2z, 1]$, and the inequality is reversed if $p > 2$. Thus again $u(0) = -1 \neq \lambda[\beta]^{p-1} \int_I u$. □

3. The inhomogeneous problem

Throughout this section we let (λ_1, u_1) be the principal eigenpair given by Theorem 1.

We start with a proof of Theorem C.

Proof of Theorem C. With $w = [u']^{p-1}$ we can rewrite (1.1) with $\lambda = 0$ subject to the initial conditions $u(0) = u'(0) = 0$ as an initial value problem

$$\begin{cases} u' = w|w|^{\frac{2-p}{p-1}}, \ x > 0, \\ w' = -f, \quad u(0) = w(0) = 0. \end{cases} \tag{3.1}$$

Then by Carathéodory's result (cf. Theorem 1.1 of Chapter 2 of [CL]), for $f \in L^1(I)$, (3.1) has a local solution on the interval $[0, \hat{x})$ for some $\hat{x} > 0$. Since

$$w(x) = -\int_0^x f \tag{3.2}$$

and $f \in L^1(I)$, we see that w is bounded on I, which in turn implies that u is bounded on I since $u(x) = \int_0^x w|w|^{\frac{2-p}{p-1}}$. Thus we can extend u so that $\hat{x} \geq 1$. Now, (3.2) implies

$$[u'(1)]^{p-1} = -\int_0^1 f.$$

Thus $\int_I f = 0$ if and only if $u'(1) = 0$. This proves the theorem. □

Remark 3.1. It is not clear yet whether the Fredholm alternative holds for the Neumann boundary condition for the other eigenvalues.

We conclude this paper with the proof of Theorem D.

Proof of Theorem D. Define u by $u(0) = \frac{|\beta|}{\beta+\varepsilon}$, $u' \equiv \frac{\text{sgn}\beta}{\beta+\varepsilon}$ on $[0, \zeta]$, where $\zeta = \varepsilon - \varepsilon^2$, $u'(x) = \frac{\varepsilon - x}{\varepsilon^2|\beta+\varepsilon|}$ on $(\zeta, \varepsilon]$ and $u' \equiv 0$ on $(\varepsilon, 1]$. Here we write sgn0 $= 1$ and we choose ε small enough so that $u'(0) > 0$. It is easily seen that $([u']^{p-1})' \in L^q$ (even L^∞ for $p \geq 2$) so $f := -\lambda_1[u]^{p-1} - ([u']^{p-1})' \in L^q$. Note that on $[\varepsilon, 1]$,

$$u \equiv \frac{\beta + \varepsilon - \frac{\varepsilon^2}{2}}{|\beta + \varepsilon|} = \text{sgn}\beta + O(\varepsilon). \tag{3.3}$$

We now estimate $\int_I fu_1 = a - b - c$ where

$$b = \lambda_1 \int_0^\varepsilon [u]^{p-1} u_1, \quad c = \lambda_1 \int_\varepsilon^1 [u]^{p-1} u_1,$$

and

$$a = -[u']^{p-1} u_1 \Big|_0^1 + \int_0^\zeta [u']^{p-1} u_1' + \int_\zeta^\varepsilon [u']^{p-1} u_1'.$$

By (3.3), $c \to \operatorname{sgn}(\beta)\lambda_1 \int_I u_1$ as $\varepsilon \to 0$, and evidently $b = O(\varepsilon)$. The first two terms in a yield $(u_1[u']^{p-1})(\zeta)$ which tends to $u_1(0)/|\beta|^{p-1}$ as $\varepsilon \to 0$ if $\beta \neq 0$ and equals $u_1'(\zeta)\varepsilon^{2-p}$ plus higher order terms in ε if $\beta = 0$. The final term in a is

$$u_1'(0) \int_\zeta^\varepsilon (\varepsilon - x)^{p-1} dx / (\varepsilon^2 |\beta + \varepsilon|)^{p-1}$$

plus higher order terms in ε. This leading term is therefore $O(\varepsilon^2)$ if $\beta \neq 0$ and $u_1'(0)\varepsilon^{3-p}$ if $\beta = 0$.

In summary, if $\beta \neq 0$, then $\int_I fu_1 \to \frac{u_1(0)}{|\beta|^{p-1}} - \operatorname{sgn}(\beta)\lambda_1 \int_I u_1 \neq 0$ by Theorem 1(iv) as $\varepsilon \to 0$. Thus for sufficiently small ε, we do indeed have $\int_I fu_1 \neq 0$. If $\beta = 0$ and $p < 2$ then $\int_I fu_1 \to -\lambda_1 \int_I u_1$ so the conclusion is the same by virtue of Theorem 1 (ii). Finally if $\beta = 0$ and $p > 2$ then $\int_I fu_1 \to +\infty$ since $u_1'(0) > 0$, and again we can ensure $\int_I fu_1 \neq 0$ for small enough ε. □

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