PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 12, December 1997, Pages 3647–3654
S 0002-9939(97)04034-3

STABLE RANK OF THE REDUCED C^*-ALGEBRAS OF NON-AMENABLE LIE GROUPS OF TYPE I

TAKAHIRO SUDO

(Communicated by Palle E. T. Jorgensen)

Abstract. In this paper we show that stable rank of the reduced C^*-algebras of connected non-compact real semi-simple Lie groups is estimated by real rank of these groups. We extend this result to the case of connected reductive Lie groups and partially even to the case of connected non-amenable real Lie groups of type I. As a corollary, we show that the product formula of stable rank holds for locally compact, σ-compact non-amenable groups of type I.

1. Introduction

The concept of stable rank of C^*-algebras, i.e. non-commutative complex dimension, was introduced by M. A. Rieffel [R]. He raised the problem concerning the determination of stable rank of the C^*-algebras of Lie groups in terms of the geometrical structure of these groups. H. Takai and the author [ST2] succeeded in the computation of stable rank of the C^*-algebras of simply-connected connected solvable Lie groups of type I as a generalization of our result [ST1]. Hence stable rank of the C^*-algebras of the radical part of simply-connected connected Lie groups of type I has been computed. We also obtained the partial results in the case of connected solvable Lie groups of type I.

In this paper, we first focus our attention on the non radical part of connected Lie groups, i.e. connected non compact real semi-simple Lie groups. They are non-amenable so that we only consider their reduced C^*-algebras. We show that stable rank of these algebras is handled by real rank of those groups. This result extends to the case of connected reductive Lie groups and partially even to the case of connected non-amenable Lie groups of type I. As a corollary, we show that the product formula of stable rank holds for the reduced C^*-algebras of locally compact, σ-compact non-amenable groups of type I.

We give the definition of stable rank of C^*-algebras as follows: Let $\mathfrak{A}$ be an unital C^*-algebra. Then we denote by $\mathrm{sr}(\mathfrak{A}) \leq n$ $(n \geq 1)$ if every element $(a_i)_{i=1}^n$ of the n-direct product $\mathfrak{A}^n$ of $\mathfrak{A}$ is approximated by an element $(b_i)_{i=1}^n$ of $\mathfrak{A}^n$ such that $\sum_{i=1}^n b_i b_i^*$ is invertible in $\mathfrak{A}$. Stable rank $\mathrm{sr}(\mathfrak{A})$ of $\mathfrak{A}$ is defined by the smallest integer with the above property. If there exist no such integers, then we let $\mathrm{sr}(\mathfrak{A}) = \infty$. If $\mathfrak{A}$ is non unital, then we define $\mathrm{sr}(\mathfrak{A}) = \mathrm{sr}(\mathfrak{A}^\sim)$ where $\mathfrak{A}^\sim$ is the unitization of $\mathfrak{A}$. We use the basic results of stable rank in [R] later.

Received by the editors April 4, 1996 and, in revised form, July 15, 1996.
1991 *Mathematics Subject Classification.* Primary 46L05, 22D25.

Next we recall the basic definitions throughout this paper as follows: Let G be a locally compact group, $\hat{G}$ its spectrum which consists of all continuous irreducible unitary representations of G up to equivalence equipped with hull-kernel topology and $\hat{G}_r$ its reduced dual which is the support of the regular representation of G. Let $C^*(G)$ be the C^*-algebra of G, which is generated by the image of the universal unitary representation of G. Let $C_r^*(G)$ be the reduced C^*-algebra of G, which is generated by the image of the regular representation of G. We identify the spectra $C^*(G)^\wedge$, $C_r^*(G)^\wedge$ of $C^*(G)$, $C_r^*(G)$ with $\hat{G}$, $\hat{G}_r$ respectively.

2. The case of semi-simple Lie groups

First of all, we give some basic properties of connected non compact real semi-simple Lie groups (refer to [Kn]).

Let G be a connected non compact real semi-simple Lie group with its Lie algebra $\mathfrak{g}$. Let θ be a Cartan involution of G, which is an automorphism of G such that $\theta^2 = 1$. Let $K = \{g \in G \,|\, \theta(g) = g\}$ be the maximal compact subgroup of G corresponding to θ. Let $d\theta$ be the differential of θ. Since $(d\theta)^2 = 1$, we have a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}$ where $\mathfrak{k}, \mathfrak{p}$ are $+1, -1$ eigenspaces of $\mathfrak{g}$ under $d\theta$ respectively.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{a}^*$ its real dual space. We identify $\mathfrak{a}^*$ with a Euclidean space. For every φ in $\mathfrak{a}^*$, let $\mathfrak{g}_\varphi$ be its root space defined by

$$\{X \in \mathfrak{g} \,|\, [Y, X] = \varphi(Y)X \text{ for every } Y \in \mathfrak{a}\}.$$

If $\mathfrak{g}_\varphi \neq \{0\}$, we call φ a root of $\mathfrak{g}$. Let Δ be the set of all roots of $\mathfrak{g}$. Fix a basis $\{\varphi_i\}_{i=1}^n$ of $\mathfrak{a}^*$. We call φ positive if $\varphi = \sum_{i=1}^n x_i\varphi_i$ with $x_i = 0 (1 \leq i \leq k)$ and $x_{k+1} > 0$ for some $k \geq 0$. Let Δ^+ be the set of all positive roots of $\mathfrak{g}$. Put $\mathfrak{n} = \sum_{\varphi \in \Delta^+} \mathfrak{g}_\varphi$ which is a nilpotent Lie subalgebra of $\mathfrak{g}$. Then $\mathfrak{g}$ decomposes into the direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$.

Let K, A and N be the Lie subgroups of G corresponding to $\mathfrak{k}, \mathfrak{a}$ and $\mathfrak{n}$ respectively. Then G has an Iwasawa decomposition $G = KAN$. Define by $\mathrm{rr}(G)$ the dimension of A, i.e. real rank of G. Let $M = Z_K(\mathfrak{a})$ which is defined by

$$\{g \in K \,|\, \mathrm{Ad}(g)X = X \text{ for every } X \in \mathfrak{a}\}.$$

It is a compact subgroup of G with its Lie algebra $\mathfrak{z}_\mathfrak{k}(\mathfrak{a})$ which is defined by

$$\{X \in \mathfrak{k} \,|\, [Y, X] = 0 \text{ for every } Y \in \mathfrak{a}\}.$$

Then $P = MAN$ is a Lie subgroup of G, which is called a minimal parabolic subgroup of G determined uniquely up to conjugacy.

Let W be the Weyl group defined by the quotient $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ where $N_K(\mathfrak{a})$ is defined by $\{g \in K \,|\, \mathrm{Ad}(g)\mathfrak{a} = \mathfrak{a}\}$. Then W acts on $\hat{M} \times \hat{A}$ as follows:

$$w \cdot \sigma(m) = \sigma(u^{-1}mu), \quad \sigma \in \hat{M}, m \in M, \quad w \cdot \chi_s(a) = \chi_s(u^{-1}au), \quad a \in A,$$

where u is any representative of w in W, s is in $\mathfrak{a}^*$ and $\chi_s(\exp X) = e^{is(X)}$ for X in $\mathfrak{a}$. We identify χ_s in $\hat{A}$ with s in $\mathfrak{a}^*$. Let (σ, s) be an element of $\hat{M} \times \hat{A}$. We denote by $[(\sigma, s)]$ the orbit of (σ, s) under W and by $(\hat{M} \times \hat{A})/W$ the orbit space of $\hat{M} \times \hat{A}$.

Then the induced representations $\mathrm{ind}_{P\uparrow G}(\sigma \otimes \chi_s)$ of $\sigma \otimes \chi_s$ to G are in $\hat{G}_r$ where $\sigma \otimes \chi_s$ are the unitary representations of P defined by $\sigma \otimes \chi_s(man) = \sigma(m)\chi_s(a)$ for m in M, a in A and n in N. Put $\pi(\sigma, s) = \mathrm{ind}_{P\uparrow G}(\sigma \otimes \chi_s)$. Then $\pi(\sigma, s)$ is unitarily equivalent to $\pi(\sigma', s')$ if and only if there exists an element w of W such that $w\cdot(\sigma,s)=(\sigma',s')$. Thus we denote by $\pi([(\sigma,s)])$ the equivalence class of $\pi(\sigma,s)$.

We refer to [L] for a topology on $\hat{G}_r$. Then the following lemma is obtained:

Lemma 2.1. *Let G be a connected non compact real semi-simple Lie group and $C_r^*(G)$ its reduced C^*-algebra. If* $\mathrm{rr}(G)\geq 2$*, then* $\mathrm{sr}(C_r^*(G))\geq 2$.

Proof. It is known that $\pi([(1_M,s)])$ is irreducible for every s in $\hat{A}$ where 1_M is the trivial representation of M [Ko]. Since $\{1_M\}\times\hat{A}$ is W-invariant clopen subset of $\hat{M}\times\hat{A}$, we see that $(\{1_M\}\times\hat{A})/W=\hat{A}/W$ is clopen in $(\hat{M}\times\hat{A})/W$. Thus there exist the direct summands $\mathfrak{I}$ and $\mathfrak{K}$ of $C_r^*(G)$ such that $C_r^*(G)=\mathfrak{I}\oplus\mathfrak{K}$, $\hat{\mathfrak{I}}=\hat{A}/W$ and $\hat{\mathfrak{K}}$ is the complement of $\hat{A}/W$ in $(\hat{M}\times\hat{A})/W$. Since $C_r^*(G)$ is liminal, so is $\mathfrak{I}$. As $\hat{A}/W$ is a locally compact T_2-space, $\mathfrak{I}$ is isomorphic to the C^*-algebra associated with the continuous fields on $\hat{\mathfrak{I}}$ [D, Theorem 10.5.4.]. We take a closed ideal $\mathfrak{L}$ of $\mathfrak{I}$, which is of continuous trace. It is also isomorphic to the C^*-algebra associated with the continuous fields on $\hat{\mathfrak{L}}$. By its local triviality [D, Theorem 10.9.5.], there exists a closed ideal $\mathcal{E}$ of $\mathfrak{L}$, which is isomorphic to $C_0(\hat{\mathcal{E}})\otimes\mathbb{K}$ where $C_0(\hat{\mathcal{E}})$ is the C^*-algebra of all continuous functions on $\hat{\mathcal{E}}$ vanishing at infinity, and $\mathbb{K}$ is the C^*-algebra of all compact operators on a countably infinite dimensional Hilbert space. Since $\dim\hat{A}\geq 2$ and W is finite, we see $\dim(\hat{A}/W)\geq 2$ so that $\dim\hat{\mathcal{E}}\geq 2$. Thus $\mathrm{sr}(\mathcal{E})=2$. Therefore $\mathrm{sr}(C_r^*(G))\geq 2$. □

We refer to [BM] for a topology on $\hat{G}_r$ in the case $\mathrm{rr}(G)=1$ Then we have the following lemma:

Lemma 2.2. *Let G be a connected non compact real semi-simple Lie group and $C_r^*(G)$ its reduced C^*-algebra. If* $\mathrm{rr}(G)=1$*, then* $\mathrm{sr}(C_r^*(G))=1$.

Proof. If $\mathrm{rr}(G)=1$, then $\hat{A}\cong\mathbb{R}$ and $W=\{1,w\}$ where w is the unique non trivial element of W. It acts on $\hat{M}\times\hat{A}$ as follows:

$$1\cdot(\sigma,s)=(\sigma,s),\quad w\cdot(\sigma,s)=(w\cdot\sigma,-s),\quad(\sigma,s)\in\hat{M}\times\hat{A}.$$

Then $(\hat{M}\times\hat{A})/W$ is a locally compact T_2-space. Let $F=\{\sigma\in\hat{M}\,|\,w\cdot\sigma=\sigma\}$. Then $(F\times\hat{A})/W=F\times[0,\infty)$. Then $F\times(0,\infty)$ is embedded in $\hat{G}_r$. Each point $(\sigma,0)$ of $F\times\{0\}$ corresponds to two irreducible representations $\{\pi_\sigma^+,\pi_\sigma^-\}$ of G. The topology on $(\{\sigma\}\times(0,\infty))\cup\{\pi_\sigma^+,\pi_\sigma^-\}$ is the usual topology except that $\{(\sigma,s)\}$ converges to $\pi_\sigma^+,\pi_\sigma^-$ as s tends to 0.

Let C be the complement of F in $\hat{M}$. Let (σ,s) be in $C\times\hat{A}$. Then we have that $(\{\sigma\}\times\hat{A}\sqcup\{w\cdot\sigma\}\times\hat{A})/W=\mathbb{R}$. It follows that $(C\times\hat{A})/W=\sqcup_{C/W}\mathbb{R}$. Then $\hat{G}_r$ decomposes in the following fashion:

$$\hat{G}_r=\hat{G}_p\cup\hat{G}_l\cup\hat{G}_d,\quad\hat{G}_p=(F\times(0,\infty))\sqcup(\sqcup_{C/W}\mathbb{R}),\quad\hat{G}_l=\sqcup_{\sigma\in F}\{\pi_\sigma^+,\pi_\sigma^-\},$$

and $\hat{G}_d$ is the discrete series of G.

We construct a finite composition series $\{\mathfrak{I}_k\}_{k=1}^3$ of $C_r^*(G)$ with $\mathfrak{I}_0=\{0\}$ and $\mathfrak{I}_3=C_r^*(G)$ as follows: $\hat{\mathfrak{I}}_1=\hat{G}_p$, $(\mathfrak{I}_2/\mathfrak{I}_1)^\wedge=\hat{G}_l$ and $(\mathfrak{I}_3/\mathfrak{I}_2)^\wedge=\hat{G}_d$. Then

$$\mathfrak{I}_1\cong(\oplus_{C/W}C_0(\mathbb{R})\otimes\mathbb{K})\oplus(\oplus_F C_0((0,\infty))\otimes\mathbb{K}),$$
$$\mathfrak{I}_2/\mathfrak{I}_1\cong\oplus_F(\mathbb{K}\oplus\mathbb{K}),\quad\mathfrak{I}_3/\mathfrak{I}_2\cong\oplus_{\hat{G}_d}\mathbb{K}.$$

Then $\{\mathfrak{I}_k/\mathfrak{I}_{k-1}\}_{k=1}^3$ have stable rank 1 and $\{\mathfrak{I}_k/\mathfrak{I}_{k-1}\}_{k=2}^3$ have connected stable rank 1 (cf. [R]). Therefore $\mathrm{sr}(C_r^*(G)) = 1$. □

Next result is useful in the computation of stable rank.

Proposition 2.3. *Let G be a locally compact, σ-compact non-amenable group of type I and $C_r^*(G)$ its reduced C^*-algebra. Then* $\mathrm{sr}(C_r^*(G)) \le 2$.

Proof. It is known that if $\hat{G} \neq \hat{G}_r$, then every element of $\hat{G}_r$ is infinite dimensional [F]. By [ST2, Proposition 3.1], the proof is complete. □

We give an application of Proposition 2.3 to show the product formula of stable rank in the case of the reduced C^*-algebras of locally compact, σ-compact non-amenable groups of type I as follows:

Corollary 2.4. *Let G, H be two connected locally compact, σ-compact non-amenable groups of type I, and $C_r^*(G)$, $C_r^*(H)$ their reduced C^*-algebras respectively. Then*

$$\mathrm{sr}(C_r^*(G) \otimes C_r^*(H)) \le \mathrm{sr}(C_r^*(G)) + \mathrm{sr}(C_r^*(H)).$$

Proof. Let e_G, e_H and $e_{G\times H}$ be the units of G, H and $G \times H$ respectively. Let $1_G, 1_H$ and $1_{G\times H}$ be their trivial representations, and λ_G, λ_H and $\lambda_{G\times H}$ their regular representations respectively. Then by [FD, Corollary 12.18, 13.6],

$$\lambda_{G\times H} \simeq \operatorname*{ind}_{\{e_{G\times H}\}\uparrow G\times H} 1_{G\times H} \simeq (\operatorname*{ind}_{\{e_G\}\uparrow G} 1_G) \otimes (\operatorname*{ind}_{\{e_H\}\uparrow H} 1_H) \simeq \lambda_G \otimes \lambda_H$$

where $\simeq$ is unitary equivalence. Thus $C_r^*(G \times H)$ is isomorphic to $C_r^*(G) \otimes C_r^*(H)$. By Proposition 2.3, $\mathrm{sr}(C_r^*(G) \otimes C_r^*(H)) \le 2$. Therefore the proof is complete. □

Combining Lemma 2.1, 2.2 and Proposition 2.3, we have the following theorem:

Theorem 2.5. *Let G be a connected non compact real semi-simple Lie group and $C_r^*(G)$ its reduced C^*-algebra. Then*

$$\mathrm{sr}(C_r^*(G)) = \mathrm{rr}(G) \wedge 2$$

where $\wedge$ means the minimum.

Remark 2.6. This result suggests that stable rank of the reduced C^*-algebras of connected non compact real semi-simple Lie groups is controlled by the real rank (i.e. the geometrical structure) of G. Note that $\mathrm{rr}(G) = 0$ if and only if G is compact. Then $\hat{G}$ is discrete. Thus $C_r^*(G)$ is isomorphic to $\oplus_{\lambda\in\hat{G}} M_{n_\lambda}(\mathbb{C})$ where $M_{n_\lambda}(\mathbb{C})$ is the C^*-algebra of all $n_\lambda \times n_\lambda$ complex matrices. Hence $\mathrm{sr}(C_r^*(G)) = 1$.

We give some examples which support Theorem 2.5 in what follows:

Example 2.7. Let G be a connected real semi-simple Lie group with $\mathrm{rr}(G) = 1$. Then it is known that G is locally isomorphic to one of the following groups (cf. [HV]):

$$\begin{aligned} &SO_0(n,1), \quad SU(n,1), \\ &Sp(n,1), \quad F_{4(-20)} \quad (n \ge 2). \end{aligned}$$

Thus their reduced C^*-algebras have stable rank 1.

Example 2.8. Let $G = SL_n(\mathbb{R})$ for $n \geq 2$. Its Iwasawa decomposition is obtained as follows: Then $K = SO_n(\mathbb{R})$. A consists of all diagonal matrices such that

$$\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$$

where $a_i > 0$ $(1 \leq i \leq n)$ and $\Pi_{i=1}^n a_i = 1$. It is isomorphic to $(\mathbb{R}_+^*)^{n-1}$ where $\mathbb{R}_+^* = \{t \in \mathbb{R} \,|\, t > 0\}$. N consists of all upper triangular matrices such that

$$\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

Thus $\mathrm{rr}(G) = 1$ if and only if $n = 2$. Therefore we obtain that

$$\mathrm{sr}(C_r^*(SL_n(\mathbb{R}))) = \begin{cases} 1 & \text{if } n = 2, \\ 2 & \text{if } n \geq 3. \end{cases}$$

Example 2.9. Let $G = \widetilde{SL_n(\mathbb{R})}$ be the universal covering group of $SL_n(\mathbb{R})$ for $n \geq 2$. It is known that G is a non linear semi-simple Lie group. Since the fundamental group of $SL_n(\mathbb{R})$ is equal to $\mathbb{Z}$ $(n = 2)$ and $\mathbb{Z}_2$ $(n \geq 3)$, we have that $G/\mathbb{Z} \cong SL_2(\mathbb{R})$ and $G/\mathbb{Z}_2 \cong SL_n(\mathbb{R})$ $(n \geq 3)$ respectively. Since $\mathbb{Z}$ and $\mathbb{Z}_2$ are amenable closed normal subgroups of G, we know that $C_r^*(SL_n(\mathbb{R}))$ is the quotient of $C_r^*(G)$ (cf. [Ka, p.1349]). By Example 2.8, $\mathrm{sr}(C_r^*(G)) \geq 2$ if $n \geq 3$. If $n = 2$, then $\mathrm{rr}(G) = 1$. Therefore we obtain that

$$\mathrm{sr}(C_r^*(\widetilde{SL_n(\mathbb{R})})) = \begin{cases} 1 & \text{if } n = 2, \\ 2 & \text{if } n \geq 3. \end{cases}$$

3. The case of reductive Lie groups

In this section, we show that Theorem 2.5 extends to the case of connected reductive Lie groups. First of all, we examine the structure of these groups.

Let G be a connected real reductive Lie group with its Lie algebra $\mathfrak{g}$ and $\tilde{G}$ its universal covering group. Then $\mathfrak{g}$ has Levi decomposition $\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}]$ where $\mathfrak{z}$ is the center of $\mathfrak{g}$. It is known that any two simply-connected Lie groups with the same Lie algebras are isomorphic (cf. [Kn, Appendix A.114]). Thus, $\tilde{G}$ is isomorphic to the direct product $Z \times S$ where Z is the Lie subgroup of $\tilde{G}$ with its Lie algebra $\mathfrak{z}$ and S is the semi-simple Lie subgroup of $\tilde{G}$ with its Lie algebra $[\mathfrak{g}, \mathfrak{g}]$. Then the center $Z_{\tilde{G}}$ of $\tilde{G}$ is of the form $Z \times Z_S$ where Z_S is the center of S. Let Γ be a discrete subgroup of $\tilde{G}$ contained in $Z_{\tilde{G}}$ such that $G = (Z \times S)/\Gamma$. Then Γ is isomorphic to the direct product $\Gamma_Z \times \Gamma_S$ where Γ_Z and Γ_S are discrete subgroups of Z and Z_S respectively. Thus we have $G = (Z/\Gamma_Z) \times (S/\Gamma_S)$.

Let $G_a = Z/\Gamma_Z$ be the abelian direct factor of G and $G_s = S/\Gamma_S$ the semi-simple one of G. Note that G_s is equal to the commutator subgroup $[G, G]$ of G. By the same reason in Corollary 2.4, $C_r^*(G)$ is isomorphic to $C_r^*(G_a) \otimes C_r^*(G_s)$. Thus $\hat{G}_r = \hat{G}_a \times (\hat{G}_s)_r$. Hence $\hat{G} \neq \hat{G}_r$ if and only if $\mathrm{rr}(G_s) \geq 1$. Since G_a is isomorphic to $\mathbb{R}^k \times \mathbb{T}^{n-k}$ for some $k \geq 0$ and $n = \dim Z \geq 0$, $C_r^*(G_a)$ is isomorphic to $C_0(\mathbb{R}^k \times \mathbb{Z}^{n-k})$.

We denote by Z_G the center of G. Then $Z_G = G_a \times Z_{G_s}$ where Z_{G_s} is the at most countable center of G_s.

Then we have the following theorem:

Theorem 3.1. *Let G be a connected non-amenable real reductive Lie group with its center Z_G and $C_r^*(G)$ its reduced C^*-algebra. Then*

$$\mathrm{sr}(C_r^*(G)) = (\mathrm{rr}([G,G]) \vee (\dim(Z_G)^\wedge + 1)) \wedge 2$$

where $\vee$ means the maximum.

Proof. If G_a is compact, and $\mathrm{rr}(G_s) = 1$, then $C_r^*(G)$ is isomorphic to $C_0(\mathbb{Z}^n) \otimes C_r^*(G_s)$ for $n = \dim(G_a)$. Using the structure of $C_r^*(G_s)$ in Lemma 2.2, and tensoring $C_0(\mathbb{Z}^n)$ with $C_r^*(G_s)$, we conclude that $\mathrm{sr}(C_r^*(G)) = 1$. On the other hand, since $\dim(Z_G)^\wedge = 0$, we have that $(\mathrm{rr}([G,G]) \vee (\dim(Z_G)^\wedge + 1)) \wedge 2 = 1$.

Next, by the methods of Lemma 2.1, $C_r^*(G_s)$ has a closed ideal $\mathfrak{I}$ which is isomorphic to $C_0(\hat{\mathfrak{I}}) \otimes \mathbb{K}$ where $\dim(\hat{\mathfrak{I}}) = \mathrm{rr}(G_s)$. Then $C_0((G_a)^\wedge) \otimes \mathfrak{I}$ is a closed ideal of $C_r^*(G)$, which is isomorphic to $C_0((G_a)^\wedge \times \hat{\mathfrak{I}}) \otimes \mathbb{K}$. If G_a is non compact and $\mathrm{rr}(G_s) = 1$, or $\mathrm{rr}(G_s) \geq 2$, then $\dim((G_a)^\wedge \times \hat{\mathfrak{I}}) \geq 2$. Thus $\mathrm{sr}(C_0((G_a)^\wedge \times \hat{\mathfrak{I}}) \otimes \mathbb{K}) = 2$. Hence $\mathrm{sr}(C_r^*(G)) \geq 2$. By Proposition 2.3, we conclude that $\mathrm{sr}(C_r^*(G)) = 2$. On the other hand, since $\dim(Z_G)^\wedge \geq 1$ or $\mathrm{rr}([G,G]) \geq 2$, we have $(\mathrm{rr}([G,G]) \vee (\dim(Z_G)^\wedge + 1)) \wedge 2 = 2$. □

Remark 3.2. We consider the case that G is amenable. If G_a is compact, and $\mathrm{rr}(G_s) = 0$, then G is compact. It follows that $\mathrm{sr}(C^*(G)) = 1$.

If G_a is non compact, and $\mathrm{rr}(G_s) = 0$, then G is of the from $\mathbb{R}^k \times \mathbb{T}^{n-k} \times G_s$ for $k \geq 1$ and $n = \dim G_a$, and G_s is compact. Then

$$\begin{aligned} C^*(G) &\cong C_0(\mathbb{R}^k) \otimes C_0(\mathbb{Z}^{n-k}) \otimes C^*(G_s) \\ &\cong (\oplus_{\mathbb{Z}^{n-k}} C_0(\mathbb{R}^k)) \otimes (\oplus_{\lambda \in \hat{G}_s} M_{n_\lambda}(\mathbb{C})) \cong \oplus_{\mathbb{Z}^{n-k}, \lambda \in \hat{G}_s} (C_0(\mathbb{R}^k) \otimes M_{n_\lambda}(\mathbb{C})). \end{aligned}$$

Thus we obtain that

$$\begin{aligned} \mathrm{sr}(C^*(G)) &= \sup_{\lambda \in \hat{G}_s} \mathrm{sr}(C_0(\mathbb{R}^k) \otimes M_{n_\lambda}(\mathbb{C})) \\ &= \sup_{\lambda \in \hat{G}_s} (\lceil (\mathrm{sr}(C_0(\mathbb{R}^k)) - 1)/n_\lambda \rceil + 1) = \sup_{\lambda \in \hat{G}_s} (\lceil ([k/2])/n_\lambda \rceil + 1) \end{aligned}$$

where $[\cdot]$ is Gauss symbol, $\lceil x \rceil = [x] + 1$ for x in $\mathbb{R} \setminus \mathbb{Z}$ and $\lceil x \rceil = x$ for x in $\mathbb{Z}$ (cf. [R]).

Next we give an example which supports Theorem 3.1 as follows:

Example 3.3. Let $G = GL_n(\mathbb{R})_0$ be the connected component of $GL_n(\mathbb{R})$ containing the unit of G for $n \geq 2$, which consists of all invertible matrices with positive determinant. We consider the mapping Φ from G to $\mathbb{R}_+^* \times SL_n(\mathbb{R})$ defined by $\Phi(g) = (\det(g), g/\det(g))$ for g in G. It is clear that Φ is a Lie group isomorphism. Since G_a is non compact, we conclude that

$$\mathrm{sr}(C_r^*(GL_n(\mathbb{R})_0)) = 2 \quad \text{for}\, n \geq 2.$$

4. The case of non-amenable Lie groups of type I

In this section, we show that Theorem 3.1 extends partially to the case of connected real Lie groups of type I.

Let G be a connected real Lie group of type I and R its radical, which is the maximal connected solvable normal Lie subgroup of G. It is known that if G/R is compact, then $\hat{G} = \hat{G}_r$ [D, Proposition 18.3.9]. Thus, if $\hat{G} \neq \hat{G}_r$, then G/R is non compact. We only consider this case. Since R is amenable, we know that $C_r^*(G/R)$ is the quotient of $C_r^*(G)$ (cf. [Ka, p.1349]). Then we have the following result:

Theorem 4.1. *Let G be a connected non-amenable real Lie group of type I with its radical R and $C_r^*(G)$ its reduced C^*-algebra. Then*

$$\mathrm{sr}(C_r^*(G)) = \begin{cases} 1 \textit{ or } 2 & \textit{if } \mathrm{rr}(G/R) = 1, \\ 2 & \textit{if } \mathrm{rr}(G/R) \geq 2. \end{cases}$$

Proof. By Proposition 2.3, we know that $\mathrm{sr}(C_r^*(G)) \leq 2$. By Lemma 2.1, if $\mathrm{rr}(G/R) \geq 2$, then $\mathrm{sr}(C_r^*(G/R)) \geq 2$. Thus $\mathrm{sr}(C_r^*(G)) \geq 2$. Therefore, we obtain $\mathrm{sr}(C_r^*(G)) = 2$. □

Remark 4.2. The above formula is the best inequality. For example, let G be the direct product $\mathbb{T} \times S$ where S is a connected real semi-simple Lie group with $\mathrm{rr}(S) = 1$. By Theorem 3.1, we know that $\mathrm{sr}(C_r^*(G)) = 1$. On the other hand, let G be the direct product $\mathbb{R} \times S$ where S is the same as before. By Theorem 3.1, we have that $\mathrm{sr}(C_r^*(G)) = 2$.

Finally, we give an example which supports Theorem 4.1 as follows:

Example 4.3. Let G be the direct product $H \times SL_n(\mathbb{R})$ for $n \geq 2$ where H is the real 3-dimensional Heisenberg group. Then G is a connected real non reductive Lie group of type I. If $n \geq 3$, then $\mathrm{rr}(G/H) \geq 2$. By Theorem 4.1, we have $\mathrm{sr}(C_r^*(G)) = 2$. Next we consider the case $n = 2$. Then $\mathrm{rr}(SL_2(\mathbb{R})) = 1$. Note that $\hat{G} = \hat{H} \times (SL_2(\mathbb{R}))^\wedge$. Thus $\hat{G}_r = \hat{H} \times (SL_2(\mathbb{R}))_r^\wedge$. It follows that $C_r^*(G) \cong C^*(H) \otimes C_r^*(SL_2(\mathbb{R}))$. It is known that $C^*(H)$ decomposes into the following exact sequence:

$$0 \to C_0(\mathbb{R} \setminus \{0\}) \otimes \mathbb{K} \to C^*(H) \to C_0(\mathbb{R}^2) \to 0.$$

Tensoring $C_r^*(SL_2(\mathbb{R}))$ with this sequence, we have that

$$0 \to C_0(\mathbb{R} \setminus \{0\}) \otimes \mathbb{K} \otimes C_r^*(SL_2(\mathbb{R})) \to C_r^*(G) \to C_0(\mathbb{R}^2) \otimes C_r^*(SL_2(\mathbb{R})) \to 0.$$

Using the structure in Lemma 2.2, we know that $C_r^*(SL_2(\mathbb{R}))$ has $\mathbb{K}$ as a quotient. Thus $C_r^*(G)$ has $C_0(\mathbb{R}^2) \otimes \mathbb{K}$ as a quotient. Hence $\mathrm{sr}(C_r^*(G)) \geq 2$. Therefore we have that

$$\mathrm{sr}(C_r^*(H \times SL_n(\mathbb{R}))) = 2 \quad \text{if } n \geq 2.$$

Acknowledgment

The author would like to thank Professor H. Takai, who suggested these topics, for many valuable conversations and warm encouragement and the referee for reading the manuscript.

References

[BM] R. Boyer and R. Martin, *The regular group C^*-algebra for real rank one groups*, Proc. Amer. Math. Soc. **59** (1976), 371–376. MR **57:**16464

[D] J. Dixmier, *C^*-Algebras*, North-Holland, Amsterdam-New York-Oxford, 1962.

[F] J. M. G. Fell, *Weak containment and Kronecker products of group representations*, Pacific J. Math. **13** (1963), 503–510. MR **27:**5865

[FD] J. M. G. Fell and R. S. Doran, *Representations of *-Algebras, Locally Compact Groups, and Banach *-Algebraic Bundles*, vol. 2, Academic Press, 1988. MR **90c:**46002

[HV] P. de la Harpe et A. Valette, *La propriété (T) de Kazhdan pour les groupes localement compacts*, Astérisque 175, Soc. Math. France, 1989. MR **90m:**22001

[Ka] E. Kaniuth, *Group C^*-algebras of real rank zero or one*, Proc. Amer. Math. Soc. **119** (1993), 1347–1354. MR **94a:**46074

[Kn] A. W. Knapp, *Representation Theory of Semisimple Groups*, An Overview Based on Examples, Princeton Univ. Press, Princeton-New Jersey, 1986. MR **87j:**22022

[Ko] B. Kostant, *On the existence and irreducibility of certain series of representations*, Bull. Amer. Math. Soc. **75** (1969), 627–642. MR **39:**7031

[L] R. L. Lipsman, *The dual topology for the principal and discrete series on semisimple groups*, Trans. Amer. Math. Soc. **152** (1970), 399–417. MR **42:**4673

[R] M. A. Rieffel, *Dimension and stable rank in the K-theory of C^*-algebras*, Proc. London Math. Soc. **46** (1983), 301–333. MR **84g:**46085

[ST1] T. Sudo and H. Takai, *Stable rank of the C^*-algebras of nilpotent Lie groups*, Internat. J. Math. **6** (1995), 439–446. MR **96b:**46083

[ST2] ———, *Stable rank of the C^*-algebras of solvable Lie groups of type I*, preprint (1996).

Department of Mathematics, Tokyo Metropolitan University, 1-1 Minami Ohsawa, Hachioji-shi, Tokyo 192-03, Japan

E-mail address: sudoh@math.metro-u.ac.jp