MALUTA’S COEFFICIENT
IN MUSIELAK-ORLICZ SEQUENCE SPACES
EQUIPPED WITH THE ORLICZ NORM

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ABSTRACT. Maluta’s coefficient of Musielak-Orlicz sequence spaces equipped with the Orlicz norm is calculated. A sufficient condition for the Schur property of these spaces is given.

1. Introduction

In the sequel \(\mathbb{N}, \mathbb{R}\) and \(\mathbb{R}_+\) stand for the set of natural numbers, the set of reals and the set of nonnegative reals, respectively. The space of all sequences \(x = (x(i))_{i=1}^{\infty}\) of reals is denoted by \(l^0\). A map \(\Phi : \mathbb{R} \to [0, +\infty)\) is said to be an Orlicz function if \(\Phi\) is convex, even, vanishing at zero, left continuous on \(\mathbb{R}_+\) and not identically equal to zero (see [9, 12, 13, 15] and [16]).

A sequence \(\Phi = (\Phi_i)_{i=1}^{\infty}\) of Orlicz functions \(\Phi_i\) is called a Musielak-Orlicz function (see [15]). By \(\Psi = (\Psi_i)\) we denote the Musielak-Orlicz function conjugate to \(\Phi = (\Phi_i)\) in the sense of Young, i.e.

\[\Psi_i(u) = \sup_{v > 0}\{|u|v - \Phi_i(v)\}\]

for each \(u \in \mathbb{R}\) and \(i \in \mathbb{N}\). Further, \(\varphi = (\varphi_i)\) is the right derivative of \(\Phi = (\Phi_i)\), i.e. \(\varphi_i\) are the right derivatives of \(\Phi_i\) for every \(i \in \mathbb{N}\).

Given a Musielak-Orlicz function \(\Phi = (\Phi_i)\) we define on \(l^0\) a convex modular \(I_{\Phi}\) by

\[I_{\Phi}(x) = \sum_{i=1}^{\infty} \Phi_i(x_i)\]

(\(\forall x = (x_i) \in l^0\)),

and the Musielak-Orlicz space \(l^\Phi\) by

\[l^\Phi = \{x \in l^0 : I_{\Phi}(\lambda x) < +\infty \text{ for some } \lambda > 0\}\].

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The functional
\[ \|x\| = \sup \left\{ \sum_{i=1}^{\infty} x(i) y(i) : I_{\Phi}(y) \leq 1 \right\} \]
is a norm in \( l^\Phi \) (called the \textit{Orlicz norm}) and the couple \((l^\Phi, \| \|)\) is a Banach space (see [15]).

Our aim in this paper is to calculate Maluta’s coefficient of Musielak-Orlicz spaces equipped with the Orlicz norm. This coefficient is connected with normal structure, which is a very important property of Banach spaces that guarantees the fixed point property for them (see [1, 2, 3, 4, 6, 8, 10, 11, 14] and [19]).

Let in the sequel \( X \) denote a reflexive infinite dimensional Banach space (which automatically does not have the Schur property) and let \( S(X) \) denote its unit sphere. For each sequence \((x_n)\) in \( X \), we define
\[
A((x_n)) = \lim_n \sup \{ \| x_i - x_j \| : i \neq j; i, j \geq n \},
\]
\[
A_1((x_n)) = \lim_n \inf \{ \| x_i - x_j \| : i \neq j; i, j \geq n \}.
\]
The \textit{weak uniform normal structure coefficient} of \( X \) is defined by (see [4])
\[
WCS(X) = \sup \{ k > 0 : \text{for each weakly convergent sequence } (x_n) \text{ in } S(X) \text{ there is some } y \in \text{conv}(x_n) \text{ such that } k \lim_n \sup \| x_n - y \| \leq A((x_n)) \}.
\]

A sequence \((x_n)\) in \( X \) is said to be an \textit{asymptotic equidistant sequence} if \( A((x_n)) = A_1((x_n)) \). This definition was introduced in [19], where it was proved that \( WCS(X) = \inf \{ A((x_n)) : (x_n) \text{ is an asymptotic equidistant sequence in } S(X) \text{ and } x_n \to 0 \text{ weakly} \} \).

Recall that \textit{Maluta’s coefficient} \( M(X) \) of a Banach space \( X \) is defined by (see [14])
\[
M(X) = \sup \left\{ \lim_n \sup \left\{ d(x_{n+1}, \text{conv}(x_j))_{j=1}^{n} : (x_n) \text{ is a bounded nonconstant sequence in } X \right\} \right\}.
\]
We have \( M(X) = 1/WCS(X) \) for each reflexive Banach space \( X \) and \( M(X) = 1 \) for each nonreflexive Banach space \( X \).

To formulate our results, we need to fix some notations. For any \( i \in \mathbb{N} \), put
\[
b_i = \sup \{ v > 0 : \Psi_i(v) < +\infty \},
\]
\[
a_i = \begin{cases} b_i & \text{if } \Psi_i(b_i) < 1 \\ \Psi_i^{-1}(1) & \text{if } \Psi_i(b_i) \geq 1, \end{cases}
\]
\[
N_x = \{ i \in \mathbb{N} : x(i) \neq 0 \}.
\]
The result that for every \( x \in l^\Phi \) with \( \sum_{i \in N_x} \Psi_i(a_i) > 1 \), we have \( \| x \| = \frac{1}{b}(1 + I_{\Phi}(kx)) \)
if and only if \( k \in [k^*_x, k^{**}_x] \), where \( k^*_x = \inf \{ k > 0 : \sum_{i=1}^{\infty} \Psi_i(\varphi_i(kx(i))) \geq 1 \} \), \( k^{**}_x = \sup \{ k > 0 : \sum_{i=1}^{\infty} \Psi_i(\varphi_i(kx(i))) \leq 1 \} \) has been proved in [18]. Moreover, for any
x ∈ ℓ^Φ there exists k > 0 such that ||x|| = \( \frac{1}{k}(1 + I_Φ(kx)) \) whenever Φ_i(u)/u → +∞ as u → +∞ for all i ∈ N.

We need to define a regularity condition for Φ = (Φ_i) called the δ^Φ_2-condition. A Musielak-Orlicz function Φ = (Φ_i) satisfies the δ^Φ_2-condition if there exist positive constants a and K and a sequence (c_i) in [0, +∞] such that \( \sum_{i=i_0}^∞ c_i < +∞ \) for some \( i_0 \in N \) and

\[ Φ_i(2u) \leq kΦ_i(u) + c_i \]

for each \( i \in N \) and each \( u \in ℝ \) satisfying \( Ψ_i(u) \leq a \). In the case when all \( c_i \) are in \( ℝ_+ \) we say that Φ satisfies the δ_2-condition (see [15]).

2. Results

We start with the following

**Lemma 1.** If \( \sum_{i=1}^∞ Ψ_i(a_i) \leq 1 \) then ℓ^Φ has the Schur property.

**Proof.** Suppose \( x_n = (x_n(i)) \in S(ℓ^Φ) \) for each \( n \in N \) and \( x_n \to x_0 \) weakly.

By \( \sum_{i=1}^∞ Ψ_i(a_i) \leq 1 \), we have \( ||x_n|| = \sum_{i=1}^∞ a_i|x_n(i)| \) (\( n = 1, 2, \ldots \)). Define \( z_n = (a_1x_n(1), a_2x_n(2), \ldots) \) and \( z_0 = (a_1x_0(1), a_2x_0(2), \ldots) \). Then \( z_n \in l^1 \) for \( n = 0, 1, \ldots \) and \( z_n \to z_0 \) weakly in \( l^1 \) (because the weak convergence in \( ℓ^Φ \) implies the weak convergence in \( l^1((a_i)) \)). Since \( l^1 \) has the Schur property, we get \( ||z_n - z_0||_1 \to 0 \). Hence, in view of the equality

\[ ||x_n - x_0|| = \sum_{i=1}^∞ a_i|x_n(i) - x_0(i)| = ||z_n - z_0||_1, \]

we get \( lim_{n} ||x_n - x_0|| = 0 \), i.e. \( ℓ^Φ \) has the Schur property.

To calculate Maluta’s coefficient for \( ℓ^Φ \) we need to define some parameter for the generating Musielak-Orlicz function Φ = (Φ_i).

In this definition there is some analogy to the definition of the parameter \( d_Φ \) in [7] (see also [17]).

For every \( m, n \in N \) and \( k > 1 \), we define

\[ c(k, m, n) = \inf \{ c_{k, x} > 0 : I_Φ \left( \frac{kx}{c_{k, x}} \right) = \frac{k - 1}{2} \text{ and } x = \sum_{i=m}^{m+n} x(i)c_i \in S(ℓ^Φ) \}. \]

The sequence \( (c(k, m, n))_{m=1}^∞ \) is nonincreasing for every \( k > 1 \) and \( m \in N \). Therefore, the limit

\[ d(k, m) = \lim_{n→∞} c(k, m, n) \]

exists. Moreover, \( d(k, m) ≤ c(k, m, n) \) for every \( k > 1 \) and \( m, n \in N \). The sequence \( (d(k, m))_{m=1}^∞ \) is nondecreasing. Hence, the limit

\[ d_k = \lim_{m→∞} d(k, m) \]

exists and \( d_k ≥ d(k, m) \) for every \( k > 1 \) and \( m \in N \).

Define

\[ d(Φ) = \inf \{ d_k : k > 1 \}. \]

We are now in a position to give the main result of the paper.
Theorem 1. Assume that $\Phi = (\Phi_i)$ is a Musielak-Orlicz function such that all $\Phi_i\ (i = 1, 2, \ldots)$ are finitely valued and $\Phi_i(u)/u \to +\infty$ as $u \to +\infty$ for all $i \in \mathbb{N}$. Then: (i) If $l^\Phi$ is nonreflexive, then $M(l^\Phi) = 1$; (ii) If $l^\Phi$ is reflexive, then $M(l^\Phi) = 1/d(\Phi)$.

Proof. Statement (i) follows immediately from the fact that $M(X) = 1$ for every nonreflexive Banach space $X$. So, we only need to prove that $WCS(L^\Phi) = d(\Phi)$ whenever $l^\Phi$ is reflexive. It is well known that the reflexivity of $l^\Phi$ is equivalent to the fact that both $\Phi$ and $\Psi$ satisfy the $\delta_0^\psi$-condition.

First, we will prove that $WCS(l^\Phi) \leq d(\Phi)$. For each $\varepsilon > 0$, by the definition of $d(\Phi)$, there is $k > 1$ such that $d(\Phi) > dk - \varepsilon$. Recall that $d_k \geq d(k, m)$ for all $k > 1$ and $m \in \mathbb{N}$. By the definition of $d(k, m)$ there is $n(m) \in \mathbb{N}$ such that

$$d(k, m) > c(k, m, n) - \varepsilon \quad \text{whenever} \quad n > n(m).$$

Finally, by the definition of $c(k, m, n)$ there exists $x_{m, n} \in S(l^\Phi)$ such that

1. $c_{k, x_{m, n}} - \varepsilon < c(k, m, n)$,

2. $I_\Phi\left(\frac{kx_{m, n}}{c_{k, x_{m, n}}}\right) = \frac{k-1}{2}$.

Take $m_1 = 1$. Then there exist $n_1 \in \mathbb{N}$, $n > n(m_1)$ and $x_{m_1, n_1}$ satisfying (1) and (2) with $m_1$ and $n_1$ in place of $m$ and $n$. Take $m_2 = m_1 + n_1 + 1$. There exists $x_{m_2, n_2}$ satisfying (1) and (2) with $m_2, n_2 > n(m_2)$, in place of $m, n$. By induction, we can construct a sequence $(x_{m_i, n_i})_{i=1}^\infty$ in $S(l^\Phi)$ with pairwise disjoint supports and satisfying (1) and (2) with $m_i$ and $n_i$, $n_i > n(m_i)$ in place of $m$ and $n$ for $i = 1, 2, \ldots$.

Define $y_k = x_{m_k, n_k}$. Then, we have $y_n \in S(l^\Phi)$ for every $n \in \mathbb{N}$. Moreover, $y_n \to 0$ weakly and for every $\nu, l \in \mathbb{N}$ there holds

$$\left\|\frac{y_{\nu} - y_l}{d(\Phi) + 2\varepsilon}\right\| \leq \frac{1}{k} \left(1 + I_\Phi\left(\frac{kx_{m_1, n_1}}{d(\Phi)} + 2\varepsilon\right)\right) = \frac{1}{k} \left(1 + I_\Phi\left(\frac{y_{\nu}}{d(\Phi)} + 2\varepsilon\right) + I_\Phi\left(\frac{y_l}{d(\Phi)} + 2\varepsilon\right)\right) \leq \frac{1}{k} \left(1 + I_\Phi\left(\frac{kx_{m_1, n_1}}{c_{k, x_{m_1, n_1}}}\right) + I_\Phi\left(\frac{kx_{m_1, n_1}}{c_{k, x_{m_1, n_1}}}\right)\right) = \frac{1}{k} \left(1 + \frac{k-1}{2} + \frac{k-1}{2}\right) = 1.$$  

Hence $A(y_n) \leq d(\Phi) + 3\varepsilon$. By the arbitrariness of $\varepsilon > 0$, we have $A(y_n) \leq d(\Phi)$. By virtue of Proposition 2 in [19] which says that for each weakly convergent sequence on the unit sphere of a Banach space $X$ there exists an asymptotic equidistant subsequence, we can now conclude that $WCS(l^\Phi) \leq d(\Phi)$.

Next, we will prove that $WCS(l^\Phi) \geq d(\Phi)$. First, we will show the equality

$$WCS(l^\Phi) = \inf\{A((x_n)) : x_n = \sum_{i=n_1+1}^{n_2} x_n(i)e_i \text{ and } (x_n)\}$$

is an asymptotic equidistant sequence in $S(l^\Phi)$. It is obvious that $WCS(l^\Phi) \leq d$, so we only need to prove that $WCS(l^\Phi) \geq d$. For any $\varepsilon > 0$, by the definition of $WCS(l^\Phi)$, there exists a sequence $(x_n)$ in $S(l^\Phi)$ being an asymptotic equidistant sequence, weakly convergent to $0$ and such that

$$A((x_n)) < WCS(l^\Phi) + \varepsilon.$$
Let \( v_1 = x_1 \). Take \( l_1 \in \mathbb{N} \) satisfying \( \| \sum_{i=l_1+1}^{\infty} v_1(i)e_i \| < \varepsilon \). Such a number \( l_1 \) exists since by the reflexivity of \( l^\Phi \), the generating function \( \Phi = (\Phi_i) \) satisfies the \( \delta_2 \)-condition. By \( x_n(i) \to 0 \) as \( n \to \infty \), \( (i = 1, 2, \ldots, l_1) \) there is \( n_0 \in \mathbb{N} \) such that

\[
\| \sum_{i=1}^{l_1} x_n(i)e_i \| < \varepsilon \text{ whenever } n > n_0.
\]

Let us fix \( N_1 > n_0 \) and set \( v_2 = x_{N_1} \). Then

\[
\| \sum_{i=1}^{l_1} v_2(i)e_i \| < \varepsilon.
\]

Take \( l_2 > l_1 \) such that \( \| \sum_{i=l_2+1}^{\infty} v_2(i)e_i \| < \varepsilon \). By \( x_n(i) \to 0 \) as \( n \to \infty \) for \( i = 1, 2, \ldots \), we can find \( N_2 > N_1 \) such that

\[
\| \sum_{i=1}^{l_2} x_n(i)e_i \| < \varepsilon \text{ whenever } n > N_2.
\]

Let us choose \( N_3 > N_2 \) and set \( v_3 = x_{N_3} \). Then

\[
\| \sum_{i=1}^{l_2} v_3(i)e_i \| < \varepsilon.
\]

Take \( l_3 > l_2 \) such that

\[
\| \sum_{i=l_3+1}^{\infty} v_3(i)e_i \| < \varepsilon.
\]

In such a way we can construct by induction a sequence \((l_n)\) of natural numbers with \( l_1 < l_2 < \ldots \) and a subsequence \((v_n)\) of \((x_n)\) satisfying \( A((v_n)) = A((x_n)) \) and

\[
\| \sum_{i=1}^{l_n} v_n(i)e_i \| < \varepsilon, \quad \| \sum_{i=l_n+1}^{\infty} v_n(i)e_i \| < \varepsilon,
\]

where \( l_0 = 0 \) by definition. Put

\[
u_n = \sum_{i=l_{n-1}+1}^{l_n} v_n(i)e_i / \| \sum_{i=l_{n-1}+1}^{l_n} v_n(i)e_i \| \quad (n = 1, 2, \ldots).
\]
Then $u_n \in S(I^p)$ for each $n \in \mathbb{N}$. Moreover, for every $m, n \in \mathbb{N}, n < m$, we have

$$\|v_n - v_m\| = \left\| \sum_{i=1}^{l_{n-1}} (v_n(i) - v_m(i))e_i + \sum_{i=l_{n-1}+1}^{l_n} (v_n(i) - v_m(i))e_i \right\|$$

$$+ \left\| \sum_{i=l_n+1}^{\infty} (v_n(i) - v_m(i))e_i + \sum_{i=l_{n-1}+1}^{l_m} (v_n(i) - v_m(i))e_i \right\|$$

$$\geq \left\| \sum_{i=l_{n-1}+1}^{l_n} v_n(i)e_i - \sum_{i=l_{n-1}+1}^{l_m} v_m(i)e_i \right\| - 4\varepsilon$$

$$\geq \|(u_n - u_m)(1 - 2\varepsilon)\| - 4\varepsilon.$$

Therefore

$$A((u_n)) \leq \frac{A((v_n))}{1 - 2\varepsilon} + \frac{4\varepsilon}{1 - 2\varepsilon} = \frac{A((x_n)) + 4\varepsilon}{1 - 2\varepsilon} \leq \frac{WCS(I^p) + 5\varepsilon}{1 - 2\varepsilon}.$$

In view of the arbitrariness of $\varepsilon > 0$, we have $d \leq WCS(I^p)$. Finally, we will prove that $d \geq d(\Phi)$. For any asymptotic equidistant sequence

$$x_n = \sum_{i=l_{n-1}+1}^{l_n} x_n(i)e_i \in S(I^p) \quad (n = 1, 2, \ldots),$$

there are $k_{m,n} > 0$ such that

$$\|(x_m - x_n)/d(\Phi)\| = \frac{1}{k_{m,n}}(1 + I_\Phi(\frac{k_{m,n}x_m - x_n}{d(\Phi)}))$$

for all $m, n \in \mathbb{N}, m \neq n$. Let us assume in the following that $m, n \in \mathbb{N}$ and $m \neq n$. We will consider now two cases.

I. If $k_{m,n} \leq 1$, then $\|x_m - x_n\| \geq d(\Phi)$.

II. If $k_{m,n} > 1$, then

$$\|(x_m - x_n)/d(\Phi)\| = \frac{1}{k_{m,n}}(1 + I_\Phi(\frac{k_{m,n}x_m}{d(\Phi)}) + I_\Phi(\frac{k_{m,n}x_n}{d(\Phi)}))$$

$$\geq \frac{1}{k_{m,n}}(1 + \frac{k_{m,n} - 1}{2} + \frac{k_{m,n} - 1}{2})$$

$$= 1,$$

whence we get again $\|x_m - x_n\| \geq d(\Phi)$. Consequently $A((x_n)) \geq d(\Phi)$. By the arbitrariness of $(x_n)$ being an asymptotic equidistant sequence in $S(I^p)$ it follows that $WCS(I^p) \geq d(\Phi)$.

The following example shows how to compute $d(\Phi)$ in some concrete cases.

Example 1. Let $\Phi_i(u) = |u|^p$ for all $i \in \mathbb{N}$ and $u \in \mathbb{R}$, where $1 < p < \infty$. If $\Phi = (\Phi_i)_{i=1}^{\infty}$, then $d(\Phi) = 2^{1/p}$.

Proof. It is obvious that $I^p = L^p$. Moreover, $\|x\| = p^{1/p}q^{1/q}\|x\|_p$ for any $x \in I^p$, where $1/p + 1/q = 1$ and $\|x\|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}$ (see [9]). Take arbitrary $k > 1$ and $x \in S(I^p)$ with finite support. It is easy to see that the number $c = c(k, x) > 0$
satisfying the equality $I_k\left(\frac{\|x\|}{c}\right) = \frac{k-1}{2}$ is equal to $2^{1/p}k(k-1)^{-1/p}p^{-1/p}q^{-1/q}$. Therefore, $d(\Phi) = \inf\{2^{1/p}k(k-1)^{-1/p}p^{-1/p}q^{-1/q} : k > 1\}$. To calculate this infimum it is enough to find $\inf\{k(k-1)^{-1/p}p^{-1/p}q^{-1/q} : k > 1\}$. Using the standard method, we get that this infimum is attained at $k_0 = q$. Since $k_0 - 1 = q/p$, we get $d(\Phi) = 2^{1/p}$.

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