ON A THEOREM OF OSSA

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Abstract. If V is an elementary abelian 2-group, Ossa proved that the connective K-theory of BV splits into copies of \( \mathbb{Z}/2 \) and of the connective K-theory of the infinite real projective space. We give a brief proof of Ossa’s theorem.

Introduction

We have been asked whether our work, [1] and [2] on the Brown-Peterson homology of BV, V an elementary p-group, gives a nice structure of the connective K-theory of BV. The answer is that the approach of [1] leads to the elegant structure theorem of Ossa [4]. Although the approach is motivated by our [1] and [2], the proof is independent of that work. In this reproof of an established theorem we shall limit our exposition to the \( p = 2 \) case. For us, the notation makes this the easiest case, but for Ossa, it was the more difficult one. With obvious modifications, the odd-primary version of our argument follows the same outline. We thank Don Davis for the Lindelevicius reference.

Notation. Let \( bu \) be the connective K-theory spectrum and let P denote \( B\mathbb{Z}/2 \) (also known as infinite real projective space). Let \( H\mathbb{Z}/2 \) be the \( \mathbb{Z}/2 \) Eilenberg-MacLane spectrum.

Theorem 1 (Ossa). With the above notation, there is a homotopy equivalence of spectra

\[
bu \wedge P \wedge P \simeq \bigvee_{0 < i,j} \Sigma^{2i+2j-2} H\mathbb{Z}/2 \vee [\Sigma^2 bu \wedge P].
\]

Eric Ossa has kindly pointed out that our proof gives this as a homotopy equivalence of BP-module spectra.

Note that \( H\mathbb{Z}/2 \wedge P \simeq \bigvee_{0 < i} \Sigma^i H\mathbb{Z}/2 \). (The proof of this is like that of Lemma 3.) Thus the theorem can be used inductively to split \( bu \wedge P \wedge \cdots \wedge P \wedge \cdots \) into suspended copies of \( H\mathbb{Z}/2 \) and one suspended copy of \( bu \wedge P \). Since \( bu \wedge BV = bu \wedge (P \times \cdots \times P) \) is a wedge sum of \( bu \wedge P \wedge \cdots \wedge P \)'s, we get the following corollary.

Corollary 2. Let V be an elementary abelian p-group. Then \( bu_\ast (BV) \) is isomorphic to a sum of suspended copies of \( \mathbb{Z}/2 \) and of \( bu_\ast (P) \). \( \square \)

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Lemma 3. There is a homotopy equivalence $bu \wedge CP^\infty \simeq \bigvee_{0<n} bu \wedge S^{2n}$. In particular, there is a projection $\rho : bu \wedge CP^\infty \longrightarrow bu \wedge S^2$.

Proof. Choose $f_n : S^{2n} \longrightarrow bu \wedge CP^\infty$ representing the $bu_*$ generators of $bu_*(CP^\infty)$. Define $f : \bigvee_{0<n} S^{2n} \longrightarrow bu \wedge CP^\infty$ by $f|S^{2n} = f_n$. We have the composition

$$F : \bigvee_{0<n} bu \wedge S^{2n} \xrightarrow{bu \wedge f} bu \wedge bu \wedge CP^\infty \xrightarrow{\mu \wedge CP^\infty} bu \wedge CP^\infty$$

where $\mu$ is the pairing of the $bu$ spectrum. $F$ induces an isomorphism in homotopy and thus is an equivalence. \hfill $\Box$

The proof of Theorem 1

Let $\pi : P \longrightarrow CP^\infty$ represent the nonzero second dimensional integral homology class of $P$. Define $g_1$ to be the composition

$$g_1 : bu \wedge P \wedge P \xrightarrow{bu \wedge \pi \wedge P} bu \wedge CP^\infty \wedge P \xrightarrow{\rho \wedge P} bu \wedge S^2 \wedge P.$$ 

Let $H^*(P \wedge P; \mathbb{Z}/2) \cong \mathbb{Z}/2[s,t]/(st)$ be the mod 2 cohomology of $P \wedge P$. For $b = s^{2i-1} \wedge t^{j-1} \in H^{2i+2j-2}(P \wedge P; \mathbb{Z}/2)$, let $g_0 : P \wedge P \longrightarrow \Sigma^{dim(b)} HZ/2$ represent $b$. Now construct the map $g_0$ by the following composition:

$$g_0 : bu \wedge P \wedge P \xrightarrow{bu \wedge \nu \wedge g_0} bu \wedge \bigvee_{0<i,j} \Sigma^{2i+2j-2}HZ/2 \xrightarrow{\nu \wedge \pi} \bigvee_{0<i,j} \Sigma^{2i+2j-2}HZ/2$$

where $\nu : bu \wedge HZ/2 \longrightarrow HZ/2$ is the standard pairing making mod 2 homology a module theory over connective $K$-theory. The map

$$g = g_0 \vee g_1 : bu \wedge P \wedge P \longrightarrow \bigvee_{0<i,j} \Sigma^{2i+2j-2}HZ/2 \vee [\Sigma^2 bu \wedge P]$$

is our candidate for the equivalence.

Let $A$ be the mod 2 Steenrod algebra and $E = E(Q_0, Q_1)$ $(Q_0 = S^1$ and $Q_1 = S^2$). Then $H^*(bu; \mathbb{Z}/2) \cong A/A(Q_0, Q_1) \cong A \otimes_{E} \mathbb{Z}/2$. In $H^*(P \wedge P; \mathbb{Z}/2)$, the classes $\{s^2 i^j : i > 0\}$ give a basis for an $E$-module $D^*$ isomorphic to $H^*(S^2 \wedge P; \mathbb{Z}/2)$. Let $M \cong H^*(P \wedge P; \mathbb{Z}/2)/D^*$. It is isomorphic to a free $E$-module with basis $\{s^{2i-1} \wedge t^j : i, j > 0\}$. Clearly in dimension 2,

$$(bu \wedge \pi)^* \circ \rho^* : H^2(bu \wedge S^2; \mathbb{Z}/2) \longrightarrow H^2(bu \wedge P; \mathbb{Z}/2)$$

is an isomorphism. Thus $g_1^*$ takes $H^*(bu \wedge S^2 \wedge P; \mathbb{Z}/2)$ isomorphically onto $A/A(Q_0, Q_1) \otimes D^*$. By the construction of the composition $g_0$, we see that $g_0^*$ takes $H^*(\bigvee_{0<i,j} \Sigma^{2i+2j-2}HZ/2; \mathbb{Z}/2)$ onto the $A$-module generated by $\{1 \wedge s^{2i-1} \wedge t^j : i, j > 0\}$. The composition of the projection of $H^*(bu \wedge P \wedge P) \longrightarrow H^*(bu \wedge P \wedge P)/(A/A(Q_0, Q_1) \otimes D^*) \cong (A \otimes_{E} \mathbb{Z}/2) \otimes M$ with $g_0^*$ gives an isomorphism. Although this is obvious it does require a proof. A generalization from the literature is Proposition 1.7 of Arunas Liulevicius [3]. Let his $N$ be $Z/2$, his $A$ our $A$, his $B$ our $E$ and his $M$ our $M$. He shows:

$$M \otimes (A \otimes_{E} \mathbb{Z}/2) \cong A \otimes_{E} M.$$

The $A$ action on the left is by the diagonal and this is isomorphic to $(A \otimes_{E} \mathbb{Z}/2) \otimes M$. The $A$ action on the right-hand side is just on $A$ and since $M$ is free this is $A$ free on the appropriate generators. Thus $g$ induces an isomorphism in mod 2 cohomology and thus is an equivalence. \hfill $\Box$
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REFERENCES


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