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SUMMABILITY OF FOURIER SERIES WITH THE METHOD OF LACUNARY ARITHMETICAL MEANS AT THE LEBESGUE POINTS

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Abstract. The existence of the 'rare' sequence of partial sums summable with the method of arithmetical means at each Lebesgue point is proved in the paper. The proof is based on the strategy of random choice.

Introduction

For a periodic function $f \in L^p(\mathbb{T}), p \geq 1$, we shall call a point x such that

$$\Phi_{x,p}(f,h) = \int_0^h |f(x+t) - 2f(x) + f(x-t)|^p dt = o(h), \quad h \to 0,$$

an $\mathcal{L}_p$-point. It is well known (see, for example, [Zg]) that $\Phi_{x,p}(f,h) = o(h)$ almost everywhere. By

$$S_k(f;x) = \sum_{|j| \leq k} \hat{f}(j) e^{ijx}$$

we denote the partial sum of the Fourier series. It is also well known ([Zg]) that the Fejer means

$$\frac{1}{N} \sum_{k=1}^{N} S_k(f;x) \to f(x), \quad N \to \infty.$$

uniformly for continuous f, and in each $\mathcal{L}_1$-point for summable f. In 1936 Zalcwasser [Zl] asked how 'rare' can be the sequence of integers $\{n_k\}$ such that

$$\frac{1}{N} \sum_{k=1}^{N} S_{n_k}(f;x) \to f(x), \quad N \to \infty.$$

This problem was completely solved for continuous functions (uniform convergence) in [Sl], [TZ], [Cl], [Bl].

If $\{n_k\}$ is convex, then the condition $\sup_N N^{-1/2} \log n_N < \infty$ is necessary and sufficient for the uniform convergence for every continuous function.

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For the convergence at $\mathcal{L}_p$-points the question is open. The statement formulated above shows that the condition $\sup_N N^{-1/2} \log n_N < \infty$ is also necessary for the summability at $\mathcal{L}_p$-points.

We prove the existence of the 'rare' sequence, such that the summability takes place at each $\mathcal{L}_1$-point.

The second theorem states the existence of the 'rare' sequence such that 'strong' summability takes place at each $\mathcal{L}_p$-point, $p > 1$.

1. Summability at $\mathcal{L}_1$-points

Let us introduce the density function for the sequence of integers $\{n_k\}$

$$\mathrm{dens}(x) = \max\{k : n_k \le x\}.$$

Theorem 1. *There exists a sequence of integers $\{n_k\}$ with density* $\mathrm{dens}(x) \simeq (\log x)^3$ *and such that*

$$\frac{1}{N}\sum_{k=1}^{N} S_{n_k}(f;x) \to f(x), \quad N \to \infty,$$

at each $\mathcal{L}_1$-point for every $f \in L^1$.

The proof is based on the following statement by Bourgain [Br1].

Lemma. *For every $M > 1$ there exist integers $n_1, ..., n_k$ such that* $M < n_j \le M^2$, $j = 1, 2, ..., k;$ $k \simeq (\log M)^3$, *and*

$$\text{(a)} \qquad \Big\| \sum_{j=1}^{k} \sin n_j x \Big\|_\infty \ll (\log M)^2.$$

Following the original proof the estimate (a) can be strengthened to

$$\text{(a}'\text{)} \qquad \sup_{1\le s\le k} \Big\| \sum_{j=1}^{s} \sin n_j x \Big\|_\infty \ll (\log M)^2.$$

Let us consider the sequence of intervals $[2^{2^k}, 2^{2^{k+1}})$ and choose in each interval the set of integers given in the Lemma. The estimate of density is obvious. Let us prove convergence at the $\mathcal{L}_1$-points. The necessary and sufficient condition for the convergence at $\mathcal{L}_1$-points is

$$\sup_N \frac{1}{N}\int_0^\pi \sup_{t<u\le\pi} \left| \sum_{k=1}^{N} \frac{\sin n_k u}{u} \right| dt < \infty$$

(see, for example [Al]). Let $2^{3m} < N \le 2^{3(m+1)}$. Hence

$$\frac{1}{N}\int_0^\pi \sup_{t<u\le\pi} \left| \sum_{k=1}^{N} \frac{\sin n_k u}{u} \right| dt \le \frac{1}{N}\int_0^\delta \sum_{k=1}^{N} n_k dt + \frac{1}{N}\int_\delta^\pi \frac{1}{t} \sup_{0\le u\le\pi} \left| \sum_{k=1}^{N} \sin n_k u \right| dt$$

$$\le \frac{\delta}{N}\sum_{k=1}^{N} n_k + \frac{\log 1/\delta}{N} \Big\| \sum_{k=1}^{N} \sin n_k u \Big\|_\infty .$$

For the first term in final expression using density estimates we have

$$\frac{\delta}{N}\sum_{j=1}^{m+1}\sum_{2^{2^j}\le n_k<2^{2^{j+1}}} n_k \le \frac{\delta}{N}\sum_{j=1}^{m+1} 2^{2^{j+1}}2^{(j+1)3} \ll \delta 2^{2^{m+2}}.$$

For the second term we apply (a) and (a$'$) and obtain

$$\frac{\log 1/\delta}{N}\sum_{j=1}^{m+1}\Big\|\sum_{2^{2^j}\le n_k<2^{2^{j+1}}}\sin n_k x\Big\| \le \frac{\log 1/\delta}{N}\sum_{j=1}^{m+1} 2^{(j+1)2} \ll \frac{\log 1/\delta}{2^m}.$$

Taking $\delta = 2^{-2^{m+2}}$ we obtain the result. □

Conjecture 1. *If $\{n_k\}$ is convex then the condition $\sup_N n^{-1/2}\log n_N < \infty$ is necessary and sufficient for the convergence at $\mathcal{L}_1$-points. The necessity follows from the corresponding result for continuous functions.*

Conjecture 2. *The right answer in Bourgain's problem is the following. For each $M>0$ there exist integers $n_1,...,n_M$ such that*

$$\Big\|\sum_{k=1}^{M}\sin n_k x\Big\|_\infty \simeq M^{1/2}.$$

2. Strong summability at $\mathcal{L}_p$ $(p>1)$ points

Let us consider the problem of the strong summability at $\mathcal{L}_p$-points, i.e.

$$\frac{1}{N}\sum_{k=1}^{N}\big|S_{n_k}(f;x)-f(x)\big|^2 \to 0, \qquad N\to\infty.$$

We consider only $p>1$ because for $p=1$ convergence at $\mathcal{L}_1$-points does not take place even for $\{n_k=k\}$, though there is convergence almost everywhere.

Theorem 2. *For given $p, 1<p\le 2$ there exists a sequence $\{n_k\}$ such that* $\mathrm{dens}(x) \simeq x^{2/p'}, (1/p+1/p'=1)$ *and*

$$\frac{1}{N}\sum_{k=1}^{N}\big|S_{n_k}(f;x)-f(x)\big|^2 \to 0, \qquad N\to\infty,$$

at every $\mathcal{L}_p$-point for all $f\in L^p(\mathbb{T})$.

Proof. Let $\{n_k\}$ be the $\Lambda(p')$-sequence with density $\mathrm{dens}(x)\simeq x^{2/p'}$ [Br2]. The project operator $A: L^p\to L^p$ from L^p on the subspace of L^p generated by the exponentials $\{e^{in_k x}\}$ is bounded. Let us consider the expression

$$\frac{1}{N}\sum_{k=1}^{N}\big|f(x)-S_{n_k}(f;x)\big|^2 \simeq \frac{1}{N}\sum_{k=1}^{N}\Big|\int_0^\pi \big[f(x+t)-2f(x)+f(x-t)\big]\frac{\sin n_k t}{t}dt\Big|^2$$

$$\ll \frac{1}{N}\sum_{k=1}^{N}\Big|\int_0^\delta \big[f(x+t)-2f(x)+f(x-t)\big]\frac{\sin n_k t}{t}dt\Big|^2$$

$$+\frac{1}{N}\sum_{k=1}^{N}\Big|\int_\delta^\pi \big[f(x+t)-2f(x)+f(x-t)\big]\frac{\sin n_k t}{t}dt\Big|^2.$$

Estimating the first integral we use the $\mathcal{L}_p$-point definition and the density function estimate. We obtain

$$\frac{1}{N}\sum_{k=1}^{N} n_k^2\delta^2 o(1) \ll \frac{o(1)\delta^2}{N}\sum_{k=1}^{N} k^{p'} \simeq o(1)\delta^2 N^{p'}.$$

To estimate the second integral we will apply Bessel's inequality to the function $A\big(\chi_{[\delta,\pi]}(t)[f(x+t)-2f(x)+f(x-t)]/t\big)$. Because $\{n_k\}$ is a $\Lambda(p')$ sequence then for every $g \in L^p$

$$\|Ag\|_2 \ll \|Ag\|_p \ll \|g\|_p.$$

We obtain

$$\frac{1}{N}\int_0^{\pi} \Big|A\Big(\chi_{[\delta,\pi]}(t)\frac{f(x+t)-2f(x)+f(x-t)}{t}\Big)\Big|^2 dt$$

$$\ll \frac{1}{N}\bigg[\int_0^{\pi} \Big|A\Big(\chi_{[\delta,\pi]}(t)\frac{f(x+t)-2f(x)+f(x-t)}{t}\Big)\Big|^p dt\bigg]^{2/p}$$

$$\ll \frac{1}{N}\bigg[\int_{\delta}^{\pi} \Big|\frac{f(x+t)-2f(x)+f(x-t)}{t}\Big|^p dt\bigg]^{2/p}$$

Integrating by parts as in the classic proof [Zg, p.182] we have

$$\frac{1}{N}\bigg[\frac{\Phi_x(t)}{t^p}\bigg|_{\delta}^{\pi} - \int_{\delta}^{\pi}\frac{\Phi_x(t)}{t^{p+1}}dt\bigg]^{2/p} \le \frac{o(\delta^{2/p})}{N\delta^{2/p'}}.$$

Let us take $\delta = N^{-p'/2}$ and the theorem is proved. □

Remark. The known necessary condition is also necessary condition for the uniform convergence of the strong arithmetical means [JLL]. I don't know if it is possible to take a sequence $\{n_k\}$ with logarithmic density.

Remark. Conjecture 2 has been disproved by S. V. Konyagin, *Estimates of maxima of sine sums*, East J. Approx., Volume 3, Number 1, March 1997, pp. 63–70.

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