THE CLASSIFICATION OF COMPLETE LIE ALGEBRAS
WITH COMMUTATIVE NILPOTENT RADICAL

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Abstract. The work in this paper is a continuation of an earlier paper of the second author (Acta Math. 34 (1991), 191–202). We discuss the properties of finite-dimensional complete Lie algebras with abelian nilpotent radical over the complex field $\mathbb{C}$. We solve the problems of isomorphism, classification and realization of complete Lie algebras with commutative nilpotent radical.

1. Introduction

A Lie algebra $\mathcal{L}$ is called a complete Lie algebra if its centre $C(\mathcal{L})$ is zero and its derivations are all inner. The definition of complete Lie algebra was given by N. Jacobson in 1962 (cf. [8]). But the first important result—the derivation tower theorem—was obtained by E. V. Schenkman in 1951 (cf. [9]). In recent years, the theory of complete Lie algebras has been developing (see [1]–[7]). In [1], the properties of complete Lie algebras with commutative nilpotent radical have been discussed. The work in this paper is a continuation of [1].

Let $\mathcal{L}$ be a finite-dimensional Lie algebra over a field of characteristic zero. Then $\mathcal{L}$ has the Levi decomposition:

$$\mathcal{L} = s + r,$$

where $s$ is a maximal semisimple subalgebra of $\mathcal{L}$ and is called the Levi subalgebra of $\mathcal{L}$, and $r$ is the maximal solvable ideal of $\mathcal{L}$ and is called the radical of $\mathcal{L}$. The ideal

$$n_0 = [\mathcal{L}, \mathcal{L}] \cap r = [\mathcal{L}, r]$$

is called the nilpotent radical of $\mathcal{L}$.

Since $[s, r] \subseteq r$, $r$ can be viewed as an $s$-module. The fact that $s$ is semisimple implies that $r$ can be decomposed into a direct sum of irreducible submodules. Let $r_0$ be the direct sum of trivial submodules, and $r_n$ the direct sum of non-trivial irreducible submodules. Denote by $C(r_0)$ the centre of $r_0$ and let $C(r_n) = \{x \in r_n \mid [x, r_n] = 0\}$. It has been proved in [1] that $\mathcal{L}$ can be decomposed into the direct sum of complete ideals as follows:

$$\mathcal{L} = (s + C(r_0) + r_n) \oplus C(r_n)$$

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and $C_{r_0}(\mathfrak{r}_n)$ is an extension of abelian Lie algebra by abelian Lie algebra, and if the base field of $L$ is algebraically closed, then $C_{r_0}(\mathfrak{r}_n)$ is a direct sum of 2-dimensional simple complete ideals. A complete Lie algebra is called a simple complete Lie algebra if it has no non-trivial complete ideals. By (1.3), if we study complete Lie algebras with commutative nilpotent radical, it is sufficient to discuss $s + C(\mathfrak{r}_0) + \mathfrak{r}_n$.

In this case, $n_0 = r_n$.

In [1], a complete Lie algebra $G = \mathfrak{g} + V + \mathfrak{a}$ over the complex field $\mathbb{C}$ was constructed in the following way. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $(\rho, V)$ a representation of $\mathfrak{g}$ which is decomposed into $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$, where $V_i (i = 1, 2, \ldots, n)$ are irreducible invariant subspaces of $\rho$. Let $I_i (i = 1, 2, \ldots, n)$ be the linear transformations of $V$ such that

$$I_i \left( \sum_{j \neq i} V_j \right) = 0, \quad I_i|_{V_i} = \text{id}|_{V_i}.$$

Let $\mathfrak{a}$ be the subalgebra of $\text{gl}(V)$ generated by $I_1, I_2, \ldots, I_n$. Then $\mathfrak{a}$ is an abelian Lie algebra. Set

$$G = \mathfrak{g} + V + \mathfrak{a}.$$ 

The bracket in $G$ is defined by

$$[x_1 + v_1 + A_1, x_2 + v_2 + A_2] = [x_1, x_2] + \rho(x_1)v_2 - \rho(x_2)v_1 - A_2(v_1) + A_1(v_2),$$

where $x_1, x_2 \in \mathfrak{g}$, $v_1, v_2 \in V$, $A_1, A_2 \in \mathfrak{a}$. Then $G$ is a complete Lie algebra with commutative nilpotent radical.

In this paper, we discuss the properties of finite-dimensional complete Lie algebras with commutative nilpotent radical $n_0 = \mathfrak{r}_n$ over the complex field $\mathbb{C}$. We deduce that if $\mathfrak{r}_n$ is the direct sum of $t$ irreducible submodules, then the dimension of $\mathfrak{r}_0$ is $t$. We prove that $\mathfrak{r}_n$ can be decomposed properly so that the action of every element of $\text{ad}_{\mathfrak{r}_0} \mathfrak{r}_0$ on each irreducible submodule is scalar. Therefore, finite-dimensional complete Lie algebras with commutative nilpotent radical $n_0 = \mathfrak{r}_n$ over $\mathbb{C}$ are in fact the Lie algebras constructed above. Hence, all finite-dimensional complete Lie algebras with commutative nilpotent radical over $\mathbb{C}$ are known.

2. Some lemmas

Let

$$\mathcal{L} = s + \mathfrak{r} = s + (\mathfrak{t}_0 + \mathfrak{r}_n)$$

be the Levi decomposition of $\mathcal{L}$. Then we have the following results.

Lemma 2.1 ([1]).

(2.2) $[s, \mathfrak{t}_0] = (0),$

(2.3) $[s, \mathfrak{r}_n] = \mathfrak{r}_n,$

(2.4) $[\mathfrak{t}_0, \mathfrak{t}_0] \subseteq \mathfrak{r}_0.$

Lemma 2.2.

(2.5) $[\mathfrak{r}_0, \mathfrak{r}_n] \subseteq \mathfrak{r}_n.$
Proof. By (2.3) and (2.2), we have

\[
[t_0, t_n] = [t_0, [s, t_n]] \subseteq [[t_0, s], t_n] + [[t_n, t_0], s] = [s, t_0 + t_n] = t_n.
\]

The lemma holds. \(\square\)

Lemma 2.3. Let \(L\) be a Lie algebra with abelian nilpotent radical \(n_0 = r_n\). Then

\[
[t_0, t_0] = [t_n, t_n] = (0).
\]

Proof. Since \(n_0\) is commutative and \(n_0 = r_n\), we have

\[
n_0^{(1)} = r_n^{(1)} = [t_n, t_n] = (0)
\]

and

\[
n_0 = [s + t_0 + t_n, t_0 + t_n] = r_n + [t_0, t_0].
\]

The lemma follows from (2.4) and Lemma 2.2. \(\square\)

Let \(a\) be an irreducible \(s\)-module. Then \(a\) is a highest weight \(s\)-module since \(s\) is semisimple and \(a\) is finite dimensional.

Let

\[
r_n = a_1 \oplus a_2 \oplus \cdots \oplus a_m
\]

be the direct sum of submodules such that

\[
a_i = a_{i1} \oplus a_{i2} \oplus \cdots \oplus a_{in_i}, \quad (i = 1, 2, \ldots, m),
\]

where \(a_{ik} (k = 1, 2, \ldots, n_i)\) are irreducible highest weight \(s\)-modules with highest weight \(\lambda_i\) and, if \(i \neq j\), then \(\lambda_i \neq \lambda_j\), \(i, j = 1, 2, \ldots, m\).

Denote by \(z_{ij} (j = 1, 2, \ldots, n_i)\) the highest weight vectors of \(a_{ij}\) \((j = 1, 2, \ldots, n_i)\) respectively, and by \(V_i\) the linear space with basis \(\{z_{i1}, \ldots, z_{in_i}\}\) \((i = 1, 2, \ldots, m)\). Then

\[
\dim V_i = n_i, \quad i = 1, 2, \ldots, m.
\]

Let \(b_0\) be a Cartan subalgebra of \(s\) and \(\Delta_0\) the root system. Let \(\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}\) be the simple root system and \(s = b_0 + \sum_{\alpha \in \Delta_0} s_\alpha\) the root space decomposition with respect to \(b_0\).

Lemma 2.4. Let \(L\) be a Lie algebra with commutative nilpotent radical, \(D \in \text{Der}(s + r_n)\) be such that \(D(s) = (0)\) and \(D(r_n) \subseteq r_n\). Then

\[
D(V_i) \subseteq V_i, \quad i = 1, 2, \ldots, m,
\]

and \(D\) is uniquely determined by \(D|_{V_i} (i = 1, 2, \ldots, m)\).

Proof. Since \(D \in \text{Der}(s + r_n)\) and \(D(s) = 0\), for any \(h \in b_0\) we have

\[
D[h, z_{ij}] = \lambda_i(h)Dz_{ij}.
\]

Note that \(z_{ij}\) is a highest weight vector of \(a_{ij}\), therefore for \(\alpha \in \Delta_0^+\) and \(e_\alpha \in s_\alpha\), we have

\[
[e_\alpha, z_{ij}] = 0.
\]

Therefore

\[
[e_\alpha, Dz_{ij}] = D[e_\alpha, z_{ij}] = 0.
\]
It is clear from (2.11) and (2.12) that $Dz_{i1}, Dz_{i2}, \ldots, Dz_{in_i}$ ($i = 1, 2, \ldots, m$) are highest weight vectors associated to highest weight $\lambda_i$ ($i = 1, 2, \ldots, m$). So $Dz_{ij} \in V_i$ ($j = 1, 2, \ldots, n_i$, $i = 1, 2, \ldots, m$). On the other hand, for $z \in a_{ij}$, $z$ has the form:

$$z = [x_1, [x_2, \ldots, [x_q, z_{ij}] \ldots]],$$

where $x_i \in s$ ($i = 1, 2, \ldots, q$). So

$$Dz = [x_1, [x_2, \ldots, [x_q, Dz_{ij}] \ldots]].$$

The lemma is proved.

\[\square\]

**Lemma 2.5.** Let $L$ be a Lie algebra with abelian nilpotent radical $n_0 = r_n$. $D \in \text{Der}(s + r_n)$ is such that $D(s) = 0$ and $D(r_n) \subseteq r_n$. Then

1) follows from Lemma 2.4. Define

$$D(a_i) \subseteq a_i, \quad i = 1, 2, \ldots, m.$$

2) Set

$$L_i = \{D \in \text{Der}(s + r_n) | D(s) = 0, \quad D(r_n) \subseteq r_n \text{ and } D|_{a_j} = 0, \quad \text{if } j \neq i\}.$$  

Then the Lie algebra $L_i$ is isomorphic to the general linear Lie algebra $\text{gl}(V_i)$ which consists of all linear transformations of $V_i$, $i = 1, 2, \ldots, m$.

**Proof.** 1) follows from Lemma 2.4. Define

$$\varphi(D) = D|_{V_i}, \quad \text{for } D \in L_i.$$  

Then $\varphi$ is a linear mapping from $L_i$ to $\text{gl}(V_i)$. For $D_1, D_2 \in L_i$, if $D_1 \neq D_2$, then from Lemma 2.4 we know $\varphi(D_1) \neq \varphi(D_2)$. Let $A \in \text{gl}(V_i)$. Define the linear transformation $D$ of $s + r_n$ by

$$D(s) = 0, \quad D|_{a_j} = 0, \quad \text{if } j \neq i.$$  

$$D[x_1, [x_2, \ldots, [x_q, z_{ij}] \ldots] = [x_1, [x_2, \ldots, [x_q, A z_{ij}] \ldots] \quad (j = 1, 2, \ldots, n_i),$$

where $x_1, x_2, \ldots, x_q \in s$. Then $D \in \text{Der}(s + r_n)$. So $\varphi$ is a bijection.

For $D_1, D_2 \in L_i$, we have

$$\varphi[D_1, D_2] = [D_1, D_2]|_{V_i}$$

$$= D_1 D_2|_{V_i} - D_2 D_1|_{V_i} = D_1|_{V_i} D_2|_{V_i} - D_2|_{V_i} D_1|_{V_i}$$

$$= [\varphi(D_1), \varphi(D_2)].$$

Hence $\varphi$ is a homomorphism from the Lie algebra $L_i$ to the Lie algebra $\text{gl}(V_i)$. The lemma holds.

\[\square\]

3. The structure of radical $\tau$

**Lemma 3.1.** Let $D$ be an inner derivation of $L$ and $D(s) = 0$. Then there exists an element $y \in \tau_0$ such that

$$D = \text{ad } y.$$

**Proof.** Since $D$ is an inner derivation of $L$, there exist $x \in s$, $y \in \tau_0$, $z \in \tau_n$ such that

$$D = \text{ad}(x + y + z).$$

$D(s) = 0$ implies that

$$[x + y + z, s] = [x, s] + [z, s] = 0.$$
From the fact that \([x, s] \subseteq s, [z, s] \subseteq r_n\), we have
\[ [x, s] = (0), \quad [z, s] = (0). \]
But \(s\) is semisimple and \(r_n\) is the direct sum of non-trivial submodules. Therefore
\[ x = z = 0. \]

\[ \square \]

**Lemma 3.2.** Let \(\mathcal{L}\) be a Lie algebra with trivial centre and commutative nilpotent radical \(n_0 = r_n\). Then
1) \(r_0\) is isomorphic to \(\text{ad}_{r_n} r_0\).
2) For \(x \in r_0\), we have
\[ \text{ad}_{r_n} x(a_i) \subseteq a_i, \quad \text{ad}_{r_n} x|_{V_i} \in \text{gl}(V_i) \quad (i = 1, 2, \ldots, m). \]

**Lemma 3.3.** Let \(\mathcal{L}\) be a complete Lie algebra with commutative nilpotent radical \(n_0 = r_n\). For \(x \in r_0\), define the linear transformations \(D_i (i = 1, 2, \ldots, m)\) of \(\mathcal{L}\) by
\[ D_i|_{s + r_0} = 0, \quad D_i|_{s_i} = \text{ad} x|_{s_i}, \quad D_i|_{a_i} = 0 \quad (j = 1, \ldots, i - 1, i + 1, \ldots, m). \]
Then there exist \(y_1, y_2, \ldots, y_m \in r_0\) such that
\[ D_i = \text{ad} y_i \quad (i = 1, 2, \ldots, m). \]

**Proof.** For \(s_1, s_2 \in s, x_1, x_2 \in r_0, y_1, y_2 \in a_i, z_1, z_2 \in a_1 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_m\), by 2) of Lemma 3.2, we have \([x_1, y_2], [y_2, y_1] \in a_i, [x_1, z_2], [x_2, z_1] \in a_1 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_m\). So by (2.2) and (2.6) we deduce that
\[ D_i(s_1 + x_1 + y_1 + z_1, s_2 + x_2 + y_2 + z_2) \]
\[ = \text{ad} x([s_1, y_2] + [x_1, y_2] + [y_1, s_2] + [y_1, x_2]) \]
\[ = [s_1, \text{ad} x(y_2)] + [x_1, \text{ad} x(y_2)] + [\text{ad} x(y_1), s_2] + [\text{ad} x(y_1), x_2], \]
\[ [D_i(s_1 + x_1 + y_1 + z_1), s_2 + x_2 + y_2 + z_2] \]
\[ + [s_1 + x_1 + y_1 + z_1, D_i(s_2 + x_2 + y_2 + z_2)] \]
\[ = [\text{ad} x(y_1), s_2] + [\text{ad} x(y_1), x_2] + [s_1 + x_1 + y_1 + z_1, \text{ad} x(y_2)] \]
\[ = [\text{ad} x(y_1), x_2] + [\text{ad} x(y_1), y_2] + [s_1, \text{ad} x(y_2)]. \]
Therefore \(D_i \in \text{Der} \mathcal{L} (i = 1, 2, \ldots, m)\). Since \(\mathcal{L}\) is a complete Lie algebra, there exist \(y_1, y_2, \ldots, y_m \in \mathcal{L}\) such that
\[ D_i = \text{ad} y_i \quad (i = 1, 2, \ldots, m). \]
By Lemma 3.1, \(y_i \in r_0 \quad (i = 1, 2, \ldots, m)\). \[ \square \]

**Theorem 3.1.** Let \(\mathcal{L}\) be a complete Lie algebra with commutative nilpotent radical \(n_0 = r_n\). Set
\[ (3.1) \quad h_i = \{ \text{ad} x|x \in r_0 \text{ and } \text{ad} x|_{a_i} = 0 \ (j = 1, \ldots, i - 1, i + 1, \ldots, m)\}, \]
\[ i = 1, 2, \ldots, m, \]
\[ (3.2) \quad H_i = h_i|_{V_i} = \varphi(h_i|_{s + r_n}), \quad i = 1, 2, \ldots, m. \]

Then
1) \(h_i|_{s + r_n}\) is a commutative subalgebra of \(\mathcal{L}_i\) and
\[ (3.3) \quad \text{ad } t_0 = h_1 \oplus h_2 \oplus \cdots \oplus h_m \simeq h_1|_{s + r_n} \oplus \cdots \oplus h_m|_{s + r_n}. \]
2) \(h_i\) is isomorphic to \(H_i\) which is an abelian subalgebra of \(\text{gl}(V_i), i = 1, 2, \ldots, m. \)

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Proof. The proof follows from Lemma 3.3 and Lemma 2.5.

**Lemma 3.4.** Let $\mathcal{L}$ be a Lie algebra with commutative nilpotent radical $\mathfrak{n}_0 = \mathfrak{r}_n$. Let $D \in \text{Der}(\mathfrak{s} + \mathfrak{r}_n)$ be such that $D(\mathfrak{s}) = 0$ and $D(\mathfrak{r}_n) \subseteq \mathfrak{r}_n$. Extend $D$ to the linear transformation of $\mathcal{L}$ by

$$D|_{\mathfrak{n}_0} = 0.$$ 

Then $D$ is a derivation of $\mathcal{L}$ if and only if

$$[D, \text{ad}_{\mathfrak{s} + \mathfrak{r}_n} \mathfrak{n}_0] = 0.$$ 

**Proof.** $D \in \text{Der} \mathcal{L}$ if and only if the formula

$$D[x_1 + y_1 + z_1, x_2 + y_2 + z_2] = [D(x_1 + y_1 + z_1), x_2 + y_2 + z_2] + [x_1 + y_1 + z_1, D(x_2 + y_2 + z_2)]$$

holds, for any $x_1, x_2 \in \mathfrak{s}$, $y_1, y_2 \in \mathfrak{n}_0$, $z_1, z_2 \in \mathfrak{r}_n$.

Note that

$$D[x_1 + y_1 + z_1, x_2 + y_2 + z_2] = D[z_1, x_1] + D[y_1, x_2] + D[x_1, z_2] + D[y_1, z_2] + D[z_1, z_2],$$

and $D \in \text{Der}(\mathfrak{s} + \mathfrak{r}_n)$ is such that $D(\mathfrak{s}) = 0$. Therefore $D \in \text{Der} \mathcal{L}$ if and only if

$$D[y, z] = [y, Dz], \quad \text{for any } y \in \mathfrak{n}_0, z \in \mathfrak{r}_n,$$

i.e.,

$$[D, \text{ad}_{\mathfrak{s} + \mathfrak{r}_n} \mathfrak{n}_0] = 0.$$ 

**Theorem 3.2.** Let $\mathcal{L}$ be a complete Lie algebra with abelian nilpotent radical $\mathfrak{n}_0 = \mathfrak{r}_n$, and let $\mathfrak{h}_i, H_i, \mathcal{L}_i, (i = 1, 2, \ldots, m)$ be as above. Then

$$(3.4) \quad C_{\mathcal{L}_i}(\mathfrak{h}_i|_{\mathfrak{s} + \mathfrak{r}_n}) = \mathfrak{h}_i|_{\mathfrak{s} + \mathfrak{r}_n}, \quad i = 1, 2, \ldots, m,$$

$$(3.5) \quad C_{\text{gl}(V_i)}(H_i) = H_i, \quad i = 1, 2, \ldots, m.$$ 

**Proof.** Since $\mathfrak{n}_0$ is commutative, we have

$$\mathfrak{h}_i|_{\mathfrak{s} + \mathfrak{r}_n} \subseteq C_{\mathcal{L}_i}(\mathfrak{h}_i|_{\mathfrak{s} + \mathfrak{r}_n}).$$

Let $D_i \in \mathcal{L}_i$ be such that $[D_i, \mathfrak{h}_i|_{\mathfrak{s} + \mathfrak{r}_n}] = 0$. Then by (3.3) we have

$$[D_i, \text{ad}_{\mathfrak{s} + \mathfrak{r}_n} \mathfrak{n}_0] = 0.$$ 

Extend $D_i$ to the linear transformation of $\mathcal{L}$ by

$$D_i|_{\mathfrak{n}_0} = 0.$$ 

Then by Lemma 3.4, $D_i$ is a derivation of $\mathcal{L}$. Note that $\mathcal{L}$ is a complete Lie algebra, therefore

$$D_i = \text{ad} \ z_i \in \mathfrak{h}_i, \quad z_i \in \mathfrak{n}_0 \quad (i = 1, 2, \ldots, m).$$

Hence $D_i|_{\mathfrak{s} + \mathfrak{r}_n} \in \mathfrak{h}_i|_{\mathfrak{s} + \mathfrak{r}_n} \quad (i = 1, 2, \ldots, m)$. This proves (3.4). (3.5) follows from the fact that $\mathcal{L}_i$ is isomorphic to $\text{gl}(V_i)$ and $\mathfrak{h}_i|_{\mathfrak{s} + \mathfrak{r}_n}$ is isomorphic to $H_i$. 

\[\square\]
Corollary 3.1. There exist elements \( \text{ad} x_i \in \mathfrak{h}_i \) \((i = 1, 2, \ldots, m)\) such that
\[
\text{ad} x_i|_{\mathfrak{a}_i} = \text{id}|_{\mathfrak{a}_i} \quad (i = 1, 2, \ldots, m).
\]
Thus \( H_i \) contains identical transformations \( I_i = \text{id}|_{\mathfrak{v}_i} \), \( i = 1, 2, \ldots, m \).

From Corollary 3.1, we have
\[
H_i = H'_i \oplus CI_i
\]
and \( H'_i \subseteq \text{sl}(V_i) \) \((i = 1, 2, \ldots, m)\), where \( \text{sl}(V_i) \) is the special linear Lie algebra on \( V_i \). Since \( \mathfrak{h}_i \) is isomorphic to \( H_i \), we have
\[
\mathfrak{h}_i = \mathfrak{h}'_i \oplus \text{C ad} x_i \quad (i = 1, 2, \ldots, m),
\]
where \( \mathfrak{h}'_i \) is isomorphic to \( H'_i \) and \( \text{ad} x_i \) is the same as in Corollary 3.1. In fact, \( \mathfrak{h}'_i|_{V_i} = H'_i \), \( \text{ad} x_i|_{V_i} = I_i \) \((i = 1, 2, \ldots, m)\).

We will show that \( H'_i \) is a maximal torus subalgebra of \( \text{sl}(V_i) \), \( i = 1, \ldots, m \).

Lemma 3.5. Let \( \mathcal{L} \) be a Lie algebra with trivial centre and nilpotent radical \( \mathfrak{n}_0 = \mathfrak{r}_n \). Let \( D_1 \in \text{Der}(\mathfrak{s} + \mathfrak{r}_n) \) be such that \( D_1(\mathfrak{s}) = (0) \) and \( D_1(\mathfrak{r}_n) \subseteq \mathfrak{r}_n \), and
\[
[D_1, \text{ad}_{\mathfrak{s} + \mathfrak{r}_n} \mathfrak{t}_0] \subseteq \text{ad}_{\mathfrak{s} + \mathfrak{r}_n} \mathfrak{t}_0.
\]
Then there exists \( D \in \text{Der} \mathcal{L} \) such that
\[
D|_{\mathfrak{s} + \mathfrak{r}_n} = D_1,
\]
and if \( [D_1, \text{ad}_{\mathfrak{s} + \mathfrak{r}_n} \mathfrak{t}_0] \neq 0 \), then \( D \) is an outer derivation of \( \mathcal{L} \).

Proof. From (3.8), for any \( x \in \mathfrak{r}_0 \) there exists \( y \in \mathfrak{r}_0 \) such that
\[
[D_1, \text{ad}_{\mathfrak{s} + \mathfrak{r}_n} x] = \text{ad}_{\mathfrak{s} + \mathfrak{r}_n} y.
\]
Define the transformation \( D_2 \) of \( \mathfrak{r}_0 \) by
\[
D_2(x) = y.
\]
By Lemma 3.2 there is no ambiguity in the definition of \( D_2 \). It is clear that \( D_2 \)
is a linear transformation of \( \mathfrak{r}_0 \). \( D_2 \) is a derivation of \( \mathfrak{r}_0 \) since \( \mathfrak{r}_0 \) is commutative. From (3.8), for \( x \in \mathfrak{r}_0 \) and \( z \in \mathfrak{r}_n \) we have
\[
D_1[x, z] = [x, D_1 z] + [D_2 x, z].
\]
Define the linear transformation \( D \) of \( \mathcal{L} \) by
\[
D|_{\mathfrak{s} + \mathfrak{r}_n} = D_1, \quad D|_{\mathfrak{r}_0} = D_2.
\]
Then for any \( x_1, x_2 \in \mathfrak{s}, y_1, y_2 \in \mathfrak{r}_0 \), \( z_1, z_2 \in \mathfrak{r}_n \), we have
\[
D[x_1 + y_1 + z_1, x_2 + y_2 + z_2] = [D_1 z_1, x_2] + D_1[z_1, y_2] + [x_1, D_1 z_2] + D_1[y_1, z_2] + D_1[z_1, z_2],
\]
\[
[D(x_1 + y_1 + z_1), x_2 + y_2 + z_2] + [x_1 + y_1 + z_1, D(x_2 + y_2 + z_2)] = [D_1 z_1, x_2] + [D_1 z_1, y_2] + [x_1, D_1 z_2] + [y_1, D_1 z_2] + [D_1 z_1, z_2] + [z_1, D_1 z_2] + [D_2 y_1, z_2] + [z_1, D_2 y_2].
\]
Since \( D_1 \in \text{Der}(\mathfrak{s} + \mathfrak{r}_n) \) is such that \( D_1(\mathfrak{s}) = (0) \) and (3.9) holds, we have
\[
D[x_1 + y_1 + z_1, x_2 + y_2 + z_2] = [D(x_1 + y_1 + z_1), x_2 + y_2 + z_2] + [x_1 + y_1 + z_1, D(x_2 + y_2 + z_2)].
\]
Hence \( D \in \text{Der} \mathcal{L} \). If \( D \in \text{ad} \mathcal{L} \), then by Lemma 3.1, we have
\[
D_1 \in \text{ad}_{\mathfrak{s} + \mathfrak{r}_n} \mathfrak{t}_0.
\]
But $\mathfrak{t}_0$ is commutative. Therefore,
\[ [D_1, \text{ad}_{s + \mathfrak{t}_n} \mathfrak{t}_0] = 0. \]
Thus we have completed the proof. 

**Theorem 3.3.** Let $\mathcal{L}$ be a complete Lie algebra with commutative nilpotent radical $\mathfrak{n}_0 = \mathfrak{t}_n$. Then $H'_i$ is a maximal torus subalgebra of $\mathfrak{sl}(V_i), i = 1, 2, \ldots, m$.

**Proof.** Let $T_i \in \mathfrak{sl}(V_i)$ be such that $[T_i, H_i] \subseteq H_i$. Then $D_i = \varphi^{-1}(T_i) \in \mathcal{L}_i$, and $D_i$ satisfies the conditions of Lemma 3.5. Thus there exists $D'_i \in \text{Der} \mathcal{L}$ such that
\[ D'_i|_{s + \mathfrak{t}_n} = D_i, \quad i = 1, 2, \ldots, m. \]
Note that $\mathcal{L}$ is a complete Lie algebra, so by Lemma 3.1, we have
\[ D'_i \in \text{ad}_\mathcal{L} \mathfrak{t}_0, \quad i = 1, 2, \ldots, m. \]
The fact that $D_i|_{\mathfrak{a}_j} = 0$ (when $j \neq i$) implies $D_i \in \mathfrak{h}_i (i = 1, 2, \ldots, m)$. Thus we have
\[ T_i = \varphi(D'_i|_{s + \mathfrak{t}_n}) = \varphi(D_i) \in H'_i, \quad i = 1, 2, \ldots, m. \]
From this we know that $H'_i$ is a self-normal subalgebra of $\mathfrak{sl}(V_i)$. Since $H'_i$ is commutative, we deduce that $H'_i$ is a Cartan subalgebra of $\mathfrak{sl}(V_i)$. But $\mathfrak{sl}(V_i)$ is a simple Lie algebra, therefore $H'_i$ is a maximal torus subalgebra of $\mathfrak{sl}(V_i)$. 

**Corollary 3.2.** Let $\mathcal{L}$ be a complete Lie algebra with abelian nilpotent radical $\mathfrak{n}_0 = \mathfrak{t}_n$. Then $H_i = H'_i + \mathfrak{c}_i$ is a maximal torus subalgebra of $\mathfrak{gl}(V_i), i = 1, 2, \ldots, m$.

**Corollary 3.3.** Let $\mathcal{L}$ be a complete Lie algebra with abelian nilpotent radical $\mathfrak{n}_0 = \mathfrak{t}_n$. Then
\begin{equation}
\dim \mathfrak{h}_i = n_i, \quad i = 1, 2, \ldots, m.
\end{equation}
Thus we have
\begin{equation}
\dim \mathfrak{t}_0 = n_1 + n_2 + \cdots + n_m.
\end{equation}
Since $H_i$ is a maximal torus subalgebra of $\mathfrak{gl}(V_i)$, we can choose a basis $\{y_{i1}, y_{i2}, \ldots, y_{im_i}\}$ of $V_i$ such that the matrix of every element of $H_i$ relative to the basis is diagonal, $i = 1, 2, \ldots, m$. On the other hand, $y_{i1}, y_{i2}, \ldots, y_{im_i}$ are highest weight vectors associated to highest weight $\lambda_i$. So the highest weight $\mathfrak{s}$-module $\mathfrak{a}'_{ij}$ whose highest weight vector is $y_{ij}$ is irreducible and
\[ \mathfrak{a}_i = \mathfrak{a}'_{i1} \oplus \mathfrak{a}'_{i2} \oplus \cdots \oplus \mathfrak{a}'_{im_i}, \quad i = 1, 2, \ldots, m. \]
From this we deduce that

**Theorem 3.4.** Let $\mathcal{L}$ be a complete Lie algebra with abelian nilpotent radical $\mathfrak{n}_0 = \mathfrak{t}_n$. Let $\mathfrak{t}_n$ be the direct sum of $t$ irreducible submodules. Then
\[ \dim \mathfrak{t}_0 = t \]
and $\mathfrak{t}_n$ can be decomposed properly into the direct sum of irreducible submodules:
\[ \mathfrak{t}_n = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \cdots \oplus \mathfrak{m}_t \]
so that
\[ \mathfrak{t}_0 \simeq \text{ad}_{\mathfrak{t}_n} \mathfrak{t}_0 = CI_1 \oplus CI_2 \oplus \cdots \oplus CI_t, \]
where $I_i$ is the linear transformation of $\tau_n$ such that
\[ I_i \left( \sum_{j \neq i} m_j \right) = (0), \quad I_i |m_1 = \text{id} |m_1, \quad (i = 1, 2, \ldots, t). \]

So $L$ is in fact the complete Lie algebra constructed in section 1.

**Theorem 3.5.** Let $L_1$ and $L_2$ be complete Lie algebras. Let $n_i$ be the nilpotent radical of $L_i$ and $s_i$ be the Levi subalgebra of $L_i$ ($i = 1, 2$). Then $L_1$ is isomorphic to $L_2$ if and only if the Lie algebra $s_1$ is isomorphic to $s_2$ and the $s_1$-module $n_1$ is isomorphic to the $s_2$-module $n_2$.

**Theorem 3.6.** Let $\mathfrak{s}$ be a semisimple Lie algebra and $\mathfrak{n}$ an $\mathfrak{s}$-module. Define
\[ [s_1 + x_1, s_2 + x_2] = [s_1, s_2] + s_1(x_2) - s_2(x_1), \tag{3.12} \]
where $s_1, s_2 \in \mathfrak{s}$, $x_1, x_2 \in \mathfrak{n}$. Then there is a unique up to isomorphism complete Lie algebra $L$ such that $\mathfrak{n}$ is its nilpotent radical and $\mathfrak{s}$ is its Levi subalgebra and its bracket satisfies (3.12).

**References**


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