EQUIVARIANT ACYCLIC MAPS

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Abstract. In this paper we apply a recently developed new version of the Bredon-Illman cohomology theory to obtain an equivariant analogue of a result of Kan and Thurston, which implies that a connected CW-complex has the homotopy type of a space obtained by applying the plus construction of Quillen to certain Eilenberg-Mac Lane spaces.

1. Statement of results

A space $X$ is acyclic if its reduced integral homology $\tilde{H}_*(X) = 0$. The universal coefficient theorem then implies that $X$ is acyclic if and only if the reduced cohomology $\tilde{H}^*(X; G) = 0$ for every coefficient group $G$. Also a map $f : X \to Y$ is acyclic if its homotopy fibre is acyclic. We say that a $G$-space $X$ is $G$-acyclic if its reduced Bredon-Illman cohomology $\tilde{H}_G^*(X; \lambda) = 0$ for every abelian $O_G$-group $\lambda$, and a $G$-map $f : X \to Y$ is $G$-acyclic if its $G$-homotopy fibre is $G$-acyclic.

Here $O_G$ denotes the category of orbit spaces $G/H$ and $G$-maps, and an $O_G$-group is a contravariant functor $O_G \to \text{Grp}$. Other notions like $O_G$-space, $O_G$-fibration, etc. have similar meaning (terminology depending on the nature of codomain of the functors). The homotopy $O_G$-group $\pi_n(X)$ of a $G$-space $X$ with a stationary point $x^0 \in X^G$ as base point is defined by $\pi_n(X)(G/H) = \pi_n(X^H, x^0)$ and $\pi_n(X)(\bar{g}) = \pi_n(g)$, where $\bar{g} : G/H \to G/K$ is a morphism in $O_G$ arising from a subconjugacy relation $g^{-1} H g \subseteq K$, and $g : X^K \to X^H$ is the left translation by $g$. A $G$-map $f : X \to Y$ induces a morphism of $O_G$-groups $\pi_n(f) : \pi_n(X) \to \pi_n(Y)$ defined by $\pi_n(f)(G/H) = \pi_n(f^H)$, where $f^H = f|_{X^H}$.

Given an $O_G$-group $\lambda$ (where $G$ is a compact Lie group) and an integer $n \geq 1$, there is a $G$-space $X$ such that $\pi_n(X) = \lambda$ and $\pi_j(X) = 0$ if $j \neq n$. This $G$-space is the classifying space for the Bredon-Illman cohomology, and is called an equivariant Eilenberg-Mac Lane space $K(\lambda, n)$ of type $(\lambda, n)$ (see [5]).

For a $G$-space $X$, there is a concept of an equivariant local coefficients system $M$ on $X$, and also of an equivariant cohomology $H^*_G(X; M)$ (see [8]). This cohomology reduces to the equivariant singular cohomology of Bredon and Illman [2], [7] when $M$ is simple in a certain sense, and to the Steenrod cohomology with the classical local coefficients system when $G$ is trivial. In Section 2 we present an alternative description of $H^*_G(X; M)$ in a way which is best suited in the context of $G$-acyclicity.
Now suppose that $G$ is finite, and consider $G$-spaces $X$ which are compactly generated weakly Hausdorff with base point $x^0 \in X^G$ such that $X$ has the $G$-homotopy type of a $G$-connected $G$-CW-complex. Then, in line of Kan and Thurston [6], our first main theorem is

**Theorem 1.1.** For a $G$-space $X$, there exist an $O_G$-group $\lambda$ with a perfect normal $O_G$-subgroup $\eta$ and a $G$-acyclic map

$$f : K(\lambda, 1) \to X,$$

which is natural with respect to $X$, such that $\text{Ker} \pi_1(f) = \eta$, and

$$f^* : H^*_G(X; M) \to H^*_G(K(\lambda, 1); f^* M)$$

is an isomorphism for every equivariant local coefficients system $M$ on $X$.

Given a $G$-space $X$ and a perfect normal $O_G$-subgroup $\eta$ of $\pi_1(X)$, it is possible to construct a $G$-space $X^+_\eta$ by applying the plus construction of Quillen [9] to each $X^H$ with respect to the group $\eta(G/H)$, and then combining the resulting spaces together by means of a functorial bar construction. It turns out that the $G$-space $X^+_\eta$ is completely determined by the pair $(\pi_1(X), \eta)$ up to $G$-homotopy equivalence. More specifically, we have the following two theorems which provide a classification of $G$-acyclic maps from a given $G$-space.

**Theorem 1.2.** If $X$ is a $G$-space and $\eta$ a perfect normal $O_G$-subgroup of $\pi_1(X)$, then there exist a $G$-space $X^+_{\eta}$ and a $G$-acyclic map $f : X \to X^+_{\eta}$ such that $\text{Ker} \pi_1(f) = \eta$.

**Theorem 1.3.** If $f : X \to Y$ and $f' : X \to Y'$ are $G$-maps, where $f$ is $G$-acyclic, then there is a $G$-map $h : Y \to Y'$ with $hf \simeq_G f'$ if and only if $\text{Ker} \pi_1(f) \subset \text{Ker} \pi_1(f')$; moreover, any two such $h$ are $G$-homotopic. In addition, if $f'$ is $G$-acyclic, then $h$ is also $G$-acyclic, and $h$ is a $G$-homotopy equivalence if and only if $\text{Ker} \pi_1(f) = \text{Ker} \pi_1(f')$.

Finally, we obtain as an application our second main theorem which is

**Theorem 1.4.** Given a $G$-space $X$, there exists an $O_G$-group $\lambda$ with a perfect normal $O_G$-subgroup $\eta$ such that $X$ has the $G$-homotopy type of $K(\lambda, 1)^+_\eta$.

We note that the condition of $G$-connectivity of $X$ is a necessary condition for each of the main theorems to be true, and therefore cannot be avoided.

The proofs of the theorems appear in Section 3.

2. **Criteria for $G$-acyclicity**

The proofs of our theorems are based on the following two propositions. The first implies that a $G$-map $f : X \to Y$ is $G$-acyclic if and only if each $f^H : X^H \to Y^H$ is acyclic, and then the second gives the cohomological assertion of Theorem 1.1.

**Proposition 2.1.** A $G$-space $X$ is $G$-acyclic if and only if each $X^H$ is acyclic.

**Proposition 2.2.** If a $G$-map $f : X \to Y$ is $G$-acyclic, then $f$ induces an isomorphism

$$f^* : H^*_G(Y; M) \to H^*_G(X; f^* M)$$

for every equivariant local coefficients system $M$ on $Y$. 
Proof of Proposition 2.1. There is a spectral sequence
\[ E_2^{p,q} = \text{Ext}^p(\overline{H}_q(X, \lambda), \lambda) \Rightarrow H_{G/H}^{p+q}(X; \lambda), \]
obtained by means of an injective resolution of the \( O_G \)-group \( \lambda \), where \( \overline{H}_q(X) \) is the \( O_G \)-group whose value at \( G/H \) is the reduced integral homology \( \overline{H}_q(X^H) \) (cf. [2, 1, §10]). Since the category of abelian \( O_G \)-groups has sufficiently many injectives, we can embed the \( O_G \)-group \( \overline{H}_q(X) \) in an injective \( O_G \)-group \( \lambda_q \). Then, we have in the corresponding spectral sequence \( E_2^{p,q} = 0 \) for \( p > 0 \). Therefore, if \( X \) is \( G \)-acyclic, then
\[ 0 = \overline{H}_G^q(X, \lambda_q) \cong \text{Ext}^0(\overline{H}_q(X), \lambda_q) = \text{Hom}(\overline{H}_q(X), \lambda_q). \]
This implies that \( \overline{H}_q(X) = 0 \) as we have already a monomorphism \( \overline{H}_q(X) \to \lambda_q \). Since this happens for every \( q \), each \( X^H \) is acyclic.

The converse follows easily again from the same spectral sequence. This completes the proof.

Turning now to Proposition 2.2, let us recall briefly from [8, §8] an alternative description of the equivariant cohomology \( H_G^*(X; M) \).

First note that an equivariant local coefficients system \( M \to X \) on \( X \) is a contravariant functor \( M : \Pi X \to \text{Ab} \), where \( \Pi X \) is the following category. An object of \( \Pi X \) is a \( G \)-map \( x_H : G/H \to X \), and a morphism \([\hat{g}, \phi] : x_H \to y_K \) is a certain equivalence class of pairs \((\hat{g}, \phi)\), where \( \hat{g} : G/H \to G/K \), \( g^{-1}Hg \subseteq K \), is a \( G \)-map, and \( \phi : G/H \to K \) is a \( G \)-homotopy from \( x_H \) to \( y_K \circ \hat{g} \).

Given \( M \), we define an \( O_G \)-group \( M_0 : O_G \to \text{Ab} \) by sending \( G/H \to M(x_H^0) \), and sending a \( G \)-map \( \hat{g} : G/H \to G/K \) to \( M([\hat{g}, k]) \), where \( x_H^0 \) is an object in \( \Pi X \) given by the constant \( G \)-map \( G/H \to x^0 \in X \), and \([\hat{g}, k] : x_H^0 \to x_K^0 \) is a morphism in \( \Pi X \) given by the constant homotopy \( k \) on \( x^0 \). Note that the bijection \( b : X^H \to M_{apG}(G/H, X), b(x)(gH) = gx \), is implicit in the definition. In fact, this makes \( M_0 \) a \( \pi_1(X) \)-module with action \( \rho : \pi_1(X) \times M_0 \to M_0 \) given by \( \rho(x)(\alpha, m) = \rho(x)(\alpha)(m), \) where \( \alpha \in \pi_1(X^H, x^0) \) and \( b(\alpha) : x_H^0 \to x_H^0 \) is an equivalence in \( \Pi X \).

Next, consider the family of universal covering spaces \( p_H : \tilde{X}^H \to X^H, H \subseteq G \). Then, for a \( G \)-map \( \hat{g} : G/H \to G/K \), the left translation \( g : X^K \to X^H \) lifts to a map \( \tilde{g} : \tilde{X}^K \to \tilde{X}^H \) which is unique up to the choice of base points over \( x^0 \) in \( \tilde{X}^K \) and \( \tilde{X}^H \).

Finally, let \( \mathbb{Z}_{\pi_1}(X) \) denote the \( O_G \)-group, where \( \mathbb{Z}_{\pi_1}(X)(G/H) \) is the integral group ring \( \mathbb{Z}_{\pi_1}(X^H, x^0) \).

Then the cohomology \( H_G^*(X; M) \) for a finite group \( G \) may be obtained by means of a cochain complex \( S_{\pi,G}^n(U ; M_0) \), where \( U \) is what we call the universal \( O_G \)-covering space of \( X \). The \( n \)th group \( S_{\pi,G}^n(U ; M_0) \) of this cochain complex is a subgroup of
\[ \bigoplus_{H \leq G} \text{Hom}(\mathbb{Z}_{\pi_1}(X)(G/H), C_n(\tilde{X}^H, M_0(G/H))) \]
consisting of elements \( c = \{ c_H \}_{H \leq G} \) which satisfy the condition: if two equivariant singular \( n \)-simplices \( \sigma : \Delta_n \to \tilde{X}^H \) and \( \tau : \Delta_n \to \tilde{X}^K \) are connected by a \( G \)-map \( \hat{g} : G/H \to G/K \) such that \( \sigma = \hat{g} \circ \tau \), then \( M_0(\hat{g})(c_K(\tau)) = c_H(\sigma) \). Note that the condition is a simplified version of a general case where \( G \) is a compact Lie group (see [8, (8.3)]).
The following definitions and notations are preparatory to our next lemma which provides yet another description of $H^*_G(X; M)$.

Let $L$ be a right $\pi_1(X)$-module which acts on $M_0$ with actions $\theta : L \times \pi_1(X) \rightarrow L$ and $\omega : L \times M_0 \rightarrow M_0$ such that $\omega \circ (\theta \times id) = \omega \circ (id \times \rho)$.

Here are two examples of $L$ which will be important in the proof of Proposition 2.2.

**Example 2.3.** Take $L = \pi_1(X)$, $\omega = \rho$ as defined above, and $\theta =$ multiplication.

**Example 2.4.** Let $f : X \rightarrow Y$ be a $G$-map and $M$ an equivariant local coefficients system on $Y$. Then $(f^*M)_0 = M_0$. Take $L = \pi_1(Y)$, and $\theta : \pi_1(Y) \times \pi_1(X) \rightarrow \pi_1(Y)$ as $\theta(G/H)(\beta, \alpha) = \beta \cdot f(H)(\alpha)$. Let $\omega : \pi_1(Y) \times M_0 \rightarrow M_0$ be as in Example 2.3, and $\rho : \pi_1(X) \times M_0 \rightarrow M_0$ be given by $\rho(G/H)(\alpha, m) = \omega(G/H)(f(H)(\alpha), m)$.

We shall denote the $L$ of this example by $f^*\pi_1(Y)$.

Consider the $O_G$-group $C_n(X; L) : O_G \rightarrow Ab$, where

$$C_n(X; L)(G/H) = L(G/H) \otimes_{\pi_1(X)(G/H)} C_n(X^H),$$

and, for a $G$-map $\tilde{g} : G/H \rightarrow G/K$, $\tilde{C}_n(X; L)(\tilde{g}) = L(\tilde{g}) \otimes C_n(\tilde{g})$. Clearly, these give rise to a chain complex $\tilde{C}_*(X; L)$ in the abelian category of abelian $O_G$-groups. Then, $Hom_L(C_*(X; L), M_0)$ becomes a cochain complex of groups whose $n$th group consists of $L$-invariant natural transformations $C_n(X; L) \rightarrow M_0$.

**Lemma 2.5.** There is an isomorphism

$$\Psi : S_{*, G}^X(U; M_0) \rightarrow Hom_L(C_*(X; L), M_0)$$

of cochain complexes.

**Proof.** Define $\Psi$ and its inverse $\Psi'$ in the following way. Let $c = \{c_H\}_{H \subseteq G} \in S_{*, G}^X(U; M_0)$, $T \in Hom_L(C_*(X; L), M_0)$, $l \in L(G/H)$, and $\sigma : \Delta_n \rightarrow X^H$ be a singular $n$-simplex. Then, set

$$\Psi(c)(G/H)(l \otimes \sigma) = \omega(G/H)(l, c_H(\sigma)), \quad \text{and} \quad (\Psi'(T)) \mu(\sigma) = T(G/H)(1 \otimes \sigma).$$

It does not pose any difficulty to verify that $\Psi$ and $\Psi'$ are cochain maps inverse to one another (cf. [8, §9]).

The point to note here is that $G$ has to be finite for $\Psi'$ to be well defined.

**Proof of Proposition 2.2.** The category of abelian $L$-invariant $O_G$-groups possesses sufficiently many injectives. Let $M^0_0$ be an injective resolution of $M_0$ in this category. Then, in view of Lemma 2.5, the bicomplex $Hom_L(C_*(X; L), M^0_0)$ provides a spectral sequence $E(X, L, M)$ in which

$$E^{p,q}_2 = Ext^p(H_q(X; L), M_0) \rightarrow H^{p+q}_G(X; M),$$

where $H_q(X; L) : O_G \rightarrow Ab$ is given by $H_q(X; L)(G/H) = H_q(X^H; L(G/H))$ which is the ordinary cohomology of $X^H$ with local coefficients $L(G/H)$.

Now if $f : X \rightarrow Y$ is a $G$-map and $M$ is an equivariant local coefficients system on $Y$, then $f$ induces a map of the spectral sequences $f^* : E(Y, \pi_1(Y), M) \rightarrow E(X, f^*\pi_1(Y), f^*M)$, where $\pi_1(Y)$ is as in Example 2.3, and $f^*\pi_1(Y)$ is as in Example 2.4. If $f$ is $G$-acyclic, then $f^*$ is an isomorphism at the $E_2$-level, by Proposition 2.1 and Proposition (4.3) of [1]. Consequently, $f^* : H^*_G(Y; M) \rightarrow H^*_G(X; f^*M)$ is an isomorphism. This completes the proof.
3. Proof of the theorems

Proof of Theorem 1.1. It is possible to convert a G-space X into an O_G-space by means of a functor R defined by R(X)(G/H) = X^H, R(X)(g) = g (left translation). Conversely, Elmendorf [5] defined a functor S: O_G-spaces → G-spaces, and a natural transformation N: R S → id such that, for each O_G-space T and each H ⊆ G, N(T)(G/H): (ST)^H → T(G/H) is a homotopy equivalence. In particular, N(R(X))(G/\{e\}) = sRX → X is a natural G-homotopy equivalence.

Now, if X is a G-space, then using the Kan-Thurston theorem [6] for each X^H, we get a group λ(G/H) with a perfect normal subgroup η(G/H), and a fibration p(G/H): K(λ(G/H), 1) → X^H satisfying the conditions that p(G/H) is acyclic, and Ker π_1(p(G/H)) = η(G/H) (note that here we are using O_G as an indexing set). By naturality, these fibrations produce an O_G-fibration p: E → B, where E = RK(λ, 1) and B = RX. Applying the Elmendorf’s functor S to it, we get a G-map S p : SE → SB so that (SE)^H and (SB)^H have the homotopy types of K(λ(G/H), 1) and X^H respectively. This gives Theorem 1.1 immediately.

Proof of Theorem 1.2. First note that the plus construction W → W^+_p, where W is a CW-space and P is a perfect normal subgroup of π_1(W), is not functorial, but functorial up to homotopy. However, it is possible to choose W^+_p from its homotopy type so that W → W^+_p becomes functorial. This may be done in the following way. Let α : \overline{W}_P → W be the covering space of W corresponding to the subgroup P so that Im π_1(α) = P, and let β : A(W_P) → W_P be the natural fibration obtained by applying the acyclic functor A of Dror [4]. Then the cofibre i : W → C_{α} o β : A(\overline{W}_P) → W, where C_{α} is the mapping cone of α, is homotopically equivalent to W → W^+_p (over W). These cofibres provide a functor which may be called the functorial plus construction.

Now if X is a G-space and η is a perfect normal O_G-subgroup of π_1(X), then applying the functorial plus construction to each X^H we get an acyclic map f(G/H) : X^H → (X^H)^{\eta(G/H)} such that Ker π_1(f(G/H)) = η(G/H). These maps give a morphism of O_G-spaces which turns into a G-map f : sRX → X^+_p by means of the Elmendorf’s functor S. Then a composition of a G-homotopy equivalence X → sRX with f gives the required G-acyclic map f : X → X^+_p. This completes the proof of Theorem 1.2.

Proof of Theorem 1.3. If h exists, then \overline{π}_1(f) = \overline{π}_1(h) o \overline{π}_1(f), and therefore Ker \overline{π}_1(f) ⊆ Ker \overline{π}_1(f'). Conversely, consider the G-push out diagram, and its restriction to each H-fixed point set

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\begin{align*}
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{f'} & & \downarrow{g'} \\
Y' & \xrightarrow{g} & Y' \cup_X Y'
\end{array}
&
\begin{array}{ccc}
X^H & \xrightarrow{f^H} & Y^H \\
\downarrow{f'^H} & & \downarrow{g'^H} \\
Y'^H & \xrightarrow{g'^H} & Y^H \cup_X Y'^H
\end{array}
\end{align*}
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The second diagram implies that g'^H is acyclic, since f'^H is so, and, by the van Kampen theorem, π_1(g'^H) is an isomorphism, since Ker π_1(f'^H) ⊆ Ker π_1(f'^H). Therefore g'^H is a homotopy equivalence, and hence g is a G-homotopy equivalence, by the equivariant Whitehead theorem [3, p. 107]. Then, if g_1 is a G-homotopy inverse of g, h = g_1 o g': Y → Y' is the required G-map with h o f ≃ g f'.
Clearly $h$ is $G$-acyclic if $f'$ is so, and, since $\pi_1(h)$ is an isomorphism if and only if $\text{Ker}\,\pi_1(f) = \text{Ker}\,\pi_1(f')$, the last assertion follows.

To see that $h$ is unique up to $G$-homotopy equivalence, suppose that $j : F \longrightarrow X$ is the $G$-homotopy fibre of $f : X \longrightarrow Y$. Then, since $f \circ j \simeq_G y^0$, $f$ extends to a $G$-map $k : X \cup_j CF \longrightarrow Y$ over the equivariant mapping cone of $j$. The $G$-map $k$ is actually a $G$-homotopy equivalence, because its restriction to each $H$-fixed point set $k^H : X^H \cup CF^H \longrightarrow Y^H$ is acyclic and $\pi_1(k^H)$ is an isomorphism. Thus we have an equivariant coexact sequence

$$F \longrightarrow X \longrightarrow Y \longrightarrow \Sigma F,$$

where $\Sigma F$ is the equivariant suspension of $F$. Since $\Sigma F^H$ is simply connected and $H_*(\Sigma F^H; \mathbb{Z}) = 0$, $\Sigma F^H$ is contractible. This implies that $\Sigma F$ is $G$-contractible by the equivariant Whitehead theorem. Thus the map $f^* : [Y, Y'^0]_G \longrightarrow [X, Y'^0]_G$ in the equivariant Barratt-Puppe sequence [3, p. 142] is injective, where $[Y, Y'^0]_G$ denotes the set of base point preserving $G$-homotopy classes of $G$-maps $Y \longrightarrow Y'$. This ensures the uniqueness of $h$, and the proof of Theorem 1.3 is complete.

The assertion of Theorem 1.4 is now straightforward.

In conclusion, we remark that the proofs appearing in this section remain valid if $G$ is a compact Lie group and $X$ is a $G$-CW-space with each $X^H$ a connected CW-space.

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