

HOMOTOPY FIXED POINTS FOR CYCLIC p -GROUP ACTIONS

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ABSTRACT. The homotopy fixed point p -compact groups for cyclic p -group actions on nonabelian connected p -compact groups are not homotopically discrete.

1. INTRODUCTION

It is a classical result that cyclic groups acting on nonabelian compact connected Lie groups have no isolated fixpoints [2, Lemme 1, p. 46]:

Theorem 1.1. *Let X be a nonabelian connected compact Lie group equipped with an action of a cyclic group G . Then the identity component of the fixed point group X^G is nontrivial.*

In this note we prove an analog for p -compact groups of this statement. First, we need a few concepts.

Suppose that X is a p -compact group [4] with classifying space BX and that G is a finite group.

Definition 1.2. A G -action on X is a sectioned fibration

$$BX \longrightarrow (BX)_{hG} \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{Ba} \end{array} BG$$

over BG with fibre BX .

If G is a finite p -group, it is known [4, 5.8] that each component of the section space $(BX)^{hG}$ is the classifying space of a p -compact group. We define the *homotopy fixed point p -compact group for the G -action* to be the p -compact group

$$X^{hG} = \Omega((BX)^{hG}, Ba)$$

whose classifying space is the component containing the section Ba .

Having introduced these concepts, we can now formulate the main result of this note. (A connected p -compact group is nontrivial if its classifying space is noncontractible.)

Theorem 1.3. *Let X be a nonabelian connected p -compact group equipped with an action of a cyclic p -group G . Then the identity component of the homotopy fixed point p -compact group X^{hG} is nontrivial.*

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The following consequence of this theorem, whose proof relies on a Lefschetz number calculation, is immediate.

Corollary 1.4. *Let $\nu: G \rightarrow Y$ be a monomorphism of a cyclic p -group into a (not necessarily connected) p -compact group Y which is not a p -compact toral group. Then the identity component of the centralizer $C_Y(G)$ of ν is nontrivial.*

The self-centralizing diagonal subgroup $(\mathbb{Z}/2\mathbb{Z})^n$ of $O(n)$ shows that noncyclic subgroups may have discrete centralizers.

Corollary 1.4 plays an important role in the proof of the main result of [9].

2. A LEFSCHETZ NUMBER CALCULATION

Let X be a connected p -compact group, $G = \mathbb{Z}/p^r, r \geq 0$, a cyclic p -group and

$$BX \longrightarrow (BX)_{hG} \begin{matrix} \xleftarrow{Ba} \\ \xrightarrow{Ba} \end{matrix} BG$$

an action of G on X . The homotopy fixed point p -compact group X^{hG} is the section space of the fibrewise looping

$$X \longrightarrow X_{hG} \begin{matrix} \xleftarrow{Ba} \\ \xrightarrow{Ba} \end{matrix} BG$$

of the G -action. Consider the associated monodromy homomorphisms

- (1) $G \rightarrow \text{Aut}_*(BX),$
- (2) $G \rightarrow \text{Aut}_*(X)$

of G into the groups of based homotopy classes of based self-homotopy equivalences of the fibres and the induced representations

- (3) $G \rightarrow \text{Aut } H_{\mathbb{Q}_p}^*(BX),$
- (4) $G \rightarrow \text{Aut } H_{\mathbb{Q}_p}^*(X)$

of G in the p -adic rational cohomology algebras. Of course, representation (4) induces yet another representation

- (5) $G \rightarrow QH_{\mathbb{Q}_p}^*(BX)$

of G in the graded vector space of indecomposables.

Let $Bg: BX \rightarrow BX$ and $g: X \rightarrow X$ be the self-homotopy equivalences induced by a generator $g \in G$. We shall compute the Lefschetz number

$$\Lambda(X; G) = \sum (-1)^i \text{trace } H_{\mathbb{Q}_p}^i(g)$$

for the action of G on X in terms of the irreducible summands of the representation (5).

Recall [4, §4] that the cyclic group G admits $r + 1$ essentially distinct irreducible representations $\rho_0, \rho_1, \dots, \rho_r$ over the p -adic numbers. Here, ρ_0 is the trivial representation and $\rho_i, 1 \leq i \leq r$, is the composition of the reduction map $G = \mathbb{Z}/p^r \rightarrow \mathbb{Z}/p^i$ with the action of \mathbb{Z}/p^i , regarded as the group of p^i th roots of unity, on the extension field $\mathbb{Q}_p(\omega_i)$ of \mathbb{Q}_p by a primitive p^i th root of unity ω_i . The dimension of $\rho_i, 1 \leq i \leq r$, is $[\mathbb{Q}_p(\omega_i) : \mathbb{Q}_p] = p^i - p^{i-1}$.

Proposition 2.1. *Suppose that the G -representation $QH_{\mathbb{Q}_p}^*(BX)$ contains the irreducible representation ρ_i with multiplicity n_i , $0 \leq i \leq r$. Then*

$$\Lambda(X; G) = \begin{cases} p^{n_1 + \dots + n_r} & \text{if } n_0 = 0, \\ 0 & \text{if } n_0 \neq 0 \end{cases}$$

is the Lefschetz number for the action of G on X . In particular, $\Lambda(X; G) = 0$ if and only if G fixes a nonzero vector of $QH_{\mathbb{Q}_p}^(BX)$.*

Proof. Note that the monodromy action (2) of G on $X = \Omega BX$ is the looping of the monodromy action (1) on BX and that the Eilenberg–Moore spectral sequence provides a functorial isomorphism between the graded object $\text{Gr}(H_{\mathbb{Q}_p}^*(X))$ associated to a filtration of $H_{\mathbb{Q}_p}^*(X)$ and the exterior algebra $E(\Sigma^{-1}QH_{\mathbb{Q}_p}^*(BX))$ on the desuspension of $QH_{\mathbb{Q}_p}^*(BX)$. Combining this with the isomorphism

$$QH_{\mathbb{Q}_p}^*(BX) \cong n_0\rho_0 \oplus n_1\rho_1 \oplus \dots \oplus n_r\rho_r$$

of G -representations induces yet another isomorphism

$$\text{Gr}(H_{\mathbb{Q}_p}^*(X)) \cong E(\Sigma^{-1}\rho_0)^{\otimes n_0} \otimes E(\Sigma^{-1}\rho_1)^{\otimes n_1} \otimes \dots \otimes E(\Sigma^{-1}\rho_r)^{\otimes n_r}$$

of G -representations. By the additivity [4, 4.12] of traces in exact sequences, then, the Lefschetz number

$$\Lambda(X; G) = \prod_{i=0}^r \Lambda_i^{n_i}$$

where Λ_i is the trace for the action of G on $E(\Sigma^{-1}\rho_i)$.

Since $E(\Sigma^{-1}\rho_0)$ is the trivial representation, $\Lambda_0 = 0$.

When $i > 0$, we pass to an algebraic closure of \mathbb{Q}_p . Then ρ_i splits into 1-dimensional representations and we see that $\Lambda_i = \Phi_i(1)$ where Φ_i is the characteristic polynomial for g acting on ρ_i or, equivalently, for ω_i acting on $\mathbb{Q}_p(\omega_i)$. Hence Φ_i is the p^i th cyclotomic polynomial so $\Phi_i(1) = p$ and the proposition follows. \square

The consequence below is evident if we recall [4, 4.5, 5.7, 5.10] that the Lefschetz number $\Lambda(X; G)$ computes the Euler characteristic of X^{hG} and that a p -compact group is homotopically discrete if it looks so in p -adic rational cohomology.

Corollary 2.2. *The following conditions are equivalent:*

- (1) X^{hG} has a nontrivial identity component.
- (2) $\chi(X^{hG}) > 0$.
- (3) $\Lambda(X; G) > 0$.
- (4) G fixes a nonzero vector of $QH_{\mathbb{Q}_p}^*(BX)$.

The proof of Theorem 1.3 has now been reduced to the following

Lemma 2.3. *Suppose that X is nonabelian (i.e. not a p -compact torus). Then G fixes a nonzero vector of $QH_{\mathbb{Q}_p}^*(BX)$.*

Proof. Let $T \rightarrow X$ be a maximal torus with Weyl group W . The dual weight lattice $L = \pi_2(BT)$ is then a $\mathbb{Z}_p[W]$ -module whose rationalization $L \otimes \mathbb{Q}$ exhibits W as a reflection group over \mathbb{Q}_p . The action of G on the symmetric invariants $\text{Sym}((L \otimes \mathbb{Q})^*)^W \cong H_{\mathbb{Q}_p}^*(BX)$ factors [4, 8.11, 9.5], [8, §3] through $N(W)/W$ where $N(W)$ is the normalizer of $W < \text{Aut}(L \otimes \mathbb{Q})$.

Suppose first that X is almost simple, i.e. [5, 1.6] that the center of X is finite and that $L \otimes \mathbb{Q}$ is a simple $\mathbb{Q}_p[W]$ -module. Then the reflection group W is one of the

irreducible reflection groups on the Shephard–Todd–Clark–Ewing list as presented e.g. in [6, p. 165]. The list provides information about the indecomposables of the invariant ring in that the degrees of each reflection group are given.

If $p > 2$, the list shows that $\dim_{\mathbb{Q}_p} QH_{\mathbb{Q}_p}^i(BX) < p - 1$ for all i . (In fact $QH_{\mathbb{Q}_p}^i(BX)$ has dimension ≤ 2 with dimension 2 occurring only in case 2a (where the degrees given in [6] are incorrect) and in case 19, neither of which are realizable for $p = 3$.) Since a nontrivial p -adic representation of a cyclic p -group requires at least $p - 1$ dimensions, G must act trivially on all of $QH_{\mathbb{Q}_p}^*(BX)$ (which is nonzero if X is nontrivial [4, 5.10]).

The case $p = 2$ requires separate treatment. The only irreducible 2-adic reflection groups are the classical Coxeter groups together with group number 24 of rank 3, $W = \mathbb{Z}/2\mathbb{Z} \times \text{GL}_3(\mathbb{F}_2)$, realized by $\text{DI}(4)$ [3]. If W is one of the classical Coxeter groups, the effect of an element of the normalizer $N(W)$ on the degree 4 invariants is multiplication by u^2 , $2u^2$, or $3u^2$, where $u \in \mathbb{Q}_2^*$ is a 2-adic unit [7, 1.7]. Since -1 doesn't have this form, the 1-dimensional G -representation $H_{\mathbb{Q}_p}^4(BX) = QH_{\mathbb{Q}_p}^4(BX)$ is the trivial one. Generators for the ring of invariant polynomials of the unique nonclassical 2-adic reflection group are [1, p. 101]

$$\begin{aligned} y_8 &= x_1x_2^3 + x_2x_3^3 + x_3x_1^3, \\ y_{12} &= \det \left(\frac{\partial^2 y_8}{\partial x_i \partial x_j} \right), \\ y_{28} &= \det \begin{pmatrix} \frac{\partial^2 y_8}{\partial x_i \partial x_j} & \frac{\partial y_{12}}{\partial x_i} \\ \frac{\partial y_{12}}{\partial x_j} & 0 \end{pmatrix}, \end{aligned}$$

where the subscript on the variable y denotes the dimension of the corresponding indecomposable cohomology class. Note that if an element of $N(W)$ takes y_8 to its opposite, then also y_{12} is taken to its opposite but y_{28} remains fixed. Thus any element of 2-power order in $N(W)/W$ must fix either y_8 or y_{28} (considered as elements of $H_{\mathbb{Q}_p}^*(BX)$).

This proves the lemma for all almost simple p -compact groups.

Next suppose that X is simply connected and nontrivial. Then there exist, by the splitting theorem [5], almost simple p -compact groups X_1, \dots, X_n with dual weight lattices L_1, \dots, L_n and Weyl groups W_1, \dots, W_n such that $X \cong X_1 \times \dots \times X_n$ and $L \cong L_1 \times \dots \times L_n$ as $W \cong W_1 \times \dots \times W_n$ -modules. The effect of Bg on $H_{\mathbb{Q}_p}^*(BX) = \bigotimes H_{\mathbb{Q}_p}^*(BX_i)$ has, cf. [8, 3.5], the form

$$H_{\mathbb{Q}_p}^*(Bg) = (A_1 \otimes \dots \otimes A_n) \circ \sigma$$

where A_i is an automorphism of $H_{\mathbb{Q}_p}^*(BX_i)$, $1 \leq i \leq n$, and σ is a permutation within the isomorphism classes of these algebras. Hence

$$QH_{\mathbb{Q}_p}^*(Bg) = (QA_1 \oplus \dots \oplus QA_n) \circ \sigma$$

on $QH_{\mathbb{Q}_p}^*(BX) = \bigoplus QH_{\mathbb{Q}_p}^*(BX_i)$. There are now essentially two distinct cases to consider. Namely, the case where σ is trivial and the case where σ is a cyclic permutation of p -power order > 1 . The first case was treated above and in the second case, $QH_{\mathbb{Q}_p}^*(Bg)$ fixes the diagonal. Hence the fixed point vector space $QH_{\mathbb{Q}_p}^*(BX)^G$ is nontrivial for any nontrivial simply connected p -compact group X .

Finally, up to isogeny any connected p -compact group has the form $X \times S$ [11, 5.4] where X is simply connected and S is a p -compact torus and any automorphism

is a product of an automorphism of X with an automorphism of S [10, 4.3]. Hence

$$QH_{\mathbb{Q}_p}^*(BX \times BS)^G \cong QH_{\mathbb{Q}_p}^*(BX)^G \oplus QH_{\mathbb{Q}_p}^*(BS)^G$$

is nontrivial if X is nontrivial. \square

We conclude this note with the easy proof of Corollary 1.4.

Proof of Corollary 1.4. Let π be the component group and X the identity component of Y . The p -compact group extension

$$X^{hG} \rightarrow C_Y(G) \rightarrow C_\pi(G)$$

shows that X^{hG} and $C_Y(G)$ have isomorphic identity components. \square

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