EQUIVARIANT ACYCLIC MAPS

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Abstract. In this paper we apply a recently developed new version of the Bredon-Illman cohomology theory to obtain an equivariant analogue of a result of Kan and Thurston, which implies that a connected CW-complex has the homotopy type of a space obtained by applying the plus construction of Quillen to certain Eilenberg-Mac Lane spaces.

1. Statement of results

A space $X$ is acyclic if its reduced integral homology $\tilde{H}_*(X) = 0$. The universal coefficient theorem then implies that $X$ is acyclic if and only if the reduced cohomology $\tilde{H}^*(X; G) = 0$ for every coefficient group $G$. Also a map $f : X \to Y$ is acyclic if its homotopy fibre is acyclic. We say that a $G$-space $X$ is $G$-acyclic if its reduced Bredon-Illman cohomology $\tilde{H}^*_G(X; \lambda) = 0$ for every abelian $O_G$-group $\lambda$, and a $G$-map $f : X \to Y$ is $G$-acyclic if its $G$-homotopy fibre is $G$-acyclic.

Here $O_G$ denotes the category of orbit spaces $G/H$ and $G$-maps, and an $O_G$-group is a contravariant functor $O_G \to \text{Grp}$. Other notions like $O_G$-space, $O_G$-fibration, etc. have similar meaning (terminology depending on the nature of codomain of the functors). The homotopy $O_G$-group $\pi_n(X)$ of a $G$-space $X$ with a stationary point $x^0 \in X^G$ as base point is defined by $\pi_n(X)(G/H) = \pi_n(X^H, x^0)$ and $\pi_n(X)(\tilde{g}) = \pi_n(g)$, where $\tilde{g} : G/H \to G/K$ is a morphism in $O_G$ arising from a subconjugacy relation $g^{-1}Hg \subseteq K$, and $g : X^K \to X^H$ is the left translation by $g$. A $G$-map $f : X \to Y$ induces a morphism of $O_G$-groups $\pi_n(f) : \pi_n(X) \to \pi_n(Y)$ defined by $\pi_n(f)(G/H) = \pi_n(f^H)$, where $f^H = f|X^H$.

Given an $O_G$-group $\lambda$ (where $G$ is a compact Lie group) and an integer $n \geq 1$, there is a $G$-space $X$ such that $\pi_n(X) = \lambda$ and $\pi_n(X) = 0$ if $j \neq n$. This $G$-space is the classifying space for the Bredon-Illman cohomology, and is called an equivariant Eilenberg-Mac Lane space $K(\lambda, n)$ of type $(\lambda, n)$ (see [5]).

For a $G$-space $X$, there is a concept of an equivariant local coefficients system $M$ on $X$, and also of an equivariant cohomology $H^*_G(X; M)$ (see [8]). This cohomology reduces to the equivariant singular cohomology of Bredon and Illman [2], [7] when $M$ is simple in a certain sense, and to the Steenrod cohomology with the classical local coefficients system when $G$ is trivial. In Section 2 we present an alternative description of $H^*_G(X; M)$ in a way which is best suited in the context of $G$-acyclicity.
Now suppose that $G$ is finite, and consider $G$-spaces $X$ which are compactly generated weakly Hausdorff with base point $x^0 \in X^G$ such that $X$ has the $G$-homotopy type of a $G$-connected $G$-CW-complex. Then, in line of Kan and Thurston [6], our first main theorem is

**Theorem 1.1.** For a $G$-space $X$, there exist an $O_G$-group $\lambda$ with a perfect normal $O_G$-subgroup $\eta$ and a $G$-acyclic map

$$f : K(\lambda, 1) \rightarrow X,$$

which is natural with respect to $X$, such that $\text{Ker} \pi_1(f) = \eta$, and

$$f^* : H_0^G(X; M) \rightarrow H_0^G(K(\lambda, 1); f^* M)$$

is an isomorphism for every equivariant local coefficients system $M$ on $X$.

Given a $G$-space $X$ and a perfect normal $O_G$-subgroup $\eta$ of $\pi_1(X)$, it is possible to construct a $G$-space $X_\eta^+$ by applying the plus construction of Quillen [9] to each $X^H$ with respect to the group $\eta(G/H)$, and then combining the resulting spaces together by means of a functorial bar construction. It turns out that the $G$-space $X_\eta^+$ is completely determined by the pair $(\pi_1(X), \eta)$ up to $G$-homotopy equivalence. More specifically, we have the following two theorems which provide a classification of $G$-acyclic maps from a given $G$-space.

**Theorem 1.2.** If $X$ is a $G$-space and $\eta$ a perfect normal $O_G$-subgroup of $\pi_1(X)$, then there exist a $G$-space $X_\eta^+$ and a $G$-acyclic map $f : X \rightarrow X_\eta^+$ such that $\text{Ker} \pi_1(f) = \eta$.

**Theorem 1.3.** If $f : X \rightarrow Y$ and $f' : X \rightarrow Y'$ are $G$-maps, where $f$ is $G$-acyclic, then there is a $G$-map $h : Y \rightarrow Y'$ with $hf \simeq_G f'$ if and only if $\text{Ker} \pi_1(f) \subseteq \text{Ker} \pi_1(f')$; moreover, any two such $h$ are $G$-homotopic. In addition, if $f'$ is $G$-acyclic, then $h$ is also $G$-acyclic, and $h$ is a $G$-homotopy equivalence if and only if $\text{Ker} \pi_1(f) = \text{Ker} \pi_1(f')$.

Finally, we obtain as an application our second main theorem which is

**Theorem 1.4.** Given a $G$-space $X$, there exists an $O_G$-group $\lambda$ with a perfect normal $O_G$-subgroup $\eta$ such that $X$ has the $G$-homotopy type of $K(\lambda, 1)^+_{\eta}$.

We note that the condition of $G$-connectivity of $X$ is a necessary condition for each of the main theorems to be true, and therefore cannot be avoided.

The proofs of the theorems appear in Section 3.

2. **Criteria for $G$-acyclicity**

The proofs of our theorems are based on the following two propositions. The first implies that a $G$-map $f : X \rightarrow Y$ is $G$-acyclic if and only if each $f^H : X^H \rightarrow Y^H$ is acyclic, and then the second gives the cohomological assertion of Theorem 1.1.

**Proposition 2.1.** A $G$-space $X$ is $G$-acyclic if and only if each $X^H$ is acyclic.

**Proposition 2.2.** If a $G$-map $f : X \rightarrow Y$ is $G$-acyclic, then $f$ induces an isomorphism

$$f^* : H_0^G(Y; M) \rightarrow H_0^G(X; f^* M)$$

for every equivariant local coefficients system $M$ on $Y$. 
Proof of Proposition 2.1. There is a spectral sequence

\[ E_2^{p,q} = \text{Ext}^p(\tilde{H}_q(X), \lambda) \Rightarrow \tilde{H}_q^{p+q}(X; \lambda), \]

obtained by means of an injective resolution of the $O_G$-group $\lambda$, where $\tilde{H}_q(X)$ is the $O_G$-group whose value at $G/H$ is the reduced integral homology $\tilde{H}_q(X^H)$ (cf. [2, 1, §10]). Since the category of abelian $O_G$-groups has sufficiently many injectives, we can embed the $O_G$-group $\tilde{H}_q(X)$ in an injective $O_G$-group $\lambda_q$. Then, we have in the corresponding spectral sequence $E_2^{p,q} = 0$ for $p > 0$. Therefore, if $X$ is $G$-acyclic, then

\[ 0 = \tilde{H}^q_G(X, \lambda_q) \cong \text{Ext}^0(\tilde{H}_q(X), \lambda_q) = \text{Hom}(\tilde{H}_q(X), \lambda_q). \]

This implies that $\tilde{H}_q(X) = 0$ as we have already a monomorphism $\tilde{H}_q(X) \rightarrow \lambda_q$. Since this happens for every $q$, each $X^H$ is acyclic.

The converse follows easily again from the same spectral sequence. This completes the proof.

Turning now to Proposition 2.2, let us recall briefly from [8, §8] an alternative description of the equivariant cohomology $H_G^*(X; M)$.

First note that an equivariant local coefficients system $M$ on $X$ is a contravariant functor $M : \Pi X \rightarrow \textbf{Ab}$, where $\Pi X$ is the following category. An object of $\Pi X$ is a $G$-map $x_H : G/H \rightarrow X$, and a morphism $[\tilde{g}, \tilde{\phi}] : x_H \rightarrow y_K$ is a certain equivalence class of pairs $(\tilde{g}, \tilde{\phi})$, where $\tilde{g} : G/H \rightarrow G/K$, $g^{-1}Hg \subseteq K$, is a $G$-map, and $\phi : G/H \times I \rightarrow X$ is a $G$-homotopy from $x_H$ to $y_K \circ \tilde{g}$.

Given $M$, we define an $O_G$-group $M_0 : O_G \rightarrow \textbf{Ab}$ by sending $G/H$ to $M(x_H^0)$, and sending a $G$-map $\tilde{g} : G/H \rightarrow G/K$ to $M([\tilde{g}, k])$, where $x_H^0$ is an object in $\Pi X$ given by the constant $G$-map $G/H \rightarrow x^0 \in X$, and $[\tilde{g}, k] : x_H^0 \rightarrow x_K^0$ is a morphism in $\Pi X$ given by the constant homotopy $k$ on $x^0$. Note that the bijection $b : X^H \rightarrow MA_{G}(G/H, X), (b(x)(gH) = gx, is implicit in the definition. In fact, this makes $M_0$ a $\prod_1(X)$-module with action $\rho : \prod_1(X) \times M_0 \rightarrow M_0$ given by $\rho(G/H)(\alpha, m) = M(b(\alpha))(m)$, where $\alpha \in \prod_1(X^H, x^0)$ and $b(\alpha) : x_H^0 \rightarrow x_K^0$ is an equivariance in $\Pi X$.

Next, consider the family of universal covering spaces $p_H : \tilde{X}^H \rightarrow X^H, H \subseteq G$. Then, for a $G$-map $\tilde{g} : G/H \rightarrow G/K$, the left translation $\tilde{g} : X^K \rightarrow X^H$ lifts to a map $\tilde{g} : \tilde{X}^K, \rightarrow \tilde{X}^H$ which is unique up to the choice of base points over $x^0$ in $\tilde{X}^K$ and $\tilde{X}^H$.

Finally, let $\mathbb{Z}_1(X)$ denote the $O_G$-group, where $\mathbb{Z}_1(X)(G/H)$ is the integral group ring $\mathbb{Z}\pi_1(X^H, x^0)$.

Then the cohomology $H_G^*(X; M)$ for a finite group $G$ may be obtained by means of a cochain complex $S^u_{\pi,G}(\mathcal{U}; M_0)$, where $\mathcal{U}$ is what we call the universal $O_G$-covering space of $X$. The $n$th group $S^u_{\pi,G}(\mathcal{U}; M_0)$ of this cochain complex is a subgroup of

\[ \bigoplus_{H \subseteq G} \text{Hom}_{\mathbb{Z}_1(X)(G/H)}(C_n(\tilde{X}^H), M_0(G/H)) \]

consisting of elements $c = \{ e_H \}_{H \subseteq G}$ which satisfy the condition : if two equivariant singular $n$-simplexes $\sigma : \Delta_n \rightarrow \tilde{X}^H$ and $\tau : \Delta_n \rightarrow \tilde{X}^K$ are connected by a $G$-map $\tilde{g} : G/H \rightarrow G/K$ such that $\sigma = \tilde{g} \circ \tau$, then $M_0(\tilde{g})(c_K(\tau)) = c_H(\sigma)$. Note that the condition is a simplified version of a general case where $G$ is a compact Lie group (see [8, (8.3)]).
The following definitions and notations are preparatory to our next lemma which provides yet another description of $H^*_G(X; M)$.

Let $L$ be a right $\pi_1(X)$-module which acts on $M_0$ with actions $\theta : L \times \pi_1(X) \to L$ and $\omega : L \times M_0 \to M_0$ such that $\omega \circ (\theta \times id) = \omega \circ (id \times \rho)$.

Here are two examples of $L$ which will be important in the proof of Proposition 2.2.

**Example 2.3.** Take $L = \pi_1(X)$, $\omega = \rho$ as defined above, and $\theta =$ multiplication.

**Example 2.4.** Let $f : X \to Y$ be a $G$-map and $M$ an equivariant local coefficients system on $Y$. Then $(f^*M)_0 = M_0$. Take $L = \pi_1(Y)$, and $\theta : \pi_1(Y) \times \pi_1(X) \to \pi_1(Y)$ as $\theta(G/H)(\beta, \alpha) = \beta \cdot f^H(\alpha)$. Let $\omega : \pi_1(Y) \times M_0 \to M_0$ be as in Example 2.3, and $\rho : \pi_1(X) \times M_0 \to M_0$ be given by $\rho(G/H)(\alpha, m) = \omega(G/H)(f^H(\alpha), m)$.

We shall denote the $L$ of this example by $f^\pi_1(Y)$.

Consider the $O_G$-group $C_n(X; L) : O_G \to \text{Ab}$, where

$$C_n(X; L)(G/H) = L(G/H) \otimes_{\pi_1(X)(G/H)} C_n(\hat{X}^H),$$

and, for a $G$-map $\hat{g} : G/H \to G/K$, $C_n(X; L)(\hat{g}) = L(\hat{g}) \otimes C_n(\hat{g})$. Clearly, these give rise to a chain complex $C_*(X; L)$ in the abelian category of abelian $O_G$-groups. Then, $\text{Hom}_L(C_*(X; L), M_0)$ becomes a cochain complex of groups whose $n$th group consists of $L$-invariant natural transformations $C_n(X; L) \to M_0$.

**Lemma 2.5.** There is an isomorphism

$$\Psi : S^n_{\pi,G}(U; M_0) \to \text{Hom}_L(C_*(X; L), M_0)$$

of cochain complexes.

**Proof.** Define $\Psi$ and its inverse $\Psi'$ in the following way. Let $c = \{c_H\}_{H \subseteq G} \in S^n_{\pi,G}(U; M_0)$, $T \in \text{Hom}_L(C_*(X; L), M_0)$, $l \in L(G/H)$, and $\sigma : \Delta_n \to \hat{X}^H$ be a singular $n$-simplex. Then, set

$$\Psi(c)(G/H)(l \otimes \sigma) = \omega(G/H)(l, c_H(\sigma)), \text{ and } (\Psi'(T))_H(\sigma) = T(G/H)(1 \otimes \sigma).$$

It does not pose any difficulty to verify that $\Psi$ and $\Psi'$ are cochain maps inverse to one other (cf. [8, §9]).

The point to note here is that $G$ has to be finite for $\Psi'$ to be well defined.

**Proof of Proposition 2.2.** The category of abelian $L$-invariant $O_G$-groups possesses sufficiently many injectives. Let $M_0^\delta$ be an injective resolution of $M_0$ in this category. Then, in view of Lemma 2.5, the bicomplex $\text{Hom}_L(C_*(X; L), M_0^\delta)$ provides a spectral sequence $E(X, L, M)$ in which

$$E^p_q = \text{Ext}_P^p(H_q(X, L), M_0^\delta) \Longrightarrow H^p+q_G(X; M),$$

where $H_q(X; L) : O_G \to \text{Ab}$ is given by $H_q(X; L)(G/H) = H_q(X^H; L(G/H))$ which is the ordinary cohomology of $X^H$ with local coefficients $L(G/H)$.

Now if $f : X \to Y$ is a $G$-map and $M$ is an equivariant local coefficients system on $Y$, then $f$ induces a map of the spectral sequences $f^* : E(Y, \pi_1(Y), M) \to E(X, \pi_1(X), f^*M)$, where $\pi_1(Y)$ is as in Example 2.3, and $f^*\pi_1(Y)$ is as in Example 2.4. If $f$ is $G$-acyclic, then $f^*$ is an isomorphism at the $E_2$-level, by Proposition 2.1 and Proposition (4.3) of [1]. Consequently, $f^* : H^*_G(Y; M) \to H^*_G(X; f^*M)$ is an isomorphism. This completes the proof.
3. Proof of the theorems

Proof of Theorem 1.1. It is possible to convert a $G$-space $X$ into an $O_G$-space by means of a functor $\mathcal{R}$ defined by $\mathcal{R}(X)(G/H) = X^H$, $\mathcal{R}(X)(g) = g$ (left translation). Conversely, Elmendorf [5] defined a functor $S : O_G$-spaces $\rightarrow$ $G$-spaces, and a natural transformation $N : \mathcal{R}S \rightarrow id$ such that, for each $O_G$-space $T$ and each $H \subseteq G$, $N(T)(G/H) : (ST)^H \rightarrow T(G/H)$ is a homotopy equivalence. In particular, $N(\mathcal{R}(X))(G/\{e\}) : SRX \rightarrow X$ is a natural $G$-homotopy equivalence.

Now, if $X$ is a $G$-space, then using the Kan-Thurston theorem [6] for each $X^H$, we get a group $\lambda(G/H)$ with a perfect normal subgroup $\eta(G/H)$, and a fibration $p(G/H) : K(\lambda(G/H), 1) \rightarrow X^H$ satisfying the conditions that $p(G/H)$ is acyclic, and $Ker\pi_1(p(G/H)) = \eta(G/H)$ (note that here we are using $O_G$ as an indexing set). By naturality, these fibrations produce an $O_G$-fibration $p : E \rightarrow B$, where $E = RK(\lambda, 1)$ and $B = RX$. Applying the Elmendorf's functor $S$ to it, we get a $G$-map $Sp : SE \rightarrow SB$ so that $(SE)^H$ and $(SB)^H$ have the homotopy types of $K(\lambda(G/H), 1)$ and $X^H$ respectively. This gives Theorem 1.1 immediately.

Proof of Theorem 1.2. First note that the plus construction $W \rightarrow W^+_p$, where $W$ is a CW-space and $P$ is a perfect normal subgroup of $\pi_1(W)$, is not functorial, but functorial up to homotopy. However, it is possible to choose $W^+_p$ from its homotopy type so that $W \rightarrow W^+_p$ becomes functorial. This may be done in the following way. Let $\alpha : \tilde{W}_p \rightarrow W$ be the covering space of $W$ corresponding to the subgroup $P$ so that $Im\pi_1(\alpha) = P$, and let $\beta : A(W_p) \rightarrow \tilde{W}_p$ be the natural fibration obtained by applying the acyclic functor $A$ of Dror [4]. Then the cofibre $i : W \rightarrow C_\alpha$ of $\alpha \circ \beta : A(\tilde{W}_p) \rightarrow W$, where $C_\alpha$ is the mapping cone of $\alpha$, is homotopically equivalent to $W \rightarrow W^+_p$ (over $W$). These cofibres provide a functor which may be called the functorial plus construction.

Now if $X$ is a $G$-space and $\eta$ is a perfect normal $O_G$-subgroup of $\pi_1(X)$, then applying the functorial plus construction to each $X^H$ we get an acyclic map $f(G/H) : X^H \rightarrow (X^H)^{\eta(G/H)}$ such that $Ker\pi_1(f(G/H)) = \eta(G/H)$. These maps give a morphism of $O_G$-spaces which turns into a $G$-map $f : SRX \rightarrow X^+_\eta$ by means of the Elmendorf's functor $S$. Then a composition of a $G$-homotopy equivalence $X \rightarrow SRX$ with $f'$ gives the required $G$-acyclic map $f : X \rightarrow X^+_\eta$. This completes the proof of Theorem 1.2.

Proof of Theorem 1.3. If $h$ exists, then $\pi_1(f') = \pi_1(h) \circ \pi_1(f)$, and therefore $Ker\pi_1(f) \subseteq Ker\pi_1(f')$. Conversely, consider the $G$-push out diagram, and its restriction to each $H$-fixed point set

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{f'} & & \downarrow{g} \\
Y' & \xrightarrow{g'} & Y' \\
\end{array}
\quad
\begin{array}{ccc}
X^H & \xrightarrow{f^H} & Y^H \\
\downarrow{f^H} & & \downarrow{g^H} \\
Y'^H & \xrightarrow{g'^H} & Y'^H \\
\end{array}
\]

The second diagram implies that $g^H$ is acyclic, since $f^H$ is so, and, by the van Kampen theorem, $\pi_1(g^H)$ is an isomorphism, since $Ker\pi_1(f^H) \subseteq Ker\pi_1(f'^H)$. Therefore $g^H$ is a homotopy equivalence, and hence $g$ is a $G$-homotopy equivalence, by the equivariant Whitehead theorem [3, p. 107]. Then, if $g_1$ is a $G$-homotopy inverse of $g$, $h = g_1 \circ g' : Y \rightarrow Y'$ is the required $G$-map with $h \circ f \simeq_G f'$.
Clearly $h$ is $G$-acyclic if $f'$ is so, and, since $\pi_1(h)$ is an isomorphism if and only if $\text{Ker}\, \pi_1(f) = \text{Ker}\, \pi_1(f')$, the last assertion follows.

To see that $h$ is unique up to $G$-homotopy equivalence, suppose that $j : F \to X$ is the $G$-homotopy fibre of $f : X \to Y$. Then, since $f \circ j \simeq_G y^0$, $f$ extends to a $G$-map $k : X \cup_j CF \to Y$ over the equivariant mapping cone of $j$. The $G$-map $k$ is actually a $G$-homotopy equivalence, because its restriction to each $H$-fixed point set $k^H : X^H \cup C F^H \to Y^H$ is acyclic and $\pi_1(k^H)$ is an isomorphism. Thus we have an equivariant coexact sequence

$$F \to X \to Y \to \Sigma F,$$

where $\Sigma F$ is the equivariant suspension of $F$. Since $\Sigma F^H$ is simply connected and $\tilde{H}_*(\Sigma F^H; \mathbb{Z}) = 0$, $\Sigma F^H$ is contractible. This implies that $\Sigma F$ is $G$-contractible by the equivariant Whitehead theorem. Thus the map $f^* : [Y, Y'^0_G] \to [X, Y'^0_G]$ in the equivariant Barratt-Puppe sequence [3, p. 142] is injective, where $[Y, Y'^0_G]$ denotes the set of base point preserving $G$-homotopy classes of $G$-maps $Y \to Y'$. This ensures the uniqueness of $h$, and the proof of Theorem 1.3 is complete.

The assertion of Theorem 1.4 is now straightforward.

In conclusion, we remark that the proofs appearing in this section remain valid if $G$ is a compact Lie group and $X$ is a $G$-CW-space with each $X^H$ a connected CW-space.

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