THE RANK STABLE TOPOLOGY OF INSTANTONS ON $\mathbb{CP}^2$

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Abstract. Let $M^n_k$ be the moduli space of based (anti-self-dual) instantons on $\mathbb{CP}^2$ of charge $k$ and rank $n$. There is a natural inclusion $M^n_k \hookrightarrow M^{n+1}_k$. We show that the direct limit space $M^\infty_k$ is homotopy equivalent to $BU(k) \times BU(k)$. Let $\ell_\infty$ be a line in the complex projective plane and let $\tilde{\mathbb{CP}^2}$ be the blow-up at a point away from $\ell_\infty$. $M^n_k$ can be alternatively described as the moduli space of rank $n$ holomorphic bundles on $\tilde{\mathbb{CP}^2}$ with $c_1 = 0$ and $c_2 = k$ and with a fixed holomorphic trivialization on $\ell_\infty$.

1. Introduction

In his 1989 paper [Ta], Taubes studied the stable topology of the based instanton moduli spaces. He showed that if $M^n_k(X)$ denotes the moduli space of based $SU(n)$-instantons of charge $k$ on $X$, then there is a map $M^n_k(X) \to M^{n+1}_k(X)$ and, in the direct limit topology, $M^\infty_k(X)$ has the homotopy type of $\text{Map}_0(X, BU(n))$.

There is also a map $M^n_k(X) \hookrightarrow M^{n+1}_k(X)$ given by the direct sum of a connection with the trivial connection on a trivial line bundle and one can consider the direct limit $M^\infty_k(X)$. For the case of $X = S^4$ with the round metric, it was shown by Kirwan and also by Sanders ([Kir],[Sa]) that the direct limit has the homotopy type of $BU(k)$.

In this note we consider the case of $X = \mathbb{CP}^2$ where $\mathbb{CP}^2$ denotes the complex projective plane with the Fubini-Study metric and the opposite orientation of the one induced by the complex structure. Our result is:

Theorem 1.1. $M^\infty_k(\mathbb{CP}^2)$ has the homotopy type of $BU(k) \times BU(k)$.

The main tool in the proof of the theorem is a construction of the moduli spaces $M^\infty_k(\mathbb{CP}^2)$ due to King [Ki]. In general, Buchdahl [Bu] has shown that, for appropriate metrics on the $N$-fold connected sum $\#_N \mathbb{CP}^2$, the moduli spaces $M^\infty_k(\#_N \mathbb{CP}^2)$ are diffeomorphic to certain spaces of equivalence classes of holomorphic bundles on $\mathbb{CP}^2$ blown-up at $N$ points. The universal $U(k) \times U(k)$ bundle that appears giving the homotopy equivalence of Theorem 1.1 can be constructed as higher direct image bundles (see section 3).

Remark 1.1. The cofibration $S^2 \to \mathbb{CP}^2 \to S^4$ gives rise to the fibration of mapping spaces $\Omega^4 BU(n) \to \text{Map}_*(\mathbb{CP}^2, BU(n)) \to \Omega^2 BU(n)$ which for K-theoretic
reasons is a trivial fibration in the limit over $n$. The total space of this fibration is homotopy equivalent to the space of based gauge equivalence classes of all connections on $\mathbb{CP}^2$. Thus, from Taubes’ result, $\mathcal{M}_k^\infty$ must have the property that taking the limit over $k$ gives $BU \times BU$. For $S^4$, similar remarks imply that $\lim_{k \to \infty} \mathcal{M}_k^\infty(S^4) \simeq BU$ and the inclusion of $\mathcal{M}_k^\infty(S^4)$ into this limit has been shown to be (up to homotopy) the natural inclusion $BU(k) \hookrightarrow BU$ ([Sa]). Theorem 1.1 and these results for $S^4$ suggest a general conjecture which is supported by the fact that the higher direct image bundle giving our homotopy equivalence generalizes in an appropriate way.

**Conjecture 1.1.** For appropriate metrics on $\#_N \mathbb{CP}^2$, $\mathcal{M}_k^\infty(\#_N \mathbb{CP}^2)$ has the homotopy type of a product $BU(k) \times \cdots \times BU(k)$ with $N + 1$ factors.

**Remark 1.2.** Combining Theorem 1.1 with Taubes’ stabilization result leads to an alternate proof of Bott periodicity for the unitary group. There is a natural map $\Phi : M \to BU$. Let $p : \mathbb{CP}^2 \to \#_N \mathbb{CP}^2$ be the map that collapses $\ell_\infty \mapsto x_0$. If $[A] \in \mathcal{M}_k^n$

2. The construction of $\mathcal{M}_k^n(\mathbb{CP}^2)$

Let $x_0 \in \mathbb{CP}^2$ be the base point. Since $\mathbb{CP}^2 \setminus \{x_0\}$ is conformally equivalent to $\mathbb{CP}^2$, the complex plane blown-up at the origin, $\mathcal{M}_k^n(\mathbb{CP}^2)$, can be regarded as instantons on $\mathbb{CP}^2$ based “at infinity”. Buchdahl [Bu] proved an analogue in this non-compact setting of Donaldson’s theorem relating instantons to holomorphic bundles: Let $\mathbb{C}_N^c$ be the complex plane blown-up at $N$ points with a Kähler metric. Then $\mathbb{C}_N^c$ has a “conformal compactification” to $\#_N \mathbb{CP}^2$ and a “complex compactification” to $\mathbb{CP}^2_N$ (the projective plane blown-up at $N$ points). We have added a point $x_0$ in the former case and a complex projective line $\ell_\infty$ in the latter.

Define $\mathcal{M}^n_{\text{alg},k}(\mathbb{CP}^2_N)$ to be the moduli space consisting of pairs $(\mathcal{E}, \tau)$ where $\mathcal{E}$ is a rank $n$ holomorphic bundle on $\mathbb{CP}^2_N$ with $c_1(\mathcal{E}) = 0$, $c_2(\mathcal{E}) = k$, and where $\tau : \mathcal{E}|_{\ell_\infty} \to \mathbb{C}^n \otimes \mathcal{O}_{\ell_\infty}$ is a holomorphic trivialization of $\mathcal{E}$ on $\ell_\infty$.

There is a natural map $\Phi : \mathcal{M}^n_k(\#_N \mathbb{CP}^2) \to \mathcal{M}^n_{\text{alg},k}(\mathbb{CP}^2_N)$ defined as follows.

Let $p : \mathbb{CP}^2_N \to \#_N \mathbb{CP}^2$ be the map that collapses $\ell_\infty \mapsto x_0$. If $[A] \in \mathcal{M}^n_k$
then the $\bar{\partial}$ operator that defines the holomorphic bundle $\mathcal{V} = \Phi(A)$ is taken to be $(d_{p^*(A)})^{(0,1)}$, the anti-holomorphic part of the covariant derivative defined by the pullback of the connection. The anti-self-duality of $A$ implies that the curvature of $p^*(A)$ is a $(1,1)$-form and so $\bar{\partial}^2 = 0$.

Buchdahl’s theorem is then

**Theorem 2.1.** The map $\Phi: \mathcal{M}_k^n(\#_N \mathbb{CP}^2) \rightarrow \mathcal{M}_{\text{alg},k}^n(\mathbb{CP}^2)$ is a diffeomorphism.

The case $N = 1$ was first proved by King [Ki]. We now restrict ourselves to that case and simply write $\mathcal{M}_k^n$ for $\mathcal{M}_k^n(\mathbb{CP}^2)$ and $\mathcal{M}_{\text{alg},k}^n(\mathbb{CP}^2)$.

King constructed $\mathcal{M}_k^n$ explicitly in terms of linear algebra data. We recall his construction. Consider configurations of linear maps:

$$
\begin{array}{ccc}
W_0 & \xrightarrow{a_1, a_2} & W_1 \\
\downarrow b & & \downarrow c \\
V_\infty & \xrightarrow{x} & 
\end{array}
$$

where $W_0$, $W_1$ and $V_\infty$ are complex vector spaces of dimensions $k$, $k$, and $n$ respectively.

A configuration $(a_1, a_2, b, c, x)$ is called integrable if it satisfies the equation

$$a_1xa_2 - a_2xa_1 + bc = 0.$$

A configuration $(a_1, a_2, b, c, x)$ is non-degenerate if it satisfies the following conditions:

$$\forall (\lambda_1, \lambda_2), (\mu_1, \mu_2) \in \mathbb{C}^2 \text{ such that } \lambda_1\mu_1 + \lambda_2\mu_2 = 0 \text{ and } (\mu_1, \mu_2) \neq (0, 0),$$

\[ \exists v \in W_1 \text{ such that } \begin{cases} xa_1v = \lambda_1v, & (\mu_1a_1 + \mu_2a_2)v = 0, \\
xa_2v = \lambda_2v, & cv = 0, \end{cases} \]

and $\exists w \in W_0^* \text{ such that } \begin{cases} x^*a_1^*w = \lambda_1w, & (\mu_1a_1^* + \mu_2a_2^*)w = 0, \\
x^*a_2^*w = \lambda_2w, & b^*w = 0. \end{cases}$

Let $A_k^n$ be the space of all integrable non-degenerate configurations. $G = Gl(W_0) \times Gl(W_1)$ acts canonically on $A_k^n$. The action is explicitly given by

$$(g_0, g_1) \cdot (a_1, a_2, b, c, x) = (g_0a_1g_1^{-1}, g_0a_2g_1^{-1}, g_0b, cg_1^{-1}, g_1xg_0^{-1}).$$

**Theorem 2.2.** The moduli space $\mathcal{M}_k^n$ is isomorphic to $A_k^n/G$.

**Proof.** King uses such configurations to determine monads that in turn determine holomorphic bundles. Configurations in the same $G$ orbit determine the same bundle. For the sake of brevity we refer the reader to [Ki] or [Br] for details. The construction identifies the vector spaces $W_0$ and $W_1$ canonically as $H^1(E(-\ell_\infty))$ and $H^1(E(-\ell_\infty + E))$ respectively, where $E \subset \mathbb{CP}^2$ is the exceptional divisor. The vector space $V_\infty$ is identified with the fiber over $\ell_\infty$. 
3. **Proof of Theorem 1.1**

We prove the theorem in two steps: We first show that the space of monad data $A^n_k$ forms a principal $G = GL(k) \times GL(k)$ bundle over $\mathcal{M}^n_k$. We then show that the induced $G$-equivariant inclusion $A^n_k \hookrightarrow A^n_{k+2k}$ is null-homotopic so that we can conclude that $A^n_\infty$ is contractible.

**Lemma 3.1.** $G$ acts freely on the space of monad data $A^n_k$.

**Proof.** This is essentially proved in [Ki] where it is implicitly shown that the non-degeneracy conditions are precisely the conditions that guarantee freeness. We point out that this also follows more conceptually from the existence of a universal family $E \to \mathcal{M}^n_k \times \overline{\mathbb{CP}}^2$ and the cohomological interpretation of $W_0$ and $W_1$:

First, the existence of a universal family can be shown via the gauge theoretic construction: Let $V$ be a smooth hermitian vector bundle on $\overline{\mathbb{CP}}^2$ with $c_2(V) = 0$ and $c_1(V) = k$. Let $A^{1,1}_0$ denote unitary connections on $V$ with curvature of pure type $(1,1)$ and that restrict to the trivial connection on $\ell_\infty$ and let $\mathcal{G}^0_0$ denote the complex gauge transformations of $V$ that are the identity restricted to $\ell_\infty$. Then $\mathcal{M}^n_k = A^{1,1}_0/\mathcal{G}^0_0$. The quotient

$$(A^{1,1}_0 \times V)/\mathcal{G}^0_0 \to \mathcal{M}^n_k \times \overline{\mathbb{CP}}^2$$

will form a universal bundle if the moduli space is smooth and no $\mathcal{E} \in \mathcal{M}^n_k$ has non-trivial automorphisms (cf. [Fr-Mo] Chapt. IV):

**Lemma 3.2.** $\mathcal{M}^n_k$ is smooth and any $\mathcal{E} \in \mathcal{M}^n_k$ has no non-trivial automorphisms preserving $\tau: \mathcal{E}|\ell_\infty \to \mathcal{C}^n \otimes \mathcal{O}_{\ell_\infty}$.

By Serre duality $H^2(\mathcal{E} \otimes \mathcal{E}^*) = H^0(\mathcal{E} \otimes \mathcal{E}^* \otimes K)^*$. Since $\mathcal{E} \otimes \mathcal{E}^*$ is trivial on $\ell_\infty$, it is also trivial on nearby lines. Any section of $\mathcal{E} \otimes \mathcal{E}^* \otimes K$ restricts to a section of $\mathcal{C}^n \otimes \mathcal{O}_{\ell_\infty}(-3)$ and so must vanish on $\ell_\infty$. Likewise, it must vanish on nearby lines and so it is 0 on an open set and must be identically 0. Thus $H^2(\mathcal{E} \otimes \mathcal{E}^*) = 0$ and smoothness follows once we show there are no automorphisms.

Suppose that there exists an automorphism $\phi \in H^0(\mathcal{E} \otimes \mathcal{E}^*)$ such that $\phi \neq 1$ and $\phi$ preserves $\tau$ so that $\phi|\ell_\infty = 1|\ell_\infty$. Then $\phi - 1$ is a non-zero section of $\mathcal{E} \otimes \mathcal{E}^*$ vanishing on $\ell_\infty$. We then get an injection $0 \to \mathcal{O}(\ell_\infty) \to \mathcal{E} \otimes \mathcal{E}^*$. Restricting this sequence to $\ell_\infty$ we get an injection $0 \to \mathcal{O}(\ell_\infty)(1) \to \mathcal{O}(\ell_\infty) \otimes \mathcal{C}^n$ which is a contradiction.

Let $\mathcal{E}$ be an $\mathcal{O}(\ell_\infty)$-vector bundle. The higher direct image sheaves $R^1\pi_*(\mathcal{E}(-\ell_\infty))$ and $R^1\pi_*(\mathcal{E}(\ell_\infty + E))$ are locally free and rank $k$. This follows from the index theorem and the vanishing of the $H^0$ and $H^2$ cohomology of $\mathcal{E}(-\ell_\infty)$ and $\mathcal{E}(\ell_\infty + E)$. The $H^0$ vanishing follows by again considering the restriction of a section of the bundles to lines nearby to $\ell_\infty$. Using Serre duality and the same argument, one gets the vanishing for $H^2$.

Consequently the vector spaces $W_0$ and $W_1$ are the fibers of the vector bundles $R^1\pi_*(\mathcal{E}(-\ell_\infty))$ and $R^1\pi_*(\mathcal{E}(\ell_\infty + E))$. The $G$-orbit of a configuration giving a bundle $\mathcal{E}$ can be identified with the group of isomorphisms $g_0: H^1(\mathcal{E}(-\ell_\infty)) \to \mathcal{C}^k$ and $g_1: H^1(\mathcal{E}(\ell_\infty + E)) \to \mathcal{C}^k$. Thus $A^n_k$ is realized precisely as the total space of
the principal $\text{Gl}(k) \times \text{Gl}(k)$ bundle associated to
\[ R^1\pi_*(E(-\ell_\infty)) \oplus R^1\pi_*(E(-\ell_\infty + E)). \]

Recall that the map $M_k^n \to M_k^{n+1}$ is defined by the direct sum with the trivial connection: $[A] \mapsto [A \oplus \theta]$. In terms of holomorphic bundles this is $E \mapsto E \oplus O$. Tracing through the monad construction, it is easy to see that the inclusion induces the $G$-equivariant map $A_k^n \to A_k^{n+1}$ given by $(a_1, a_2, x, b, c) \mapsto (a_1, a_2, x, b', c')$ where $b'$ is $b$ with an extra first column of zeroes and $c'$ is $c$ with an extra first row of zeroes.

Define $A_k^\infty$ to be the direct limit $\lim_{n \to \infty} A_k^n$ so that there is a homeomorphism between $M_k^\infty$ and $A_k^\infty / G$.

**Lemma 3.3.** $A_k^\infty$ is a contractible space.

**Proof.** Since the $A_k^n$‘s are algebraic varieties and the maps $A_k^n \to A_k^{n+1}$ are algebraic, they admit triangulations compatible with the maps. Thus $A_k^\infty$ inherits the structure of a CW-complex and so it is sufficient to show that all of its homotopy groups are zero. To this end we prove that for any $k$ and $l$ there is an $r > l$ such that the natural inclusion from $A_k^r \to A_k^l$ is homotopically trivial.

Consider the homotopy $H_t : A_k^n \to A_k^{2k+n}$ defined as follows:
\[ H_t((a_1, a_2, x, b, c)) = ((1-t)a_1, (1-t)a_2, (1-t)x, b_t, c_t) \]
where
\[ c_t = \begin{pmatrix} tI_k \\ 0_{k,k} \\ (1-t)c \end{pmatrix}, \quad b_t = (0_{k,k}, tI_k, (1-t)^2b), \]
$I_k$ is the $k \times k$ identity matrix and $0_{k,k}$ is the $k \times k$ zero matrix. To see that $H_t(v) \in A_k^{n+2k}$ for any $v \in A_k^n$, we check that the integrability and non-degeneracy conditions are satisfied for all $0 \leq t \leq 1$. Integrability holds because $b_tc_t = (1-t)^3bc$. Non-degeneracy is satisfied for all $t \neq 0$ because there is a full rank $k \times k$ block, $tI_k$, in both $c_t$ and $b_t$. Furthermore, $H_0$ is just the inclusion $A_k^n \to A_k^{n+2k}$, so non-degeneracy also holds when $t = 0$. Finally, note that $H_1$ is a constant map.

These lemmas show that $A_k^\infty$ is a contractible space acted on freely by $G = \text{Gl}(k) \times \text{Gl}(k)$ and $A_k^\infty / G = M_k^\infty$. Thus $M_k^\infty$ is homotopic to $BG$ which in turn has the homotopy type of $BU(k) \times BU(k)$. We end by remarking that the proof shows that the universal $U(k) \times U(k)$ bundle is the bundle that restricts to any of the finite $M_k^n$‘s as $R^1\pi_*(E(-\ell_\infty)) \oplus R^1\pi_*(E(-\ell_\infty + E))$.

**References**


[Sa] Sanders, M., Classifying spaces and dirac operators coupled to instantons, Trans. of the A.M.S. Vol. 347, No. 10 1995. MR 96m:58030


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