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# NORM ATTAINING FUNCTIONALS ON $C(T)$

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Abstract. It is shown that for any infinite compact Hausdorff space $T$, the Bishop-Phelps set in $C(T)^*$ is of the first Baire category when $C(T)$ has the supremum norm.

For a Banach space $X$ with closed unit ball $B(X)$, the *Bishop-Phelps set* is the set of all functionals in the dual $X^*$ which attain their supremum on $B(X)$; that is, the set $\{\mu \in X^*\colon$ there exists $x \in B(X)$ with $\mu(x) = \|\mu\|\}$. It is well known that the Bishop-Phelps set is always dense in the dual $X^*$. It is more difficult, however, to decide whether the Bishop-Phelps set is residual as this would require a concrete characterisation of those linear functionals in $X^*$ which attain their supremum on $B(X)$. In general it is difficult to characterise such functionals, however it is possible when $X = C(T)$, for some compact $T$, with the supremum norm. The aim of this note is to show that in this case the Bishop-Phelps set is a first Baire category subset of $C(T)^*$.

A problem of considerable interest in differentiability theory is the following.

*If the Bishop-Phelps set is a residual subset of the dual $X^*$, is the dual norm necessarily densely Fréchet differentiable?*

It is known, for example, that this is the case if the dual $X^*$ is weak Asplund, [1, Corollary 1.6(i)], or if $X$ has an equivalent weakly mid-point locally uniformly rotund norm and the weak topology on $X$ is $\sigma$-fragmented by the norm, [2, Theorems 3.3 and 4.4]. It is also known that each dual norm on $X^*$ is densely Fréchet differentiable on $X^*$ provided that the Bishop-Phelps set of each equivalent norm on $X$ is residual in $X^*$, [6]. No counterexamples are known, and a consequence of our result is that no space of the form $X = C(T)$ can be a counterexample to the problem above.

Let $T$ be an infinite compact Hausdorff space and $C(T)$ be the space of all continuous real-valued functions on $T$ with the supremum norm $\|x\| \equiv \sup\{|x(t)| : t \in T\}$. We will denote by $B$ the unit ball of $C(T)$; that is, $B \equiv \{x \in C(T) : \|x\| \leq 1\}$.

For every pair of real-valued functions on $T$ (not necessarily continuous) we write $x \leq y$ if and only if $x(t) \leq y(t)$ for every $t \in T$. We denote by $1_U$ the characteristic function of a subset $U$ of $T$; that is,

$$1_U(t) = \begin{cases} 1 & \text{if } t \in U, \\ 0 & \text{if } t \notin U. \end{cases}$$

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Given $\lambda \in C(T)^*$ and an open $U \subset T$ we define

$$\lambda_+^*(U) \equiv \sup\{\lambda(h) : h \in C(T), 0 \le h \le 1_U\}.$$

Clearly $\lambda_+^*(U) \ge 0$. Similarly we define

$$\begin{aligned}\lambda_-^*(U) &\equiv \sup\{\lambda(-h) : h \in C(T), 0 \le h \le 1_U\}\\ &= \sup\{(-\lambda)(h) : h \in C(T), 0 \le h \le 1_U\}\\ &= (-\lambda)_+^*(U).\end{aligned}$$

For our first result we will need to consider the following closed sets,

$$\begin{aligned}S_+^\lambda &\equiv \{t \in T : \lambda_+^*(U) > 0 \text{ for every open neighbourhood } U \text{ of } t\},\\ S_-^\lambda &\equiv \{t \in T : \lambda_-^*(U) > 0 \text{ for every open neighbourhood } U \text{ of } t\}.\end{aligned}$$

**Theorem 1.** *A linear functional $\lambda \in C(T)^*$ attains its supremum on $B$ at $x_0$ if and only if*

$$x_0(t) = \begin{cases} 1 & \text{for every } t \in S_+^\lambda,\\ -1 & \text{for every } t \in S_-^\lambda.\end{cases}$$

*That is, $\lambda$ is in the Bishop-Phelps set if and only if $S_+^\lambda \cap S_-^\lambda = \varnothing$.*

*Proof.* Suppose $\lambda$ attains its supremum over $B$ at $x_0$ but there exists a $t_0 \in S_+^\lambda$ such that $x_0(t_0) < r < 1$. Then there is an open neighbourhood $U_0$ of $t_0$ such that $x_0(t) < r < 1$ for every $t \in U_0$, by the continuity of $x_0$. Since $t_0 \in S_+^\lambda$ we have $\lambda_+^*(U_0) > 0$. Then there exists some $h \in C(T)$, $0 \le h \le 1_{U_0}$, with $\lambda(h) > 0$. Take some $0 < \epsilon < 1 - r$ and consider the function $x_0 + \epsilon h$. Then

$$(x_0 + \epsilon h)(t) = \begin{cases} x_0(t) + \epsilon h(t) < r + \epsilon < 1 & \text{for } t \in U_0,\\ x_0(t) & \text{for } t \notin U_0.\end{cases}$$

Remembering $h \ge 0$ we conclude that $x_0 + \epsilon h \in B$. On the other hand

$$\lambda(x_0 + \epsilon h) = \lambda(x_0) + \epsilon\lambda(h) > \lambda(x_0)$$

contradicting the assumption that $\lambda$ attained its supremum on $B$ at $x_0$. Hence $x_0(t_0) = 1$ for every $t_0 \in S_+^\lambda$.

Similarly, or by noting that $\lambda_-^* = (-\lambda)_+^*$, we conclude that $x_0(t_0) = -1$ for every $t_0 \in S_-^\lambda$.

Conversely, suppose

$$x_0(t) = \begin{cases} 1 & \text{for every } t \in S_+^\lambda,\\ -1 & \text{for every } t \in S_-^\lambda,\end{cases}$$

and $y_0 \in B$. Set

$$(y_0 - x_0)^+(t) = \max\{(y_0 - x_0)(t), 0\}$$

and

$$(y_0 - x_0)^-(t) = \min\{(y_0 - x_0)(t), 0\},$$

so $(y_0 - x_0) = (y_0 - x_0)^+ + (y_0 - x_0)^-$. Consider $T_\epsilon = \{t \in T : (y_0 - x_0)^+(t) \geq \epsilon\}$, a compact subset of $T$ disjoint from $S_+^\lambda$. For each $t \in T_\epsilon$ there is an open neighbourhood $U$ of $t$ for which $\lambda_+^*(U) = 0$. These form an open cover of $T_\epsilon$, and so there is a finite subcover $U_i$ of $T_\epsilon$ for $i = 1, 2, \ldots, n$, which together with $U_0 \equiv (T_\epsilon)^c$ form an open cover of $T$. Let $p_i$ be a partition of unity subordinate to this cover, [5, p. 171]. Thus

$$\begin{aligned}\lambda((y_0 - x_0)^+) &= \lambda\left(\sum_{i=0}^{n} p_i (y_0 - x_0)^+\right) \\ &\leq \lambda(p_0(y_0 - x_0)^+) \\ &\leq \|\lambda\| \, \|p_0((y_0 - x_0)^+)\| \\ &\leq \|\lambda\|\epsilon.\end{aligned}$$

Similarly $\lambda((y_0 - x_0)^-) \leq \|\lambda\|\epsilon$, and so we conclude that $\lambda(y_0 - x_0) \leq 2\|\lambda\|\epsilon$. Since this is true for any $\epsilon > 0$ we have $\lambda(y_0) \leq \lambda(x_0)$ for all $y_0 \in B$, so $\lambda$ attains its supremum over $B$ at $x_0$. □

We need one more definition to formulate our next proposition. Given $\lambda \in C(T)^*$ and $t \in T$, define

$$\begin{aligned}r_+^\lambda(t) &= \inf\{\lambda_+^*(U) : t \in U,\ U \text{ open}\} \text{ and} \\ r_-^\lambda(t) &= \inf\{\lambda_-^*(U) : t \in U,\ U \text{ open}\}.\end{aligned}$$

**Theorem 2.** *For every $\lambda \in C(T)^*$ the sets $\{t \in T : r_+^\lambda(t) > 0\}$ and $\{t \in T : r_-^\lambda(t) > 0\}$ are countable. Moreover, $\sum_{t\in T} r_+^\lambda(t) < \infty$ and $\sum_{t \in T} r_-^\lambda(t) < \infty$.*

*Proof.* Let $t_i \in T$ for $i = 1, 2, \ldots, n$, be distinct points in $T$ and let $\epsilon > 0$. There exist disjoint open sets $U_i$ such that $t_i \in U_i$. Since $\lambda_+^*(U_i) \geq r_+^\lambda(t_i) > r_+^\lambda(t_i) - \epsilon/2^i$, there exists some $h_i \in C(T)$, with $0 \leq h_i \leq 1_{U_i}$, such that $\lambda(h_i) > r_+\lambda(t_i) - \epsilon/2^i$. Then $h \equiv \sum_{i=1}^n h_i$ is in $B$ since the open sets $U_i$ are a disjoint family. Thus

$$\|\lambda\| \geq \lambda(h) = \sum_{i=1}^{n} \lambda(h_i) > \sum_{i=1}^{n} r_+^\lambda(t_i) - \epsilon \sum_{i=1}^{n} \frac{1}{2^i}.$$

Consequently $\|\lambda\| + \epsilon > \sum_{i=1}^n r_+^\lambda(t_i)$, which shows that $\sum_{t\in T} r_+^\lambda(t) < \infty$, and so the set $\{t \in T : r_\lambda(t) > 0\}$ must be countable. That $\sum_{t\in T} r_-^\lambda(t) < \infty$, and the set $\{t \in T : r_-^\lambda(t) > 0\}$ is countable, follow similarly. □

We note that the points $t \in T$ where $r_+^\lambda(t) > 0$ ($r_-^\lambda(t) > 0$) are points with positive (negative) $\lambda$-measure. They are related to the "$l_1$-part" of the functional $\lambda$, an $l_1$-weighted sum of point evaluations.

**Lemma 3.** *Given $\lambda \in C(T)^*$ and $t \in T$ where $r_+^\lambda(t) > 0$ ($r_-^\lambda(t) > 0$) then there exists an open neighbourhood $A$ of $\lambda$ such that $r_+^\mu(t) > 0$ ($r_-^\mu(t) > 0$) for each $\mu \in A$.*

*Proof.* We consider the case $r_+^\lambda(t) > 0$. Choose $\epsilon > 0$ such that $r_+^\lambda(t) > 3\epsilon$, and consider any $\mu \in A \equiv \{\mu \in C(T)^* : \|\lambda - \mu\| < \epsilon\}$. Let $U$ be any open neighbourhood $t$, then $\lambda_+^*(U) > 3\epsilon$. Then there exists an $h \in B,\ 0 \leq h \leq 1_U$, such that $\lambda(h) > 2\epsilon$. Since $\|\lambda - \mu\| < \epsilon$ we conclude that $\mu(h) > \epsilon$. Hence $r_+^\mu(t) \geq \epsilon > 0$.

The case $r_-^\lambda(t) > 0$ follows similarly. □

All is in readiness for our main theorem.

**Theorem 4.** *For any infinite compact Hausdorff space $T$, the Bishop-Phelps set is of the first Baire category in $C(T)^*$.*

*Proof.* The proof splits into two cases, $T$ is scattered or not.

*Case* 1. *$T$ scattered.* In this case $T$ contains a convergent sequence $\{t_i\}$ of distinct points with limit point $t_\infty \in T$, [3, Lemma 5.3]. Consider the sets

$$G_i \equiv \{\mu \in C(T)^* : \text{ for some } j > i, r_+^\mu(t_j) > 0\},$$
$$H_i \equiv \{\mu \in C(T)^* : \text{ for some } j > i, r_-^\mu(t_j) > 0\}.$$

Now these are open by Lemma 3. They are also dense since for any $\lambda \in C(T)^*$ and any $\epsilon > 0$ there exists a $j > i$ such that $r_-^\lambda(t_j) < \epsilon$ ($r_+^\lambda(t_j) < \epsilon$) and so for $\mu(x) \equiv \lambda(x) + \epsilon x(t_j)$ ($\mu(x) \equiv \lambda(x) - \epsilon x(t_j)$) we have $r_+^\mu(t_j) > 0$ ($r_-^\mu(t_j) > 0$), and so $\mu \in G_i$ ($\mu \in H_i$). But then for any $\lambda \in \bigcap_{i=1}^\infty (G_i \cap H_i)$, we have $t_\infty \in S_+^\lambda \cap S_-^\lambda$, and by Theorem 1, $\lambda$ does not attain its supremum over $B$. That is, the Bishop-Phelps set lies in the complement of a dense $G_\delta$ set, and so is of the first Baire category in $C(T)^*$.

*Case* 2. *$T$ not scattered.* In this case there exists a continuous function $p$ mapping $T$ onto $[0,1]$ [4, §8.5.4, (i)⇒(ii)]. By Zorn's Lemma there exists a minimal compact subset $T'$ of $T$ such that $p(T') = [0,1]$. Let $\{V_i\}$ be a countable open base for the interval $[0,1]$ and define the sets

$$G_i \equiv \{\mu \in C(T)^* : \text{ for some } t \in p^{-1}(V_i) \cap T', r_+^\mu(t) > 0\},$$
$$H_i \equiv \{\mu \in C(T)^* : \text{ for some } t \in p^{-1}(V_i) \cap T', r_-^\mu(t) > 0\}.$$

Again by Lemma 3 these sets are open. Since each set $p^{-1}(V_i) \cap T'$ is uncountable, for any $\lambda \in C(T)^*$ there exists a $t \in p^{-1}(V_i) \cap T'$ such that $r_+^\lambda(t) = 0$ ($r_-^\lambda(t) = 0$). So for any $\epsilon > 0$, and $\mu(x) \equiv \lambda(x) + \epsilon x(t)$ ($\mu(x) \equiv \lambda(x) - \epsilon x(t)$), we have $r_+^\mu(t) > 0$ ($r_-^\mu(t) > 0$), and so $\mu \in G_i$ ($\mu \in H_i$). That is, $G_i$ and $H_i$ are dense.

Consider any $\lambda \in \bigcap_{i=1}^\infty (G_i \cap H_i)$. Let $t \in T'$ and $N$ be an open neighbourhood of $t$. Since $T'$ is minimal, $p(T' \backslash N) \neq [0,1]$, and so we may choose $V_n \subset [0,1] \backslash p(T' \backslash N)$. Thus $p^{-1}(V_n) \cap T' \subset N \cap T' \subset N$. As $\lambda \in G_n(H_n)$ there exists an element $t_n \in p^{-1}(V_n) \cap T' \subset N$ such that $r_+^\lambda(t_n) > 0$ ($r_-^\lambda(t_n) > 0$). In particular, $N \cap S_+^\lambda \neq \varnothing$ and $N \cap S_-^\lambda \neq \varnothing$. Now this holds for arbitrary open sets $N$, and the sets $S_+^\lambda$ and $S_-^\lambda$ are both closed. We conclude that $t \in S_+^\lambda$ and $t \in S_-^\lambda$. That is, $T' \subset S_+^\lambda \cap S_-^\lambda$, and so by Theorem 1, $\lambda$ does not attain its supremum over $B$. That is, the Bishop-Phelps set lies in the complement of a dense $G_\delta$ set, and so is of the first Baire category in $C(T)^*$. □

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