FACTORIZATION OF AN INTEGRALLY CLOSED IDEAL IN TWO-DIMENSIONAL REGULAR LOCAL RINGS

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Abstract. Let \((R, m, k)\) be a two-dimensional regular local ring with algebraically closed residue field \(k\) and \(I\) be an \(m\)-primary integrally closed ideal in \(R\). Let \(T(I)\) be the set of Rees valuations of \(I\) and \(k(v)\) be the residue field of the valuation ring \(V\) associated with \(v \in T(I)\). Assume that \((a, b)\) is any minimal reduction of \(I\). We show that if \(I\) is the product of the distinct simple \(m\)-primary integrally closed ideals in \((R, m, k)\), then \(k(v)\) is generated by the image of \(a/b\) over \(k\) for all \(v \in T(I)\), and the converse of this is also true.

1. Introduction

Throughout this paper \((R, m, k)\) will denote a 2-dimensional regular local ring (RLR for short) with residue field \(k\) and quotient field \(K\). Let \(I\) be an \(m\)-primary integrally closed ideal in \((R, m, k)\). Concerning the structure of the integrally closed ideals in a 2-dimensional RLR \((R, m, k)\), O. Zariski proved three beautiful theorems which are the main background for this paper. Zariski’s Product Theorem ([8], Appendix 5, Theorem 2’’) says that any product of integrally closed ideals is integrally closed. Hence the set of Rees valuations of \(I\) is

\[ T(I) = \bigcup_{\substack{a \neq 0 \, \forall \ell \in I}} \{ v \mid v \text{ is the valuation of } (R[I/a])_q, q \in \Min(aR[I/a]) \}. \]

By Zariski’s Unique Factorization Theorem ([8], Appendix 5, Theorem 3), \(I = I_1^{\mu_1} \cdots I_l^{\mu_l}, \mu_i \geq 1\), where \(I_1, \ldots, I_l\) are distinct simple \(m\)-primary integrally closed ideals in \((R, m, k)\). Zariski also set up an one-to-one correspondence between the set of simple integrally closed ideals of \(R\) and the set of prime divisors of the second kind on \(R\) ([8], Appendix 5, Theorem E). Therefore, if \(\Min(mR[I/a]) = \{ q_1, \ldots, q_\lambda \} \) then \(\lambda = l\) and, upon reordering, \(v_i\) is the valuation of \((R[I/a])_{q_i}\), where \(v_i\) is the prime divisor associated to the ideal \(I_i\) for \(i = 1, \ldots, l\).

We denote by \(k(v)\) the residue field of the valuation ring \(V\) associated with \(v\). Assume that \(k\) is an algebraically closed field. Then \(k(v)\) is a simple transcendental field extension of \(k\) for all \(v \in T(I)\). Moreover, if \((a, b)\) is any minimal reduction of \(I\), then for all \(v \in T(I)\), the image of \(a/b\) in \(k(v)\) is transcendental over \(k\). We show that if \(I = I_1 \cdots I_l\), where \(I_1, \ldots, I_l\) are distinct simple \(m\)-primary integrally closed ideals in \((R, m, k)\), then \(k(v)\) is generated by the image of \(a/b\) over \(k\) for
all \( v \in T(I) \), and the converse of this is also true. Section 2 is devoted to some preliminaries. In section 3, we will prove main results.

2. Preliminaries

Let \((A, n)\) be a local ring and \(I\) an ideal of \(A\). An ideal \(J\) contained in \(I\) is called a reduction of \(I\) if \(JI^s = I^{s+1}\) for some integer \(s \geq 0\). A reduction \(J\) of \(I\) is called a minimal reduction of \(I\) if \(J\) is minimal with respect to being a reduction of \(I\). The order \(o(I)\) of an ideal \(I\) of a local ring \((A, n)\) is \(r\) if \(I \subseteq n^r\) but \(I \nsubseteq n^{r+1}\). We will use notation \(e(I)\) to denote the multiplicity of an \(n\)-primary ideal \(I\) of \(A\). Recall that an ideal is simple if it is not the unit ideal and has no nontrivial factorization.

An element \(a \in A\) is said to be integral over an ideal \(I\) of \(A\) if \(a\) satisfies an equation of the form

\[
a^n + r_1a^{n-1} + \cdots + r_n = 0, \quad r_i \in I^i.
\]

The set of all elements in \(A\) which are integral over an ideal \(I\) forms an ideal, denoted by \(I\) and called the integral closure of \(I\). An ideal \(I\) is said to be integrally closed (or equivalently “complete”) if \(I = I\).

In a \(d\)-dimensional local domain \((A, n, l)\) with quotient field \(L\), by a prime divisor of the second kind on \(A\) (or equivalently prime divisor of \((A, n)\)) we mean a discrete valuation \(v\) of \(L\) on \(A\) which is non-negative on \(A\) and has center \(n\) on \(A\) and whose residual transcendence degree (denoted by \(\text{tr.deg}(k(v))\)) is \(d - 1\).

**Lemma 2.1.** Let \((A, n)\) be a quasilocal normal domain with quotient field \(L\). If \(u \in L\backslash A\) is such that \(u^{-1} \notin A\), then \(nA\{u\}\) is a prime ideal in \(A\{u\}\) and \(A\{u\}/nA\{u\} \cong (A/n)[X]\), a polynomial ring in one-variable over the field \(A/n\).

**Proof.** Define the canonical homomorphism \(\phi\) from \(A\{X\}\) onto \(A\{u\}\) with \(\phi(X) = u\). \(\text{Ker(}\phi\text{)}\) is a prime ideal in \(A\{X\}\) since \(A\{u\}\) is a domain. By Theorem 11.13. in [7], \(\text{Ker(}\phi\text{)}\) is generated by linear polynomials \(cX - d\) with \(u = d/c\), where \(c, d \in A\). It is not difficult to see that \(c\) and \(d\) are in the maximal ideal \(n\) of \(A\). Hence we have the following exact sequence:

\[
0 \rightarrow nA\{X\} \overset{\phi}{\rightarrow} A\{X\} \overset{\overline{\phi}}{\rightarrow} A\{u\}/nA\{u\} \rightarrow 0,
\]

where \(\overline{\phi}\) is the map induced by \(\phi\). Hence we have \(A\{u\}/nA\{u\} \cong A\{X\}/nA\{X\}\) is a polynomial ring one-variable over the field \(A/n\), and so \(nA\{u\}\) is a prime ideal in \(A\{u\}\).

3. Main results

The following lemma will play a key role in the proofs of the first main result.

**Lemma 3.1.** Let \(I\) be an \(m\)-primary integrally closed ideal in \((R, m, k)\). Assume that \(\text{Min}(mR[I]) = \{p_1, \cdots, p_l\}\) and that \(k\) is an algebraically closed field. Let \(v_i\) be the valuation of \(R[I]_{p_i} \cap K\) for \(i = 1, \cdots, l\). Then the following conditions are equivalent.

1. \(R[I]/p_i\) is regular for \(i = 1, \cdots, l\).
2. \(I = I_1 \cdots I_l\), where \(I_1, \cdots, I_l\) are distinct simple \(m\)-primary integrally closed ideals in \(R\).
(3) For any reduction \((a, b)\) of \(I\), there exist elements \(c_{i_1}, \ldots, c_{i_n}\) in \(I\) such that 
\[a, b, c_{i_1}, \ldots, c_{i_n}\] is a minimal generating set of \(I\) and \(v_i(c_{i_j}) > v_i(a) = v_i(b)\) for \(i = 1, \ldots, l\) and \(j = 1, \ldots, n\).

(4) There exist a reduction \((a, b)\) of \(I\) and elements \(c_{i_1}, \ldots, c_{i_n}\) in \(I\) such that 
\[a, b, c_{i_1}, \ldots, c_{i_n}\] is a minimal generating set of \(I\) and \(v_i(c_{i_j}) > v_i(a) = v_i(b)\) for \(i = 1, \ldots, l\) and \(j = 1, \ldots, n\).

**Proof.** \((1) \iff (2)\) See ([5], Theorem 3.1).
\((1) \iff (3) \iff (4)\) See ([5], Theorem 3.3).

**Theorem 3.2.** Let \(I\) be an \(m\)-primary integrally closed ideal in \((R, m, k)\) with an algebraically closed field \(k\) and \((a, b)\) be any minimal reduction of \(I\). Assume that 
\(I = I_1 \cdot \ldots \cdot I_l\), where \(I_1, \ldots, I_l\) are distinct \(m\)-primary simple integrally closed ideals in \(R\). Then \(k(v)\) is generated by the image of \(a/b\) over \(k\) for all \(v \in T(I)\).

**Proof.** Suppose that \(I = I_1 \cdot \ldots \cdot I_l\), where \(I_1, \ldots, I_l\) are distinct simple \(m\)-primary integrally closed ideals in \(R\). Let \(v_i\) be a prime divisor of \(R\) associated to \(I_i\) for \(i = 1, \ldots, l\). Let \(\Min(mR[I/b]) = \{q_1, \ldots, q_k\}\). By Zariski’s One-to-One Correspondence Theorem, we have that \(\lambda = l\) and, upon reordering, \(v_i\) is the discrete valuation of \(R[I/b]/q_i\) for \(i = 1, \ldots, l\), i.e., \(T(I) = \{v_1, \ldots, v_l\}\). By Lemma 3.1, there exist elements \(c_{i_1}, \ldots, c_{i_n}\) in \(I\) such that \(a, b, c_{i_1}, \ldots, c_{i_n}\) is a minimal generating set of \(I\) and 
\[v_i(c_{i_j}) > v_i(a) = v_i(b)\] for \(i = 1, \ldots, l\) and \(j = 1, \ldots, n\). For each \(i = 1, \ldots, l\), let \(J_i = (m, c_{i_1}/b, \ldots, c_{i_n}/b)\). Then \(J_i \subseteq q_i\) since \(v_i(c_{i_j}) > v_i(a) = v_i(b)\) for \(j = 1, \ldots, n\). And we have 
\[R[I/b]/J_i = R[a/b]/mR[a/b] \cong (R/m)[X]\] by Lemma 3.1.

Let \((a/b)_i^*\) be the image of \(a/b\) in \(R[I/b]/J_i\) for \(i = 1, \ldots, l\). Since \(J_i\) is a prime ideal in \(R[I/b]\) and \(\dim(R[I/b]/J_i) = \dim(R[I/b]/q_i) = 1\), we have \(J_i = q_i\) for \(i = 1, \ldots, l\). Thus \(R[I/b]/q_i = (R/m)((a/b)_i^*)\) for \(i = 1, \ldots, l\), which is a polynomial ring in one-variable over \(k\). Localizing at \(q_i\), we have \(k(v_i) = (R/m)((a/b)_i^*)\) for \(i = 1, \ldots, l\).

**Corollary 3.3 ([3], Remark 3.5).** Let \(I\) be an \(m\)-primary integrally closed ideal in \((R, m, k)\) with an algebraically closed field \(k\) and \((a, b)\) be any minimal reduction of \(I\). Assume that \(I\) is simple. Let \(v\) be a prime divisor of \(R\) associated to \(I\). Then \(k(v)\) is generated by the image of \(a/b\) over \(k\).

We remark that Theorem 3.2 does not extend, in general, to the case where \(I\) is a power of a simple ideal.

**Example 3.4.** Let \(R = k[x, y]_{(x,y)}\) and \(m = (x,y)R\) with an algebraically closed field \(k\). Let \(I = m^2\). Then \(T(I) = \{o\}\), where \(o\) is the order valuation, i.e., the Rees valuation of \(I\) is the \(m\)-adic prime divisor of \(R, R[x/y]_{mR[x/y]}\). \((x^2, y^2)\) is a minimal reduction of \(I\). Since \(m = (x,y)\) and \(T(m) = \{o\}\), by Theorem 3.2, \(k(o) = k(\theta)\), where \(\theta\) is the image of \(x/y\) in \(R[x/y]/mR[x/y]\). Since \(o(x^2) = o(y^2) = 2\), we have that \(x^2/y^2\) is unit in \(R[x/y]/mR[x/y]\), and hence the image of \(x^2/y^2\) in \(R[x/y]/mR[x/y]\) is \(\theta^2\). But \(k(\theta^2) \subseteq k(\theta)\).

**Lemma 3.5 (Lipman, [2]).** Let \(I\) be an \(m\)-primary integrally closed ideal in \((R, m, k)\) with an algebraically closed field \(k\). Assume that \(I = I_1^{\mu_1} \cdot \ldots \cdot I_l^{\mu_l}\) is the
unique factorization of $I$ as a product simple integrally closed ideals $I_1, \cdots , I_l$. Then

$$e(I) = \sum_{i=1}^{l} \mu_iv_i(I),$$

where $v_i$ is the prime divisor associated to $I_i$ for $i = 1, \cdots , l$.

Lemma 3.6 ([4], Theorem 1.1). Let $(A, n)$ be a local ring with infinite residue field and $J = (x_1, \cdots , x_d)A$ an ideal generated by a system of parameters of $A$. Set

$$T = A[x_1/x_1, \cdots , x_{d-1}/x_d]_{nA[x_1/x_1, \cdots , x_{d-1}/x_d]}.$$

Then $e(J) = e(JT)$.

Theorem 3.7. Let $I$ be an $m$-primary integrally closed ideal in $(R, m, k)$ with an algebraically closed field $k$. Assume that $T(I) = \{v_1, \cdots , v_l\}$. Then the following conditions are equivalent.

1. $I = I_1 \cdots I_l$, where $I_1, \cdots , I_l$ are distinct simple $m$-primary integrally closed ideals in $R$.
2. For any reduction $(a, b)$ of $I$, $k(v_i) = k((a/b)^*_i)$, where $(a/b)^*_i$ is the image of $a/b$ in $k(v_i)$ for $i = 1, \cdots , l$.
3. For some reduction $(a, b)$ of $I$, $k(v_i) = k((a/b)^*_i)$, where $(a/b)^*_i$ is the image of $a/b$ in $k(v_i)$ for $i = 1, \cdots , l$.

Proof. (1)$\Rightarrow$ (2): This follows immediately from Theorem 3.2.

(2)$\Rightarrow$ (3): It is clear.

(3)$\Rightarrow$ (1): Assume that (3) is true. Let $J = (a, b)$. By Zariski’s Unique Factorization and One-to-One Correspondence Theorems, $I = I_1^{\mu_1} \cdots I_l^{\mu_l}$, where $I_1, \cdots , I_l$ are distinct simple $m$-primary integrally closed ideals in $R$ and $\mu_i \geq 1$ for $i = 1, \cdots , l$, and $\text{Min}(mR[I/b]) = \{q_1, \cdots , q_l\}$ and, upon reordering, $v_i$ is the discrete valuation of $R[I/b]_{q_i}$ for $i = 1, \cdots , l$. $mR[a/b]$ is a prime ideal in $R[a/b]$ of $ht(mR[a/b]) = 1$ since $a, b$ are a regular sequence on $R$. Set $T = R[a/b]_{mR[a/b]}$. Then we have

$$e(I) = \sum_{i=1}^{l} \mu_iv_i(I) \quad \text{by Lemma 3.5}$$

$$= e(J) \quad \text{($J$ is a reduction of $I$)}$$

$$= e(JT) \quad \text{by Lemma 3.6.}$$

Let $T'$ be the integral closure of $T$. Then $T' = R[I/b]_{R[a/b]}mR[a/b]$ is an one-dimensional Noetherian normal domain since $R[a/b] \subseteq R[I/b]$ is integral and $R[I/b]$ is normal. By Corollary of the Krull-Akizuki Theorem ([6], Corollary of Theorem 11.7), there are just a finite number of maximal ideals of $T'$ lying over $mT$.

Claim. $mR[a/b] = q_i \cap R[a/b]$ for $i = 1, \cdots , l$.

Let $Q_i = q_i \cap R[a/b]$ for $i = 1, \cdots , l$. Then $Q_i \supseteq mR[a/b]$. It is enough to show that $ht(Q_i) = 1$ for $i = 1, \cdots , l$. Suppose that $ht(Q_i) = 2$. Then $R[a/b]/Q_i \cong k[X]/Q_i'$, where $Q_i'$ is the image of $Q_i$ in $k[X]$, since $Q_i \supseteq mR[a/b]$. Hence the $0$-dimensional domain $R[a/b]/Q_i$ is a finitely generated $k$-algebra. By the Nullstellensatz ([1], Corollary 5.24), $R[a/b]/Q_i$ is a finite algebraic extension of $k$. Since $k$ is an algebraically closed field, $R[a/b]/Q_i = k$. But $(a/b)^* \in R[a/b]/Q_i$, where $(a/b)^*$ is the image of $a/b$ in $R[a/b]/Q_i$, which is a contradiction since $(a/b)^*$ is transcendental over $k$. The proof of the claim is complete.
By the above claim, \( q_1 T', \ldots, q_l T' \) are all the maximal ideals of \( T' \) lying over \( mT \) and
\[
T' = \bigcap_{i=1}^{l}(T')_{q_i T'} = \bigcap_{i=1}^{l} R[I/b]_{q_i}.
\]
By the projection formula ([8], Corollary 1, p. 299),
\[
e(JT) = \sum_{i=1}^{l} [T'/q_i T' : T/mT] e(J(T')_{q_i T'}).
\]
Since \( R[a/b]/mR[a/b] \subseteq R[I/b]/q_i \) have the same quotient fields by the assumption, \( T/mT = T'/q_i T' \) for all \( i \). \( e(J(T')_{q_i T'}) = v_i(J) \) since \( (T')_{q_i T'} = R[I/b]_{q_i} \) is a Rees valuation ring of \( I \) with associated valuation \( v_i \) for \( i = 1, \ldots, l. \ v_i(J) = v_i(I) \) for all \( i \) since \( J \subseteq I \) is a reduction of \( I \). We have
\[
e(JT) = \sum_{i=1}^{l} v_i(I).
\]
Hence we have
\[
\sum_{i=1}^{l} \mu_i v_i(I) = \sum_{i=1}^{l} v_i(I).
\]
That is,
\[
\sum_{i=1}^{l} (\mu_i - 1) v_i(I) = 0.
\]
Since \( v_i(I) \) is positive value and \( \mu_i \geq 1 \) for all \( i \), \( \mu_i = 1 \) for all \( i \). The proof is complete.

References

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