CESÀRO TRANSFORMS OF FOURIER COEFFICIENTS OF $L^\infty$-FUNCTIONS

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Abstract. In this note, we show that Cesàro transforms of Fourier cosine or sine coefficients of any $L^\infty(0, \pi)$-function are Fourier cosine or sine coefficients of some $BMO(0, \pi)$-function.

Let $p \in [1, \infty)$ and $L^p(0, \pi)$ denote the space of Lebesgue measurable functions $f : (0, \pi) \to (-\infty, \infty)$ with the usual norm $\|f\|_p < \infty$. As is well known, $L^\infty(0, \pi)$, the space of essentially bounded functions $f : (0, \pi) \to (-\infty, \infty)$ with the usual norm $\|f\|_\infty < \infty$, is viewed as a limit space $L^p(0, \pi)$ as $p \to \infty$ in sense of duality. However, in the situation of Hardy space, $L^\infty(0, \pi)$ is substituted by $BMO(0, \pi)$—the space of functions $f \in L^1(0, \pi)$ with bounded mean oscillation: $\|f\|_* = \sup_{I} \frac{1}{|I|} \int_I |f(x) - f_I| dx < \infty$, where the supremum is taken over all subintervals $I$ of $(0, \pi)$, $f_I$ stands for the mean value of $f$ on $I$: $1/|I| \int_I f(x) dx$ and $|I|$ denotes the length of $I$: $|I| = \int_I dx$.

The following inclusion chain is helpful for us to understand the relation between those spaces mentioned above:

$$L^\infty(0, \pi) \subseteq BMO(0, \pi) \subseteq \bigcap_{1 \leq p < \infty} L^p(0, \pi).$$

Now, suppose that $f \in L^1(0, \pi)$ and $a = \{a_n\}$ or $b = \{b_n\}$ is the sequence of Fourier cosine or sine coefficients of $f$ extended to $(-\pi, \pi)$ as an even or odd function, namely,

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx, \quad n = 0, 1, 2, \ldots,$$

or

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx, \quad n = 1, 2, \ldots.$$
In other words, the even or odd extension of \( f \in L^1(0, \pi) \) has a Fourier series below:

\[
 f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx
\]

or

\[
 f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx
\]

The Cesàro transform of \( a = \{a_n\} \) or \( b = \{b_n\} \) is defined by \( Ca = \{A_n\} \) or \( Cb = \{B_n\} \), where

\[
 A_0 = a_0, \quad A_n = \frac{\sum_{k=1}^{n} a_k}{n}, \quad n = 1, 2, \ldots,
\]

or

\[
 B_n = \frac{\sum_{k=1}^{n} b_k}{n}, \quad n = 1, 2, \ldots
\]

A very natural question is raised here: If \( f \in L^p(0, \pi) \) with Fourier cosine or sine coefficients \( a = \{a_n\} \) or \( b = \{b_n\} \) then must \( Ca = \{A_n\} \) or \( Cb = \{B_n\} \) be Fourier cosine or sine coefficients of a function also in \( L^p(0, \pi) \)?

In 1928, Hardy gave a positive answer for the question in the case: \( p \in [1, \infty) \). Since then, there have been some further generalizations, [1, 2]. But there has been no satisfactory result about the case: \( p = \infty \), just like the case \( p \in [1, \infty) \), [4]. For instance, if taking a bounded function \( f(x) = \cos x \) with Fourier cosine coefficients \( a = \{0, 1, 0, 0, \ldots\} \), then we immediately find that \( Ca = \{0, 1, 1/2, 1/3, \ldots\} \) is the sequence of Fourier cosine coefficients of function \( F(x) = \log 1/(2 \sin(x/2)) \).

However, this \( F \) is unbounded, i.e., \( F \notin L^\infty(0, \pi) \). Through a careful observation, we, on the other hand, discover that the function \( F \) is of BMO property, that is to say, \( F \in BMO(0, \pi) \). More importantly, we are motivated by the above argument to enable us to answer the question in the case of \( p = \infty \).

**Theorem.** Let \( f \in L^\infty(0, \pi) \) with Fourier cosine or sine coefficients \( a = \{a_n\} \) or \( b = \{b_n\} \). Then \( Ca = \{A_n\} \) or \( Cb = \{B_n\} \) are Fourier cosine or sine coefficients of some function \( F \in BMO(0, \pi) \).

**Proof.** It is sufficient to verify this fact for Fourier cosine coefficients.

First of all, we define a linear operator \( \sigma \) on \( L^\infty(0, \pi) \), which may be called the Cesàro operator on \( L^\infty(0, \pi) \), and is exactly given by

\[
 (\sigma f)(x) = \int_x^{\pi} \frac{f(t)}{\tan \frac{t}{2}} \, dt, \quad f \in L^\infty(0, \pi).
\]

Also let

\[
 (\lambda f)(x) = 2 \int_x^{\pi} \frac{f(t)}{t} \, dt, \quad f \in L^\infty(0, \pi).
\]

Then \( \lambda f - \sigma f \) is a bounded function, i.e., \( \lambda f - \sigma f \in L^\infty(0, \pi) \). In fact, for \( f \in L^\infty(0, \pi) \), it is easy to get that 

\[
 |(\lambda f)(x) - (\sigma f)(x)| \leq C_1 ||f||_{\infty}, \quad \text{where} \ C_1 = 2 \int_0^\pi \left| \frac{1}{\tan \frac{t}{2}} - \frac{1}{t} \right| \, dt < \infty.
\]

Next, assuming that \( K(t) = -\log |2 \sin(t/2)| \) for \( t \in (-\pi, \pi) \), we get that \( c = \{c_n\} \), where \( c_0 = 0, c_n = a_n/n, n = 1, 2, \ldots \), is the sequence of Fourier cosine
coefficients of function $g(x) = \frac{1}{2} \int_{-\pi}^{\pi} f(x+t)K(t)dt$ [7, p.180]. As Hardy showed in [3], $\sigma a$ is the sequence of Fourier cosine coefficients of function $F(x) = ((\sigma f)(x) + g(x))/2$.

Finally, we prove that $F \in BMO(0, \pi)$. For this end, it suffices to check that $\lambda f$ is in $BMO(0, \pi)$ since $\lambda f - \sigma f \in L^\infty(0, \pi)$ and $\|g\|_\infty \leq C_2\|f\|_\infty$, where $C_2 = \frac{1}{\pi} \int_{-\pi}^{\pi} |\log|2\sin \frac{t}{2}||dt < \infty$. At this time, taking any interval $I = (\alpha, \beta) \subset (0, \pi)$ and $C_I = (\lambda f)(\beta)$ for $f \in L^\infty(0, \pi)$, we obtain that $|I| = \beta - \alpha$ and

$$
\int_I |(\lambda f)(x) - C_I|dx = \int_{\alpha}^{\beta} |\int_{x}^{\beta} \frac{f(t)}{t}dt|dx
\leq \|f\|_\infty \int_{\alpha}^{\beta} \log \frac{\beta}{x}dx
\leq |I|\|f\|_\infty.
$$

Furthermore,

$$
\frac{1}{|I|} \int_I |(\lambda f)(x) - (\lambda f)_I|dx \leq \frac{2}{|I|} \int_I |(\lambda f)(x) - C_I|dx
\leq 2\|f\|_\infty.
$$

That is to say, $\lambda f \in BMO(0, \pi)$. Hence the proof is completed.

Remarks. 1. $L^\infty(0, \pi)$ in Theorem cannot be replaced by $BMO(0, \pi)$. Otherwise, it will follow that $\log^2 |x|$ is a function in $BMO(0, \pi)$, which results in a contradiction. Indeed, if $f \in BMO(0, \pi)$ then the statement that $Ca$ or $Cb$ is a sequence of Fourier cosine or sine coefficients of some function in $BMO(0, \pi)$ holds if and only if the operator $\lambda$ is bounded from $BMO(0, \pi)$ to $BMO(0, \pi)$. Yet, if picking $f(x) = \log(\pi/x)$ then we see that $(\lambda f)(x) = \log^2(\pi/x)$ is outside $BMO(0, \pi)$ due to the unboundedness of $\log(\pi/x)$ on $(0, \pi)$ and Stegenga’s multiplier theorem applied to this $(\lambda f)$, [5]. Of course, we have here used a fact that $F \in BMO(0, \pi)$ once $f \in BMO(0, \pi)$, which is easily derived. Actually, if we write $H^1_R(-\pi, \pi)$ and $BMO(-\pi, \pi)$ as the real Hardy space and BMO (bounded mean oscillation) space on $(-\pi, \pi)$ respectively then Fefferman’s duality theorem tells us that $[H^1_R(-\pi, \pi)]^* = BMO(-\pi, \pi)$ under the inner pair: $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$. [6]. To show that $F \in BMO(0, \pi)$, we only need to prove that the following function

$$
F_*(x) = \begin{cases} 
F(x), x \in (0, \pi), \\
0, x \in (-\pi, 0),
\end{cases}
$$

is in $BMO(-\pi, \pi)$. For this, by Fubini’s theorem we find a constant $C_3$ such that for any $G \in H^1_R(-\pi, \pi)$,

$$
|\int_{-\pi}^{\pi} F_*(x)G(x)dx| = \frac{1}{\pi} |\int_{0}^{\pi} [\int_{-\pi}^{\pi} f(x+t)G(x)dx]K(t)dt|
\leq C_3\|f\|_1\|G\|_1 \int_{0}^{\pi} |K(t)|dt.
$$

Equivalently, $F_* \in BMO(-\pi, \pi)$ and hence $F \in BMO(0, \pi)$. 
2. From the above proof it is turns out that $\sigma$ is bounded linear operator from $L^\infty(0, \pi)$ (not from $BMO(0, \pi)$) to $BMO(0, \pi)$. This operator looks very much like the conjugate operator below:

$$\tilde{f}(x) = \frac{1}{\pi} \int_0^\pi \frac{f(x-t) - f(x+t)}{2\tan \frac{t}{2}} dt.$$  

Nevertheless, we should note that $\tilde{f} \in BMO(0, \pi)$ if $f \in BMO(0, \pi)$.

References

3. G.H. Hardy, Notes on some points in the integral calculus LXVI, Messenger Math. 58 (1928), 50-52.
5. D.A. Stegenga, Bounded Toeplitz operators on $H^1$ and applications on the duality between $H^1$ and the functions of bounded mean oscillation, Amer. J. Math. 98 (1976), 573-589. MR 54:8340

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