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INFINITELY RENORMALIZABLE DIFFEOMORPHISMS OF THE DISK AT THE BOUNDARY OF CHAOS

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Abstract. We show that, among area contracting embeddings of the 2-disk, infinitely renormalizable maps with a bounded geometry either have positive topological entropy or correspond to a cascade of period doubling.

1. Introduction

Let f be a C^1 map from the unit m-dimensional disk D^m into itself and $(a_n)_{n\geq 0}$ a sequence of integers greater than or equal to 2. We say that the map f is $(a_n)_{n\geq 0}$-*infinitely renormalizable* if there exists a sequence of nested m-dimensional disks $D^m \supset \mathcal{D}_0(f) \supset \mathcal{D}_1(f) \supset \cdots \supset \mathcal{D}_n(f) \cdots$ such that, for each n:

$$f^i(\mathcal{D}_n(f)) \cap \mathcal{D}_n(f) = \varnothing, \quad \text{for } 1 \leq i \leq a_0 \cdot a_1 \cdots a_n - 1,$$

and

$$f^{a_0 \cdot a_1 \cdots a_n}(\mathcal{D}_n(f)) \subset \mathcal{D}_n(f).$$

The sets $f^i(\mathcal{D}_n)$, for $0 \leq i \leq a_0 \cdot a_1 \cdots a_n - 1$, are called *the atoms of generation n of f*. Maps which satisfy this property but only for a finite sequence $(a_n)_{m-1\geq n\geq 0}$ are called $(a_n)_{m-1\geq n\geq 0}$-*renormalizable* or m-times renormalizable when there is no ambiguity. We say that an infinite renormalizable map is *of bounded combinatorial type* if the sequence $(a_n)_{n\geq 0}$ is bounded. Notice that, in this case, the sequence $(a_n)_{n\geq 0}$ has an accumulation point, i.e. there is an integer that appears infinitely many times in the sequence. This type of map occurs very naturally in one-dimensional dynamics: actually for any sequence $(a_n)_{n\geq 0}$ there exists a value of the parameter a for which the quadratic map $x \mapsto 1-ax^2$ is an $(a_n)_{n\geq 0}$-infinitely renormalizable map. Since any continuous map on the interval which possesses a periodic orbit whose period is not a power of 2 has positive topological entropy [BF], it follows easily that the only infinitely renormalizable maps with topological entropy zero are the ones for which each element of the sequence $(a_n)_{n\geq 0}$ is a power of 2. The aim of this paper is to prove a similar result for area contracting maps of the 2-disk. But, before that, let us emphasize some recent results about infinitely renormalizable maps on the interval.

To an $(a_n)_{n\geq 0}$-infinitely renormalizable map f we can associate another map that we call *the renormalized map* of f, denoted by $\mathcal{R}(f)$, and defined by

$$\mathcal{R}(f) = \xi^{-1}(f) \circ f^{a_0} \circ \xi(f),$$

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where $\xi(f)$ is an affine scaling which maps D^1 onto $\mathcal{D}_0(f)$. The renormalized map $\mathcal{R}(f)$ is again a $(b_n)_{n\geq 0}$-infinitely renormalizable map with $b_n = a_{n+1}$ for all n, and its corresponding sequence of nested intervals is given by

$$\mathcal{D}_n(\mathcal{R}(f)) = \xi^{-1}(f)(\mathcal{D}_{n+1}(f)) \quad \text{for all } n \geq 0.$$

From the independent discovery and explanation by Coullet and Tresser ([CT], [TC]) and Feigenbaum [Fe] in 1978, that infinitely renormalizable maps exhibit universal geometrical behaviors, to the culminating work of Sullivan [Su] in 1992, a huge amount of work, both numerical and theoretical, has been done in this field. It is not our intention to give here a panorama of the actual knowledge. For this purpose, we refer the reader to [MS] and the references quoted therein. We just want to focus on a key point of Sullivan's theory, often referred to as "real bounds", that we present here for sake of simplicity, in a very weakened form:

Consider a smooth map, f, with a quadratic singularity: more precisely, we consider the set $\mathcal{U}^{1+1}$ of the maps $f\colon [0,1] \to [0,1]$ that can be written as

$$f = \phi \circ \mathcal{Q} \circ \psi,$$

where $\psi\colon\ [0,1] \to [\psi(0),1]$ is an orientation preserving diffeomorphism such that $\psi(0)$ is in $(-1,0)$, $\mathcal{Q}\colon [\psi(0),1] \to [0,1]$ is the quadratic map $\mathcal{Q}(x) = x^2$, $\phi\colon\ [0,1] \to [0,1]$ is an orientation reversing diffeomorphism, and the two maps ϕ and ψ are C^{1+1}, that is to say, C^1 and the derivatives satisfy a Lipschitz condition. Sullivan proves the following "beau" theorem:

Theorem 1.1 ([Su]). *Let $f \in \mathcal{U}^{1+1}$ be an infinitely renormalizable map with combinatorial type bounded by N. Then, for all $n \geq 0$:*

(1) *The renormalized maps $\mathcal{R}^n(f)$ belong to $\mathcal{U}^{1+1}$, their C^1-norm and the Lipschitz constants are bounded by a constant which depends only on f.*

(2) *There exist two constants a_f and b_f which depend only on f such that, if $\mathcal{I}$ is an atom of the generation m and $\mathcal{J} \subset \mathcal{I}$ is an atom of the generation $m+1$ of $\mathcal{R}^n(f)$, then $0 < a_f \leq |\mathcal{J}|/|\mathcal{I}| \leq b_f < 1$ (where $|\cdot|$ stands for the diameter).*

All these bounds are "beau" (bounded and eventually universally (bounded)), that is to say, that for n big enough, these bounds can be chosen so that they depend only on N.

In dimension 2, infinitely renormalizable maps are also frequently observed. For instance, they appear naturally in the infinitely dissipative situation for a map $(x,y) \mapsto (g(x),0)$, where g is an infinitely renormalizable map on the interval, and also in the area preserving case of a map exhibiting resonant islands.

To an $(a_n)_{n\geq 0}$-infinitely renormalizable map f of the 2-disk, we can again associate a renormalized map of f, defined by

$$\mathcal{R}(f) = \xi^{-1}(f) \circ f^{a_0} \circ \xi(f),$$

where $\xi(f)$ is a C^1 scaling which maps D^2 onto $\mathcal{D}_0(f)$.

The renormalized map $\mathcal{R}(f)$ is a $(b_n)_{n\geq 0}$-infinitely renormalizable map with $b_n = a_{n+1}$ for all n, and its corresponding sequence of nested disks is given by

$$\mathcal{D}_n(\mathcal{R}(f)) = \xi^{-1}(f)(\mathcal{D}_{n+1}(f)) \quad \text{for all } n \geq 0.$$

Definition. By analogy with the one-dimensional case, we say that a C^{1+1} infinitely renormalizable map of the 2-disk has a *bounded geometry* if it satisfies the following three conditions:

(1) For all $n \geq 0$, the renormalized maps $\mathcal{R}^n(f)$, the scaling maps $\xi(\mathcal{R}^n(f))$, and their inverse $\xi(\mathcal{R}^n(f))^{-1}$, are C^{1+1} and their C^1 norm and their Lipschitz constants are bounded by a constant K_f which only depends on f.

(2) There exist constants $0 < a_f < b_f < 1$ which depend only on f such that, for all $n \geq 0$, if $\mathcal{I}$ is an atom of the generation m and $\mathcal{J} \subset \mathcal{I}$ is an atom of the generation $m+1$ of $\mathcal{R}^n(f)$, then $a_f \leq |\mathcal{J}|/|\mathcal{I}| \leq b_f$ (where $|\cdot|$ stands for the diameter).

(3) There exists a constant $0 < c_f$ which depends only on f such that, for all $n \geq 0$, the distance from $\mathcal{R}^n(f)(D^2)$ to the boundary of the disk D^2 is bigger than c_f.

Remark. To have a bounded geometry is a very strong assumption. An infinitely renormalizable map $f \in \mathcal{U}^{1+1}$ with bounded combinatorial type, satisfies this assumption, and recently it has been proved that this is also the case for other one-dimensional maps with finitely many critical points (see [Hu]). However there is no result of this type for two-dimensional maps.

Unlike in dimension 1, for any sequence $(a_n)_{n\geq 0}$, we can find $(a_n)_{n\geq 0}$-infinitely renormalizable C^2 diffeomorphisms of the 2-disk with topological entropy zero; moreover if the sequence $(a_n)_{n\geq 0}$ is bounded, these maps can be constructed with a bounded geometry [GT2].

However, for an area contracting map of the 2-disk the situation seems to be much more rigid. On the one hand, the only known examples of area contracting infinitely renormalizable embeddings with topological entropy zero are such that the sequence $(a_n)_{n\geq 0}$ is a sequence of powers of 2 [GST]. Actually, these maps are the only known area contracting embeddings of the 2-disk with topological entropy zero, that can be transformed by an arbitrary small C^1-perturbation into maps with positive topological entropy. On the other hand, there is some numerical evidence that shows that for the Hénon model, maps which belong to the C^1-boundary of positive topological entropy, are geometrically bounded infinitely renormalizable maps of the 2-disk such that the sequence $(a_n)_{n\geq 0}$ is eventually a sequence of powers of 2 [GT1].

The central result of this paper may be seen as a step towards an explanation of this numerical evidence and can be stated as follows:

Theorem 1.2. *Let f be a $(a_n)_{n\geq 0}$-infinitely renormalizable map of the 2-disk with a bounded geometry, which uniformly contracts the area. Then:*

- *either, the topological entropy of f is positive,*
- *or, eventually the sequence $(a_n)_{n\geq 0}$ is a sequence of powers of 2.*

It would be nice to have some simple assumptions on the dynamics that imply that a map is infinitely renormalizable. In dimension one, in the case of multimodal maps, we are helped by the kneading theory. However, kneading invariants give us necessary but not sufficient conditions for the existence of infinitely renormalizable maps. The only positive result in this direction is the following theorem due to Hu and Tresser:

Theorem 1.3 ([HT]). *Let f be a real polynomial map on the interval such that the set of periods of f is the set of all powers of 2. Then, f is $(a_n)_{n\geq 0}$-infinitely renormalizable.*

Notice that this theorem is also true if, instead of assuming that the map is a real polynomial map, we make the weaker assumption that it is a multimodal map with no wandering intervals, no plateaux and no more than finitely many periodic attractors (see [HT]). It seems reasonable to make a similar conjecture for maps in dimension 2.

Conjecture. *Let f be a real polynomial map on the 2-disk, which uniformly contracts the area and such that the set of periods of f is the set of all powers of 2. Then, f is $(a_n)_{n\geq 0}$-infinitely renormalizable.*

2. Proof of the Theorems

The techniques we use in this paper were first introduced in [Ca].

Let us start with some notations. For any positive K, we denote by $\mathcal{U}(K)$ the set of C^{1+1} maps from the disk $\mathbf{D}^2$ into itself, whose derivatives are Lipschitz, with Lipschitz constant smaller than K. Thanks to the Arzèla-Ascoli theorem, any sequence of maps in $\mathcal{U}(K)$ has a converging subsequence in the C^1 topology. All along the proof, we shall frequently make use of this property.

Consider now an $(a_n)_{n\geq 0}$-infinitely renormalizable map f of the 2-disk with a bounded geometry. Since the sequence $(\mathcal{R}^n(f))_{n\geq 0}$ remains in $\mathcal{U}(K_f)$, it possesses an accumulation point in $\mathcal{U}(K_f)$.

Lemma 2.1. *Let p_0 be an accumulation point of the sequence $(a_n)_{n\geq 0}$. Then, there is a map g_0 in $\mathcal{U}(K_f)$ which is an accumulation point of the sequence $(\mathcal{R}^n(f))_{n\geq 0}$ and which satisfies:*

(i) *g_0 is 1-time renormalizable, more precisely, there exists a differentiable disk $\mathcal{D}_0(g_0) \subset \mathbf{D}^2$ such that $\mathcal{D}_0(g_0), g_0(\mathcal{D}_0(g_0)), \ldots, g_0^{p_0-1}(\mathcal{D}_0(g_0))$ are disjoint and $g_0^{p_0}(\mathcal{D}_0(g_0)) \subset \mathcal{D}_0(g_0)$.*

(ii) *Every atom $\mathcal{J}$ of the first generation of g_0 satisfies $|\mathcal{J}| \leq 2b_f$ (where b_f is the bound given in the above definition).*

Proof. Let p_0 be an accumulation point of the sequence $(a_n)_{n\geq 0}$. Then, there exists a subsequence $(a_{\phi(n)})_{n\geq 0}$ which is constant and equal to p_0. From the sequences $\mathcal{R}^{\phi(n)}(f)$ and $\xi(\mathcal{R}^{\phi(n)}(f))$, we can extract subsequences $\mathcal{R}^{\psi(n)}(f)$ and $\xi(\mathcal{R}^{\psi(n)}(f))$ which respectively converge to maps g_0 and ξ_0 in $\mathcal{U}(K_f)$.

For each $n \geq 0$, we have

$$\mathcal{D}_0(\mathcal{R}^{\psi(n)}(f)) = \xi(\mathcal{R}^{\psi(n)}(f))(\mathbf{D}^2).$$

Since the maps $\mathcal{R}^{\psi(n)}(f)$ are 1-time renormalizable, we get

$$(\mathcal{R}^{\psi(n)}(f))^i(\mathcal{D}_0(\mathcal{R}^{\psi(n)}(f))) \cap \mathcal{D}_0(\mathcal{R}^{\psi(n)}(f)) = \varnothing \quad \text{for } 1 \leq i \leq p_0 - 1,$$

and

$$(\mathcal{R}^{\psi(n)}(f))^{p_0}(\mathcal{D}_0(\mathcal{R}^{\psi(n)}(f))) \subset \mathcal{D}_0(\mathcal{R}^{\psi(n)}(f)).$$

By setting

$$\mathcal{D}_0(g_0) = \xi_0(\mathbf{D}^2),$$

we get by continuity

$$g_0^i(\mathcal{D}_0(g_0)) \cap \mathcal{D}_0(g_0) = \varnothing \quad \text{for } 1 \leq i \leq p_0 - 1,$$

and

$$g_0^{p_0}(\mathcal{D}_0(g_0)) \subset \mathcal{D}_0(g_0).$$

The fact that $\mathcal{D}_0(g_0)$ is a differentiable disk comes from the uniform estimates on the norm of the derivative of the inverse scaling functions.

Since f has bounded geometry, we know that, for each $n \geq 0$ and for each atom J of the first generation of $\mathcal{R}^n(f)$, we have:

$$|J| \leq 2b_f.$$

By continuity we get the same estimate for g_0. □

Lemma 2.1 is actually the first step of an inductive process:

Lemma 2.2. *Let p_0 be an accumulation point of the sequence $(a_n)_{n\geq 0}$. Then, there exists a sequence of maps $(g_l)_{l\geq 0}$ in $\mathcal{U}(K_f)$ which are accumulation points of the sequence $(\mathcal{R}^n(f))_{n\geq 0}$ and which satisfy, for each $l \geq 0$,*

(i) *g_l is $l+1$ times renormalizable. More precisely, there exists a sequence $(a_{l,n})_{l\geq n\geq 0}$ with $a_{l,l} = p_0$, such that g_l is $(a_{l,n})_{l\geq n\geq 0}$-renormalizable.*

(ii) $\mathcal{R}(g_l) = g_{l-1}$.

(iii) *Every atom $\mathcal{J}$ of the nth generation of g_l, $0 \leq n \leq l+1$, satisfies $|\mathcal{J}| \leq 2b_f^n$.*

Proof. Let p_0 be an accumulation point of the sequence $(a_n)_{n\geq 0}$. There exists a subsequence $(a_{\phi(n)})_{n\geq 0}$ which is constant and equal to p_0. The subsequence $(a_{\phi(n)-1})_{n\geq 0}$ also has an accumulation point, say p_1. By iterating this process l times, we can find a subsequence $(a_{\phi_l(n)})_{n\geq 0})$ which is such that:

- $a_{\phi_l(n)}$ is constant and equal to p_l,
- $a_{\phi_l(n)+1}$ is constant and equal to p_{i-1},
- $\cdots$
- $a_{\phi_l(n)+l}$ is constant and equal to p_0.

By a diagonal process, we can extract from the sequences $\mathcal{R}^{\phi_l(n)}(f)$ and $\xi(\mathcal{R}^{\phi_l(n)}(f))$, subsequences $\mathcal{R}^{\psi_l(n)}(f)$ and $\xi(\mathcal{R}^{\psi_l(n)}(f))$ which respectively converge to maps g_l and ξ_l in $\mathcal{U}(K_f)$ and g_l is such that $\mathcal{R}(g_l) = g_{l-1}$. From this point, we can use the same techniques as in the proof of Lemma 2.1, to terminate the proof of Lemma 2.2. □

Lemma 2.3. *For each $l \geq 0$, there exist an atom J_l, of the lth generation of g_l and a point x_l in J_l such that $\|dg_l(x_l)\| \geq 1$.*

Proof. We know that

$$g_l^{p_1\cdots p_l}(\mathcal{D}_l(g_l)) \subset \mathcal{D}_l(g_l),$$

where

$$\mathcal{D}_l(g_l) = \xi_{l-1}\cdots\xi_0(\mathbf{D}^2),$$

and that $g_l^{p_1\cdots p_l}$ possesses in $\mathcal{D}_l(g_l)$ a periodic orbit with period p_0. It follows that there exists a point y_l in $\mathcal{D}_l(g_l)$ such that

$$\|dg_l^{p_1\cdots p_l}(y_l)\| \geq 1.$$

Consequently, in one of the $p_1\cdots p_l - 1$ first images of $\mathcal{D}_l(g_l)$, that is to say, in an atom J_l of the lth generation of g_l, there is a point x_l, image of y_l by some iterate of f, such that $\|dg_l(x_l)\| \geq 1$. □

Let us now assume that the map f uniformly contracts the area, i.e. there exists b such that $|\det(df(x))| \leq b < 1$ for all x in $\mathbf{D}^2$. Then we have the following result:

Lemma 2.4. *Any accumulation point g_0 of the sequence $(\mathcal{R}^n(f))_{n\geq 0}$ is a singular map, i.e. $\det(dg_0(x)) = 0$ for all $x \in \mathbf{D}^2$.*

Proof. We have

$$\mathcal{R}^n(f) = \xi^{-1}(\mathcal{R}^{n-1}(f)) \circ \cdots \circ \xi^{-1}(f) \circ f^{a_0 \cdots a_{n-1}} \circ \xi(f) \circ \cdots \circ \xi(\mathcal{R}^{n-1}(f)).$$

Since for any linear map A in dimension two we have $|\det A| \leq \|A\|^2$, it follows that

$$|\det d\mathcal{R}^n(f)(x)| \leq K_f^{4n} b^{a_0 \cdots a_{n-1}},$$

and this quantity goes to 0 when n goes to ∞. Thus, by continuity, any accumulation point of the sequence $(\mathcal{R}^n(f))_{n\geq 0}$ is a singular map. $\square$

Consider now the sequence (g_l, x_l) defined in Lemmas 2.2 and 2.3. We can extract from it a subsequence $(g_{\theta(l)}, x_{\theta(l)})$ which converges to some (g, x), where g is a map in $\mathcal{U}(K_f)$ which is an accumulation point of the sequence $(\mathcal{R}^n(f))_{n\geq 0}$, and x is an accumulation of the sequence $(x_l)_{l\geq 0}$. By continuity, we get $\|dg(x)\| \geq 1$. It follows that there exist a connected neighbourhood $\mathcal{V}$ of x and an integer l_0 such that, for all $y \in \mathcal{V}$, and for all $l \geq l_0$,

(i) $\|dg_{\theta(l)}(y)\| \geq 1/2$, and

(ii) $\mathcal{V}$ contains $J_{\theta(l)}$ (the atom of the $\theta(l)$th generation of $g_{\theta(l)}$ containing $y_{\theta(l)}$).

Thanks to Lemma 2.4, we know that the maps g_l and g are singular:

$$\det(dg_l(y)) = \det(dg(y)) = 0, \quad \forall y \in \mathbf{D}^2.$$

Therefore, for all $l \geq l_0$, and for all $y \in \mathcal{V}$, the dimension of the kernel of $Dg_{\theta(l)}$ is 1. Thus, $\ker Dg_{\theta(l)}$ defines a Lipschitz field of directions on $\mathcal{V}$, and consequently a Lipschitz foliation. The image of each leaf is a point because the derivative of $g_{\theta(l)}$ along the leaves is zero. Consider now a trivialization of this foliation in $\mathcal{V}$. That is to say, a map $\tau\colon\ \mathcal{V} \to \mathbf{R}^2$ which is a Lipschitz homeomorphism onto its image and that maps each leaf of the foliation in $\mathcal{V}$ into a horizontal line. Recall now that $g_{\theta(l)}$ maps its atom of the $\theta(l)$th generation, $J_{\theta(l)}$, into itself after $q_{\theta(l)} = p_1 \cdots p_{\theta(l)}$ iterations:

$$g_{\theta(l)}^{q_{\theta(l)}}(J_{\theta(l)}) \subset J_{\theta(l)},$$

and that $g_{\theta(l)}^{q_{\theta(l)}}$ possesses in $J_{\theta(l)}$, and thus in $\mathcal{V}$, a periodic orbit with period p_0. Using the conjugacy by the trivialization τ, the map $f_{\theta(l)} = \tau \circ g_{\theta(l)}^{q_{\theta(l)}} \circ \tau^{-1}\colon\ \tau(\mathcal{V}) \to \mathbf{R}^2$ reads

$$f_{\theta(l)}(x_1, x_2) = (f_{1,\theta(l)}(x_1), f_{2,\theta(l)}(x_2)),$$

where (x_1, x_2) are the canonical coordinates in $\mathbf{R}^2$ and f_1 and f_2 are two continuous maps from some interval to the reals.

Since the map $g_{\theta(l)}^{q_{\theta(l)}}$ has a periodic orbit with period p_0 in $\mathcal{V}$, we get that the map $f_{2,\theta(l)}$ also possesses a periodic orbit with period p_0.

We are now in a good position to prove our theorem. Assume that we started with an $(a_n)_{n\geq 0}$-infinitely renormalizable map f of the 2-disk with a bounded geometry. Assume also that f uniformly contracts the area. If the sequence $(a_n)_{n\geq 0}$ is not eventually a sequence of powers of 2, it has an accumulation point p_0 which is not a power of 2. The construction we made above yields an interval map $f_{2,\theta(l)}$ which has a periodic orbit whose period p_0 is not a power of 2, and thus has positive topological entropy. This means [Mi] that there exist an interval I where $f_{2,\theta(l)}$ is

defined, two disjoint subintervals $I_0 \subset I$ and $I_1 \subset I$, and an integer n such that:
– $f^n_{2,\theta(l)}(I_0) \supset I$, and
– $f^n_{2,\theta(l)}(I_1) \supset I$.
It follows that the map $f^n_{\theta(l)}$ maps the two horizontal strips $D_0 = (\mathbf{R} \times I_0) \cap \tau(\mathcal{V})$ and $D_1 = (\mathbf{R} \times I_1) \cap \tau(\mathcal{V})$ on two lines whose projections on the vertical axis (parallel to the horizontal one) cover the interval I.

Consider now a continuous map $g\colon\ \tau(\mathcal{V}) \to \tau(\mathcal{V})$. If g is C^0 close enough to the map $f_{\theta(l)}$, it will map the two strips D_0 and D_1 on two strips whose projections on the vertical axis (parallel to the horizontal one) cover the interval I and such that $g^n(\mathbf{R} \times \partial I_0) \cap \tau(\mathcal{V})$ and $g^n(\mathbf{R} \times \partial I_1) \cap \tau(\mathcal{V})$ do not intersect $\mathbf{R} \times I$, where ∂I stands for the boundary of I. It follows that g^n has as invariant set in $\tau(\mathcal{V})$ such that g^n, restricted to this invariant set, is semi-conjugate to the shift on two symbols. Thus g^n, and consequently g, have positive topological entropy. Since the map $g_{\theta(l)}$ is an accumulation point of the sequence $(\mathcal{R}^n(f))_{n\geq 0}$, we get that some iterates of maps in this sequence accumulate (in the C^1-topology) on the map $g^{q_{\theta(l)}}_{\theta(l)}$. Thus, their images by the conjugacy τ accumulate, in the C^0-topology, on $f_{\theta(l)}$. Consequently the renormalized maps $(\mathcal{R}^n(f))_{n\geq 0}$ and then f have positive topological entropy. This achieves the proof of our theorem.

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