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ON A THEOREM OF OSSA

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Abstract. If V is an elementary abelian 2-group, Ossa proved that the connective K-theory of BV splits into copies of $\mathbf{Z}/2$ and of the connective K-theory of the infinite real projective space. We give a brief proof of Ossa's theorem.

Introduction

We have been asked whether our work, [1] and [2] on the Brown-Peterson homology of BV, V an elementary p-group, gives a nice structure of the connective K-theory of BV. The answer is that the approach of [1] leads to the elegant structure theorem of Ossa [4]. Although the approach is motivated by our [1] and [2], the proof is independent of that work. In this reproof of an established theorem we shall limit our exposition to the $p = 2$ case. For us, the notation makes this the easiest case, but for Ossa, it was the more difficult one. With obvious modifications, the odd-primary version of our argument follows the same outline. We thank Don Davis for the Liulevicius reference.

Notation. Let bu be the connective K-theory spectrum and let P denote $B\mathbf{Z}/2$ (also known as infinite real projective space). Let $H\mathbf{Z}/2$ be the $\mathbf{Z}/2$ Eilenberg-Mac Lane spectrum.

Theorem 1 (Ossa)**.** *With the above notation, there is a homotopy equivalence of spectra*

$$bu \wedge P \wedge P \simeq [\bigvee_{0<i,j} \Sigma^{2i+2j-2} H\mathbf{Z}/2] \vee [\Sigma^2 bu \wedge P].$$

Eric Ossa has kindly pointed out that our proof gives this as a homotopy equivalence of BP-module spectra.

Note that $H\mathbf{Z}/2 \wedge P \simeq \bigvee_{0<i} \Sigma^i H\mathbf{Z}/2$. (The proof of this is like that of Lemma 3.) Thus the theorem can be used inductively to split $bu \wedge P \wedge \cdots \wedge P$ into suspended copies of $H\mathbf{Z}/2$ and one suspended copy of $bu \wedge P$. Since $bu \wedge BV = bu \wedge (P \times \cdots \times P)$ is a wedge sum of $bu \wedge P \wedge \cdots \wedge P$'s, we get the following corollary.

Corollary 2. *Let V be an elementary abelian p-group. Then $bu_*(BV)$ is isomorphic to a sum of suspended copies of $\mathbf{Z}/2$ and of $bu_*(P)$.* □

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Lemma 3. *There is a homotopy equivalence* $bu \wedge \mathbf{C}P^\infty \simeq \bigvee_{0<n} bu \wedge S^{2n}$. *In particular, there is a projection* $\rho : bu \wedge \mathbf{C}P^\infty \longrightarrow bu \wedge S^2$.

Proof. Choose $f_n : S^{2n} \longrightarrow bu \wedge \mathbf{C}P^\infty$ representing the bu_* generators of $bu_*(\mathbf{C}P^\infty)$. Define $f : \bigvee_{0<n} S^{2n} \longrightarrow bu \wedge \mathbf{C}P^\infty$ by $f|S^{2n} = f_n$. We have the composition

$$F : \bigvee_{0<n} bu \wedge S^{2n} \xrightarrow{bu\wedge f} bu \wedge bu \wedge \mathbf{C}P^\infty \xrightarrow{\mu\wedge \mathbf{C}P^\infty} bu \wedge \mathbf{C}P^\infty$$

where μ is the pairing of the bu spectrum. F induces an isomorphism in homotopy and thus is an equivalence. $\square$

The proof of Theorem 1

Let $\pi : P \longrightarrow \mathbf{C}P^\infty$ represent the nonzero second dimensional integral homology class of P. Define g_1 to be the composition

$$g_1 : bu \wedge P \wedge P \xrightarrow{bu\wedge\pi\wedge P} bu \wedge \mathbf{C}P^\infty \wedge P \xrightarrow{\rho\wedge P} bu \wedge S^2 \wedge P.$$

Let $H^*(P \wedge P; \mathbf{Z}/2) \cong \mathbf{Z}/2[s,t](st)$ be the mod 2 cohomology of $P \wedge P$. For $b = s^{2i-1} \wedge t^{2j-1} \in H^{2i+2j-2}(P \wedge P; \mathbf{Z}/2)$, let $g_b : P \wedge P \longrightarrow \Sigma^{dim(b)} H\mathbf{Z}/2$ represent b. Now construct the map g_0 by the following composition:

$$g_0 : bu \wedge P \wedge P \xrightarrow{bu\wedge\vee_b g_b} bu \wedge [\bigvee_{0<i,j} \Sigma^{2i+2j-2} H\mathbf{Z}/2] \xrightarrow{\vee_b \nu} [\bigvee_{0<i,j} \Sigma^{2i+2j-2} H\mathbf{Z}/2]$$

where $\nu : bu \wedge H\mathbf{Z}/2 \longrightarrow H\mathbf{Z}/2$ is the standard pairing making mod 2 homology a module theory over connective K-theory. The map

$$g = g_0 \vee g_1 : bu \wedge P \wedge P \longrightarrow [\bigvee_{0<i,j} \Sigma^{2i+2j-2} H\mathbf{Z}/2] \vee [\Sigma^2 bu \wedge P]$$

is our candidate for the equivalence.

Let A be the mod 2 Steenrod algebra and $E = E[Q_0, Q_1]$ ($Q_0 = Sq^1$ and $Q_1 = Sq^3 + Sq^2Sq^1$). Then $H^*(bu; \mathbf{Z}/2) \cong A/A(Q_0, Q_1) \cong A \otimes_E \mathbf{Z}/2$. In $H^*(P \wedge P; \mathbf{Z}/2)$, the classes $\{s^2 \wedge t^i : i > 0\}$ give a basis for an E-module D^* isomorphic to $H^*(S^2 \wedge P; \mathbf{Z}/2)$. Let $M \cong H^*(P \wedge P; \mathbf{Z}/2)/D^*$. It is isomorphic to a free E-module with basis $\{s^{2i-1} \wedge t^{2j-1} : i, j > 0\}$. Clearly in dimension 2,

$$(bu \wedge \pi)^* \circ \rho^* : H^2(bu \wedge S^2; \mathbf{Z}/2) \longrightarrow H^2(bu \wedge P; \mathbf{Z}/2)$$

is an isomorphism. Thus g_1^* takes $H^*(bu \wedge S^2 \wedge P; \mathbf{Z}/2)$ isomorphically onto $A/A(Q_0, Q_1) \otimes D^*$. By the construction of the composition g_0, we see that g_0^* takes $H^*(\bigvee_{0<i,j} \Sigma^{2i+2j-2} H\mathbf{Z}/2; \mathbf{Z}/2)$ onto the A-module generated by $\{1 \wedge s^{2i-1} \wedge t^{2j-1} : i, j > 0\}$. The composition of the projection of

$$H^*(bu \wedge P \wedge P) \longrightarrow H^*(bu \wedge P \wedge P)/(A/A(Q_0, Q_1) \otimes D^*) \cong (A \otimes_E \mathbf{Z}/2)) \otimes M$$

with g_0^* gives an isomorphism. Although this is obvious it does require a proof. A generalization from the literature is Proposition 1.7 of Arunas Liulevicius [3]. Let his N be $\mathbf{Z}/2$, his A our A, his B our E and his M our M. He shows:

$$M \otimes (A \otimes_E \mathbf{Z}/2) \cong A \otimes_E M.$$

The A action on the left is by the diagonal and this is isomorphic to $(A \otimes_E \mathbf{Z}/2) \otimes M$. The A action on the right-hand side is just on A and since M is E free this is A free on the appropriate generators. Thus g induces an isomorphism in mod 2 cohomology and thus is an equivalence. $\square$

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