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ON AN IDENTITY OF RAMANUJAN

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ABSTRACT. An identity stated and utilized by Ramanujan in his now classic study of certain arithmetical functions is proved.

1. Introduction

The following identity, which is valid for each complex number x such that $|x| < 1$, was first stated by Ramanujan [4, p. 144] in his now classical paper "On certain arithmetical functions":

$$x\left\{\sum_0^\infty x^{n(n+1)/2}\right\}^8 = \sum_1^\infty \frac{n^3x^n}{1-x^{2n}}. \tag{1.1}$$

As usual, he gave no proof. The identity played a crucial role in providing the first rather crude estimate of the order of magnitude of values of the function τ, defined by the expansion

$$x\prod_1^\infty (1-x^n)^{24} = \sum_1^\infty \tau(n)x^n \qquad (|x|<1).$$

In [1] the author used the identity to give a formula for τ. Regarding this formula see also [5, pp. 267–269]. In view of these applications, as well as possible others, it seems desirable that an elementary proof of this identity should be presented. This we propose to do in the present note. Details are given in section 2.

2. Proof of (1.1)

Our proof is based on the identities:

$$\prod_1^\infty (1-x^{2n})(1+tx^{2n-1})(1+t^{-1}x^{2n-1}) = \sum_{-\infty}^\infty x^{n^2}t^n \tag{2.1}$$

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and

(2.2)
$$\prod_1^\infty (1-x^{2n})^2(1+abx^{2n-1})(1+a^{-1}b^{-1}x^{2n-1})(1+ab^{-1}x^{2n-1})(1+a^{-1}bx^{2n-1})$$
$$= \sum_{-\infty}^{\infty} x^{2m^2}a^{2m}\sum_{-\infty}^{\infty} x^{2n^2}b^{2n} + x\sum_{-\infty}^{\infty} x^{2m(m+1)}a^{2m+1}\sum_{-\infty}^{\infty} x^{2n(n+1)}b^{2n+1},$$

which are respectively valid for each pair t, x and each triple a, b, x of complex numbers such that $t \neq 0$, $a \neq 0$, $b \neq 0$ and $|x| < 1$. Identity (2.1) is a celebrated result, due to Gauss and Jacobi. E.g. see [3, p. 282]. Identity (2.2), due to the author, is proved in [2].

To prove identity (1.1) we first appeal to identity (2.1) to express each series on the right side of identity (2.2) as an infinite product.

(2.3)
$$\prod_1^\infty (1-x^{2n})^2(1+abx^{2n-1})(1+a^{-1}b^{-1}x^{2n-1})(1+ab^{-1}x^{2n-1})(1+a^{-1}bx^{2n-1})$$
$$= \prod_1^\infty (1-x^{4n})^2(1+a^2x^{4n-2})(1+a^{-2}x^{4n-2})(1+b^2x^{4n-2})(1+b^{-2}x^{4n-2})$$
$$+ (a+a^{-1})(b+b^{-1})x\prod_1^\infty (1-x^{4n})^2(1+a^2x^{4n})(1+a^{-2}x^{4n})(1+b^2x^{4n})(1+b^{-2}x^{4n}).$$

Next, in (2.3) let $a = b$ and subsequently let $a \to ia$ to get

$$\prod_1^\infty (1-x^{2n})^2(1+x^{2n-1})^2(1-a^2x^{2n-1})(1-a^{-2}x^{2n-1})$$
$$= \prod_1^\infty (1-x^{4n})^2(1-a^2x^{4n-2})^2(1-a^{-2}x^{4n-2})^2 \tag{2.4}$$
$$- (a-a^{-1})^2x\prod_1^\infty (1-x^{4n})^2(1-a^2x^{4n})^2(1-a^{-2}x^{4n})^2.$$

In (2.4) let $x \to -x$, multiply the resulting identity and (2.4), and then let $x \to x^{1/2}$ to get

$$\prod_1^\infty (1-x^n)^4(1-x^{2n-1})^2(1-a^4x^{2n-1})(1-a^{-4}x^{2n-1})$$
$$= \prod_1^\infty (1-x^{2n})^4(1-a^2x^{2n-1})^4(1-a^{-2}x^{2n-1})^4 \tag{2.5}$$
$$- (a-a^{-1})^4x\prod_1^\infty (1-x^{2n})^4(1-a^2x^{2n})^4(1-a^{-2}x^{2n})^4.$$

With D_a denoting derivation with respect to a, put $\theta_a := aD_a$. We then (i) operate on both sides of (2.5) with θ_a^4, (ii) put $a = 1$, (iii) divide each side of the resulting identity by the product

$$\prod_{1}^{\infty}(1-x^n)^4(1-x^{2n-1})^4,$$

and (iv) simplify to get

$$x\left\{\prod_{1}^{\infty}\frac{1-x^{2n}}{1-x^{2n-1}}\right\}^8=\sum_{1}^{\infty}\frac{n^3x^n}{1-x^{2n}}.$$

Finally, we appeal to the following well-known identity of Gauss [3, p. 284] to prove identity (1.1):

$$\prod_{1}^{\infty}\frac{1-x^{2n}}{1-x^{2n-1}}=\sum_{0}^{\infty}x^{n(n+1)/2} \qquad (|x|<1).$$

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