NORMALIZERS OF NEST ALGEBRAS

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Abstract. For a nest $\mathcal{N}$ with associated nest algebra $A_{\mathcal{N}}$, we define $S_{\mathcal{N}}$, the normalizer of $A_{\mathcal{N}}$. We develop a characterization of elements of $S_{\mathcal{N}}$ based on certain order homomorphisms of $\mathcal{N}$ into itself. This characterization enables us to prove several structure theorems.

A normalizer of a subalgebra $A$ of $B(H)$ can be defined as the set of operators $T$ such that $T^*AT \subseteq A$ and $TA T^* \subseteq A$. Normalizers of diagonal algebras (which are typically defined so as to comprise only partial isometries) have played an important role in the study of certain limit algebras [P]. In this paper, we examine normalizers of nest algebras.

Theorem 2, the main theorem of this paper, establishes a characterization of an element of the normalizer of a nest algebra in terms of certain order homomorphisms of the nest into itself. We show that the normalizer is strongly closed, and that the order homomorphisms defined in Theorem 2 are related to the order homomorphisms defined in [EP]. We also develop a simplified characterization in the special case where $\mathcal{N}$ is continuous. The latter part of the paper examines the theory of finite rank operators. Theorem 12 establishes that every finite rank element of the normalizer is a sum of rank one elements in the normalizer.

We first recall some basic concepts of the theory of nests and nest algebras, which can be found in greater detail in [D].

For $\mathcal{H}$ a Hilbert space, a nest $\mathcal{N}$ is defined to be a complete totally ordered lattice of (self-adjoint) projections. Where there is no possibility of confusion, we identify a projection with its range so that for $N \in \mathcal{N}$, the statement “$x \in H$ such that $N x = x$” is shortened to “$x \in N$”. We actually make use of the identification in its strongest form: [D, Theorem 2.13] states that a nest of subspaces with the order topology is homeomorphic to the corresponding nest of projections with the strong operator topology.

For $M, N \in \mathcal{N}$, $M \leq N$, the interval $(M, N)$ refers to the set $\{L \in \mathcal{N}: M < L < N\}$. $N - M$ is called an interval projection. We define $N_- \in N$ to be $\vee\{M \in \mathcal{N}: M \leq N\}$, and $N_+$ to be $\wedge\{M \in \mathcal{N}: M > N\}$. Since $\mathcal{N}$ is a complete lattice, both $N_-$ and $N_+$ lie in $\mathcal{N}$. If $N_- \neq N$, we say that $N_-$ is the immediate predecessor of $N$. If $N_+ \neq N$, we say that $N_+$ is the immediate successor of $N$. If $N - N_-$ is nonzero, then it is a minimal projection, or atom, in the core of $\mathcal{N}$, the von Neumann algebra generated by $\mathcal{N}$. A nest $\mathcal{N}$ is said to be continuous if it contains no atoms; it is said to be purely atomic if its atoms span $\mathcal{H}$.
The nest algebra \( A_N \) is defined to be the algebra \( \{ A \in B(H) : (I - N)AN = 0 \text{ for all } N \in \mathcal{N} \} \), where \( I \) is the identity operator. This is equivalent to the algebra of all operators that leave invariant each subspace \( N \) of \( \mathcal{N} \).

We are now ready to formally define the normalizer of a nest algebra.

**Definition 1.** Given a nest \( \mathcal{N} \), the normalizer of \( A_N \) is the set \( S_N \) of operators \( T \) in \( B(H) \) with the property that \( T^*A_NT \subseteq A_N \) and \( TA_NT^* \subseteq A_N \).

Note that \( S_N \) is a semigroup (but not an algebra).

**Theorem 2.** Let \( T \in B(H) \). Then \( T \in S_N \) if and only if there are order homomorphisms \( \Phi, \Phi' : \mathcal{N} \to \mathcal{N} \) such that for every \( N \in \mathcal{N} \), \( TN = \Phi(N)T \) and \( T^*N = \Phi'(N)T^* \).

**Proof.** Suppose \( T \in B(H) \) and there are order homomorphisms \( \Phi, \Phi' : \mathcal{N} \to \mathcal{N} \) such that for every \( N \in \mathcal{N} \), \( TN = \Phi(N)T \) and \( T^*N = \Phi'(N)T^* \). Let \( A \in A_N \). Then for every \( N \in \mathcal{N} \), \( (T^*ATN)H = (T^*A\Phi(N)T)H \subseteq (T^*\Phi(N))H = (NT^*)H \subseteq N \). Thus, \( T^*A_NT \subseteq A_N \). Similarly, \( TA_NT^* \subseteq A_N \), so \( T \in S_N \).

Assume \( T \in S_N \). Define \( \Phi_T : \mathcal{N} \to \mathcal{N} \) in the following way: \( \Phi_T(N) = \wedge \{ L \in \mathcal{N} : TN \subseteq L \} \), \( N \in \mathcal{N} \). Then \( \Phi_T \) is an order homomorphism of \( \mathcal{N} \) into itself such that \( TN = \Phi_T(N)TN \) for all \( N \in \mathcal{N} \). To show that \( TN = \Phi_T(N)T \), we will show that \( \Phi_T(N)T(I - N) = 0 \).

Suppose there is \( N \in \mathcal{N} \) such that \( \Phi_T(N)T(I - N) \neq 0 \). Assume first that \( \Phi_T(N) \) has no immediate predecessor. Then there is \( L \in \mathcal{N}, L < \Phi_T(N) \) such that \( LT(I - N) \neq 0 \). By the definition of \( \Phi_T(N) \), \( (\Phi_T(N) - L)TN \neq 0 \). Let \( x \in N \) be such that \( (\Phi_T(N) - L)TNx \neq 0 \). Since \( LT(I - N) \neq 0 \), we have that \( (I - N)T^*L \neq 0 \). Let \( w \in L \) be such that \( (I - N)T^*Lw = z \neq 0 \). Since \( w \in L, y \in (I - L^*) \), \( [R, \text{Lemma 3.3}] \) implies that \( y^* \otimes w \in A_N \), where \( (y^* \otimes w)(v) = \langle v, y \rangle w \) for \( v \in H \). Since \( L, \Phi_T(N) \) are both in \( \mathcal{N} \), \( A = L(y^* \otimes w)(\Phi_T(N) - L) \in A_N \). Thus \( T^*AT \in A_N \) by hypothesis. But \( (I - N)T^*ATNx = ||y||^2z \neq 0 \), a contradiction.

Assume now that \( \Phi_T(N)T(I - N) \neq 0 \), and that \( \Phi_T(N) \) exists. Then there is \( x \in N \) such that \( (\Phi_T(N) - \Phi_T(N) \)TNx = y ≠ 0, \( ||y|| = 1 \). Now, either \( (\Phi_T(N) - \Phi_T(N)T(I - N) \neq 0 \) or \( (\Phi_T(N) - \Phi_T(N)T(I - N) \neq 0 \). If

\[
(\Phi_T(N) - \Phi_T(N)T(I - N) \neq 0,
\]

the argument of the previous paragraph (with \( L \) replaced by \( \Phi_T(N) \)) will give a contradiction. If

\[
(\Phi_T(N) - \Phi_T(N)T(I - N) \neq 0,
\]

then \( (I - N)T^*(\Phi_T(N) - \Phi_T(N) \) \neq 0 \). Letting \( z \in (\Phi_T(N) - \Phi_T(N) \) be such that \( (I - N)T^*z \neq 0 \), we have by \( [R, \text{Lemma 3.3}] \) that \( y^* \otimes z \in A_N \), but \( (I - N)T^*(y^* \otimes z)TN \neq 0 \), again giving a contradiction. We conclude that for every \( N \in \mathcal{N} \), \( \Phi_T(N)T(I - N) = 0 \), so that \( TN = \Phi_T(N)TN \).

A similar argument with \( T \) replaced by \( T^* \) establishes the existence of an order homomorphism \( \Phi' : \mathcal{N} \to \mathcal{N} \) such that for every \( N \in \mathcal{N} \), \( T^*N = \Phi'(N)T^* \).  

The maps \( \Phi_T, \Phi' \) defined above are left continuous in the order topology on \( \mathcal{N} \), that is, if \( \{ N_\lambda \} \) is a net of projections in \( \mathcal{N} \), \( N_\lambda < N \in \mathcal{N} \), and \( N_\lambda \) converges to \( N \) in the order topology, then \( \Phi_T(N_\lambda) \) converges to \( \Phi_T(N) \), and \( \Phi'(N_\lambda) \) converges to \( \Phi'(N) \). This fact can be shown directly, but it is also a consequence of Lemma 3 below.
In [EP], Erdos and Power define a left order continuous order homomorphism
\[ \Theta_\mathcal{U} : \mathcal{N} \to \mathcal{N} \]
associated with a norm closed \( \mathcal{A}_\mathcal{N} \) bimodule \( \mathcal{U} \) in the following way:
\[ \Theta_\mathcal{U}(N) = \bigvee \{ \text{ran}(XN) : X \in \mathcal{U} \}, \quad N \in \mathcal{N}. \]

For \( T \in B(H) \), let \( (T) \) denote the strongly closed \( \mathcal{A}_\mathcal{N} \) bimodule generated by \( T \).

**Lemma 3.** For \( T \in B(H) \), \( \Phi_T = \Theta_{(T)} \).

**Proof.** Let \( N \in \mathcal{N} \). Since \( \text{ran}(TN) \subseteq \Theta_{(T)}(N) \), we have \( \Phi_T(N) \leq \Theta_{(T)}(N) \).

For the other direction, note that if \( M \in \mathcal{N} \) is such that \( \text{ran}(TN) \subseteq M \), then \( \text{ran}(ATA') \subseteq M \) for all \( A, A' \in \mathcal{A}_\mathcal{N} \). Therefore \( \text{ran}(\sum_{i=1}^n A_iTA'_i) \subseteq M \) for \( A_i, A'_i \in \mathcal{A}_\mathcal{N}, 1 \leq i \leq n \in \mathbb{N} \). But elements of the form \( \sum_{i=1}^n A_iTA'_i \) strongly generate \((T)\), so that \( X \in (T) \) implies that \( \text{ran}(XN) \subseteq M \). This implies that \( \Theta_{(T)}(N) \leq M \) for \( M \in \mathcal{N} \) such that \( \text{ran}(TN) \subseteq M \). But \( \text{ran}(TN) \subseteq \Phi_T(N) \), so that \( \Theta_{(T)}(N) \leq \Phi_T(N) \). \( \blacksquare \)

In [EP, Theorem 1.5], it is shown that if \( \mathcal{U} \) is a strongly closed \( \mathcal{A}_\mathcal{N} \) bimodule, then \( \mathcal{U} = \{ X \in B(H) : XN = \Theta_\mathcal{U}(N)XN \text{ for all } N \in \mathcal{N} \} \). Corollary 4 is an immediate consequence of Lemma 3 and this fact.

**Corollary 4.** For \( T \in \mathcal{S}_\mathcal{N} \), \( (T) = \{ X \in B(H) : XN = \Phi_T(N)X \text{ for every } N \in \mathcal{N} \} \).

Since the adjoint operation is not strongly continuous, it is not immediate that \( \mathcal{S}_\mathcal{N} \) is strongly closed. This fact can be established using Theorem 2.

**Proposition 5.** \( \mathcal{S}_\mathcal{N} \) is strongly closed.

**Proof.** Let \( \{ T_\lambda \}_{\lambda \in \Lambda} \) be a net in \( \mathcal{S}_\mathcal{N} \) and let \( T \in B(H) \) with \( T_\lambda \) converging strongly to \( T \in B(H) \). Suppose there is \( N \in \mathcal{N} \) with \( \Phi_T(N)T(I - N) \neq 0 \). Then there is \( x \in (I - N) \) and \( \Phi_T(N)Tx = y \neq 0 \). Since \( T_\lambda x \to Tx \), there is \( \lambda_0 \in \Lambda \) such that \( \lambda \geq \lambda_0 \) implies that \( \Phi_T(N)T_\lambda x = y_\lambda \neq 0 \). But, \( \Phi_T(N)T_\lambda x = 0 \) for all \( \lambda \in \Lambda \), so \( \Phi_T(N)x < \Phi_T(N) \) for all \( \lambda \geq \lambda_0 \).

If \( P \in \mathcal{N}, P < \Phi_T(N) \), there exists a \( z \in N \) such that \( Tz \in \Phi_T(N), Tz \notin P \). Since \( T_\lambda z \to Tz \), there is \( \lambda_1 > \lambda_0 \) such that \( \lambda \geq \lambda_1 \) implies \( T_\lambda z \notin P \), so that \( \Phi_T(N) > P \). We thus conclude that \( \Phi_T(N) \to \Phi_T(T) \) in the order topology, which is equivalent to convergence in the strong operator topology [D, Theorem 2.13]. Since \( \Phi_T(N) \) is a projection for every \( \lambda \), the family \( \{ \Phi_T(N) \}_{\lambda \in \Lambda} \) is uniformly bounded in norm by 1. [KR, Remark 2.5.10] then implies that
\[ \Phi_T(N)T_\lambda(I - N) \to \Phi_T(T)(I - N) \]
in the strong operator topology. But \( \Phi_T(N)T_\lambda(I - N) = 0 \) for \( \lambda \in \Lambda \), forcing \( \Phi_T(T)(I - N) = 0 \) to be 0, contradicting the hypothesis.

Thus, \( \Phi_T(T)(I - N) = 0 \) so that \( TN = \Phi_T(T)T \) for every \( N \in \mathcal{N} \). A similar argument shows that \( T^*N = \Phi_T(T)T^* \) for every \( N \in \mathcal{N} \), so \( T \in \mathcal{S}_\mathcal{N} \) and \( \mathcal{S}_\mathcal{N} \) is strongly closed. \( \blacksquare \)

In general, the condition \( TN = \Phi_T(T)T \) for all \( N \in \mathcal{N} \) is not sufficient to guarantee that \( T \in \mathcal{S}_\mathcal{N} \). The proof of Theorem 2 shows that \( TN = \Phi_T(T)T \) for all \( N \in \mathcal{N} \) is equivalent only to \( T^*A_NT \subseteq \mathcal{A}_\mathcal{N} \). We close this section by showing that in the special case when \( N \) is a continuous nest, then \( TN = \Phi_T(T)T \) for all \( N \in \mathcal{N} \) is sufficient to guarantee that \( T \in \mathcal{S}_\mathcal{N} \).
Proposition 6. If \( \mathcal{N} \) is a continuous nest, then \( T \in \mathcal{S}_\mathcal{N} \) if and only if \( TN = \Phi_T(N)T \) for every \( N \in \mathcal{N} \).

Proof. Suppose \( TN = \Phi_T(N)T \) for all \( N \in \mathcal{N} \). Define the order homomorphism \( \Psi_T : \mathcal{N} \to \mathcal{N} \) by the following equation:

\[
\Psi_T(N) = \vee \{ M : \Phi_T(M) \leq N \}, \quad N \in \mathcal{N}.
\]

We will show that \( T^*N = \Psi_T(N)T^* \), by which Theorem 2 will imply that \( T \in \mathcal{S}_\mathcal{N} \).

Let \( N \in \mathcal{N} \) and note that \( T\Psi_T(N) = \Phi_T(\Psi_T(N))T \), so that \( T^*\Phi_T(\Psi_T(N)) = \Psi_T(N)T^* \). Since \( \Psi_T(N) = \vee \{ M : \Phi_T(M) \leq N \} \) and \( \Phi_T \) is left continuous, we see that \( \Phi_T(\Psi_T(N)) \leq N \). Expand \( T^*N \) as follows:

\[
T^*N = T^*\Phi_T(\Psi_T(N)) + T^*(N - \Phi_T(\Psi_T(N)))
\]

\[(*)\]

So to show that \( T^*N = \Psi_T(N)T^* \), we show that \( T^*(N - \Phi_T(\Psi_T(N))) = 0 \). We first claim that \( \text{ran}(T^*N) = \text{ran}(\Psi_T(N)T^*) \). Since \( T\Psi_T(N) = \Phi_T(\Psi_T(N))T \), we have

\[
\text{ran}(T^*N) \supseteq \text{ran}(T^*\Phi_T\Psi_T(N)) = \text{ran}(\Psi_T(N)T^*).
\]

To get containment in the other direction, note that if \( M > \Psi_T(N) \), then \( \Phi_T(M) \geq N \), so that

\[
\text{ran}(T^*N) \subseteq \text{ran}(T^*\Phi_T(M)) = \text{ran}(MT^*).
\]

Since this is true for every \( M > \Psi_T(N) \), continuity of \( \mathcal{N} \) implies that \( \text{ran}(T^*N) \subseteq \text{ran}(\Psi_T(N)T^*) \). With the inclusion established both ways, we conclude that \( \text{ran}(T^*N) = \text{ran}(\Psi_T(N)T^*) \).

Let \( x \in N - \Phi_T(\Psi_T(N)) \), \( T^*x = z \). Since \( z \in \text{ran}(T^*N), z \in \text{ran}(\Psi_T(N)T^*) \), that is, \( z \in \Psi_T(N) \). So \( \langle Tx, z \rangle = \|z\|^2, z \in \Psi_T(N), \) and \( x \in N - \Phi_T(\Psi_T(N)) \). But

\[
\Phi_T(\Psi_T(N)) = \measuredangle \{ M : \text{ran}(T\Psi_T(N)) \subseteq M \},
\]

so \( \|z\|^2 \neq 0 \) implies that \( x \in \Phi_T(\Psi_T(N)) \). Thus, \( z = 0 \) for arbitrary \( x \in N - \Phi_T(\Psi_T(N)) \), so that \( T^*(N - \Phi_T(\Psi_T(N))) = 0 \) in \((*)\) above, and \( T^*N = \Psi_T(N)T^* \).

Note that in light of the proof of Theorem 2, Proposition 6 can be restated as follows: For \( \mathcal{N} \) continuous, \( T^*A_NT \subseteq A_N \) if and only if \( TA_NT^* \subseteq A_N^* \)

Recall that the diagonal of \( \mathcal{N} \) is the algebra \( A_N \cap A_N^* \), which is also the commutant \( \mathcal{N}' \) of \( \mathcal{N} \). Since \( \mathcal{N} \) is abelian, \( \mathcal{N} \subseteq \mathcal{N}' \) so that \( \mathcal{N}'' \subseteq \mathcal{N}' \), i.e., the core of \( \mathcal{N} \) is contained in the diagonal of \( \mathcal{N} \). If \( A \) is an operator in the diagonal, then \( A, A^* \in A_N \) so that \( ABA^*, A^*BA \in A_N \) for all \( B \in A_N \) and we have that the diagonal of \( \mathcal{N} \) is contained in \( \mathcal{S}_\mathcal{N} \).

If \( \mathcal{N} \) is continuous, then \( \mathcal{S}_\mathcal{N} \) contains no compact operators, for if \( T \in \mathcal{S}_\mathcal{N} \), then \( T^*T \) is in the diagonal of \( \mathcal{N} \), and the diagonal of \( \mathcal{N} \) contains no compacts. If \( \mathcal{N} \) is purely atomic, then the finite rank elements of \( \mathcal{S}_\mathcal{N} \) strongly generate \( \mathcal{S}_\mathcal{N} \): There is a net \( \{ J_\lambda \} \) of finite rank projections in the diagonal of \( \mathcal{N} \) converging strongly to \( I \) and so \( \{ J_\lambda T \} \) lies in \( \mathcal{S}_\mathcal{N} \) and converges strongly to \( T \) for any \( T \in \mathcal{S}_\mathcal{N} \). The remainder of this paper culminates in Proposition 11 and Theorem 12, which give decomposition
results for finite rank elements of $S_N$. In all that follows, $T$ is assumed to be a nonzero element of $S_N$.

**Lemma 7.** Let $N_1 \leq N_2 \leq M_1 \leq M_2$ be elements of $N$, and let $P = N_2 - N_1$, $Q = M_2 - M_1$. Then for $T \in S_N$, $\text{ran}(TP) \perp \text{ran}(TQ)$.

**Proof.** $QT^*TP = 0$, so

$$\text{ran}(TP) \perp (\text{ker}(QT^*))^\perp$$

and $(\text{ker}(QT^*))^\perp$ contains ran($TQ$).

**Lemma 8.** Let $T \in S_N$ have rank $n < \infty$. Then there exist $N_\lambda$, $N_\mu$, $N_\nu$ in $N$ such that

(i) $N_\lambda$ is maximal among all $N \in N$ such that $TN = 0$.

(ii) $N_\mu$ is minimal among all $N \in N$ such that $TN = T$.

(iii) $N_\nu$ is minimal among all $N \in N$ such that $TN \neq 0$.

**Proof.** (i) and (ii) follow easily from the fact that $N$ is a complete lattice. For (iii), suppose there does not exist $N_\nu$ minimal among all $N$ such that $TN \neq 0$. For every neighborhood $(J,K)$ of $N_\lambda$, there is $L \in N$ with $N_\lambda < L < K$ (by assumption, $N_\lambda$ has no immediate successor). We associate to each neighborhood such a projection $L$ and partially order the neighborhoods by reverse inclusion. This gives a net of projections $\{L_t\}$ in $N$ converging strongly to $N_\lambda$ such that $L_t > N_\lambda$ for every $t$.

We pick $t_0$ and $x \in L_{t_0}$ such that $TL_{t_0}x = y \neq 0$. We then pick $t_1 > t_0$ such that $TL_{t_1}x \neq y$. Letting $P_1 = L_{t_0} - L_{t_1}$, we have that $TP_1 \neq 0$. Continuing inductively, we obtain $n + 1$ pairwise orthogonal interval projections $P_1, P_2, \ldots, P_{n+1}$ such that $TP_i \neq 0$, $i = 1, 2, \ldots, n + 1$. But $T(\sum_{i=1}^{n+1} P_i) \in S_N$, so Lemma 7 implies that $\text{ran}(TP_i) \perp \text{ran}(TP_j)$ for $i \neq j$. But this implies that $T(\sum_{i=1}^{n+1} P_i)$ has rank at least $n + 1$, a contradiction. The result follows.

In Lemma 7, if $P$, $Q$ are distinct they are orthogonal, and $PB(H)Q \subseteq A_N$. If $P = Q$ is a minimal interval projection, then $PB(H)P \subseteq A_N$. Further, if $P$, $Q$ are any two minimal interval projections, it is not difficult to see that $PB(H)Q \subseteq S_N$: Let $S \in PB(H)Q$. Since $Q$ is minimal, $Q = N - N_-$ for some $N \in N$. If $L \subseteq N_-$, $S^*A_NSL = 0 \subseteq L$. If $L > N_-$, then $S^*A_NSL = S^*A_NSQ \subseteq Q \subseteq L$. Similarly, $S_A^*A_NS^*L \subseteq L$ for all $L \in N$. 

**Lemma 9.** Let $T \in S_N$ have rank $m < \infty$, and suppose that there are minimal interval projections $P, Q$ such that $T = PTQ$. Then there exist rank one operators $T_1, T_2, \ldots, T_m$ in $S_N$ such that $T = \sum_{i=1}^{m} T_i$.

**Proof.** Since $T \in PB(H)Q$ is of rank $m$, there are $m$ rank one operators $T_1, T_2, \ldots, T_m$ in $PB(H)Q$ such that $T = \sum_{i=1}^{m} T_i$. By the observation immediately preceding this lemma, each of these rank one operators is in $S_N$.

**Lemma 10.** Let $T \in S_N$ and suppose that $Q$ is a minimal interval projection and that $TQ \neq 0$. Then there exists a minimal interval projection $P$ such that $TQ = PTQ$.

**Proof.** For $QT^* = (TQ)^* \in S_N$, define $N_\lambda, N_\mu,$ and $N_\nu$ as in Lemma 8. Then $N_\mu - N_\nu, N_\nu - N_\lambda$ are orthogonal projections such that $QT^* = QT^*(N_\mu - N_\mu) + QT^*(N_\nu - N_\lambda)$. Now, $TQB(H)QT^* \subseteq A_N$, so that

$$0 = (N_\mu - N_\nu)TQB(H)QT^*(N_\mu - N_\lambda).$$
This implies that at most one of \((N_\mu - N_\nu)TQ\), \(QT^*(N_\nu - N_\lambda)\) is nonzero. By construction, \(QT^*(N_\nu - N_\lambda) \neq 0\), so \(QT^* = QT^*(N_\nu - N_\lambda)\), or \(TQ = (N_\nu - N_\lambda)TQ\). The result follows with \(P = N_\nu - N_\lambda\).

**Proposition 11.** Let \(T \in Spec\) have rank \(n < \infty\). Then there exist projections \(N_1 < N_2 < \cdots < N_k\), \(M_1 < M_2 < \cdots < M_k\), \(k \leq n\), in \(Spec\) such that \(T = \sum_{i=1}^{k} P_iTQ_i\), where \(P_i = (M_i - (M_i)_-)\) and \(Q_i = (N_i - (N_i)_-)\).

**Proof.** For the given \(T\), let \(N_1 = N_\nu\) as in Lemma 8. Then \((N_1)_- = (N_\nu)_- = N_\lambda\) and \(T = T(N_1 - (N_1)_-) + T(I - N_1)\), with both \(T(N_1 - (N_1)_-)\) and \(T(I - N_1)\) in \(Spec\). By Lemma 10, there is an \(M_1 \in Spec\) such that

\[
T(N_1 - (N_1)_-) = (M_1 - (M_1)_-)T(N_1 - (N_1)_-) \neq 0.
\]

By Lemma 7, \((M_1 - (M_1)_-) \perp ran(T(I - N_1))\).

If \(T(I - N_1) \neq 0\), we repeat the above construction to obtain an \(M_2\) and an \(N_2\) such that \(T = (M_1 - (M_1)_-)T(N_1 - (N_1)_-) + (M_2 - (M_2)_-)T(N_2 - (N_2)_-) + T(I - N_2)\), where at least the first two terms in this sum are nonzero, \(N_1 < N_2\), and \((M_2 - (M_2)_-)\), \((M_1 - (M_1)_-)\), \(ran(T(I - N_2))\) are pairwise orthogonal. Further, \(M_1 < M_2\): Let \(\Phi_T\) be defined as in Theorem 2. By construction, \(M_1 \leq \Phi_T(N_1)\). Now,

\[
0 = TN_1(N_2 - (N_2)_-) = \Phi_T(N_1)T(N_2 - (N_2)_-) = \Phi_T(N_1)(M_2 - (M_2)_-)T.
\]

Since \((M_2 - (M_2)_-)T\) is nonzero, we have that

\[
\Phi_T(N_1)(M_2 - (M_2)_-)T \neq (M_2 - (M_2)_-)T,
\]

so that \(\Phi_T(N_1) < M_2\).

Repeating the above process \(k\) times, we have \(N_1 < N_2 < \cdots < N_k\), \(M_1 < M_2 < \cdots < M_k\) such that

\[
T = \sum_{i=1}^{k} ((M_i - (M_i)_-)T((N_i - (N_i)_-) + T(I - N_k),
\]

where the support (range) of any term in the sum is orthogonal to the support (range) of any other term. Since \(T\) has rank \(n\), there exists a \(k \leq n\) such that \(T(I - N_k) = 0\).

**Theorem 12.** Let \(T \neq 0\) in \(Spec\) have rank \(n < \infty\). Then there exist \(n \leq \infty\) rank one operators \(T_1, T_2, \ldots, T_n\) in \(Spec\) such that \(T = \sum_{i=1}^{n} T_i\).

**Proof.** We use Proposition 11 to write \(T\) as \(\sum_{i=1}^{k} P_iTQ_i\). By construction, the rank of \(T\) is the sum of the ranks of the individual terms, and by Lemma 9 each term of rank \(m\) can be written as a sum of \(m\) rank one operators in \(Spec\), for any \(m\).

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REFERENCES


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