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NORMALIZERS OF NEST ALGEBRAS

KEITH J. COATES

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Abstract. For a nest $\mathcal{N}$ with associated nest algebra $\mathcal{A}_{\mathcal{N}}$, we define $\mathcal{S}_{\mathcal{N}}$, the normalizer of $\mathcal{A}_{\mathcal{N}}$. We develop a characterization of elements of $\mathcal{S}_{\mathcal{N}}$ based on certain order homomorphisms of $\mathcal{N}$ into itself. This characterization enables us to prove several structure theorems.

A normalizer of a subalgebra $\mathcal{A}$ of $B(H)$ can be defined as the set of operators T such that $T^*\mathcal{A}T \subseteq \mathcal{A}$ and $T\mathcal{A}T^* \subseteq \mathcal{A}$. Normalizers of diagonal algebras (which are typically defined so as to comprise only partial isometries) have played an important role in the study of certain limit algebras [P]. In this paper, we examine normalizers of nest algebras.

Theorem 2, the main theorem of this paper, establishes a characterization of an element of the normalizer of a nest algebra in terms of certain order homomorphisms of the nest into itself. We show that the normalizer is strongly closed, and that the order homomorphisms defined in Theorem 2 are related to the order homomorphisms defined in [EP]. We also develop a simplified characterization in the special case where $\mathcal{N}$ is continuous. The latter part of the paper examines the theory of finite rank operators. Theorem 12 establishes that every finite rank element of the normalizer is a sum of rank one elements in the normalizer.

We first recall some basic concepts of the theory of nests and nest algebras, which can be found in greater detail in [D].

For H a Hilbert space, a nest $\mathcal{N}$ is defined to be a complete totally ordered lattice of (self-adjoint) projections. Where there is no possibility of confusion, we identify a projection with its range so that for $N \in \mathcal{N}$, the statement "$x \in H$ such that $Nx = x$" is shortened to "$x \in N$". We actually make use of the identification in its strongest form: [D, Theorem 2.13] states that a nest of subspaces with the order topology is homeomorphic to the corresponding nest of projections with the strong operator topology.

For $M, N \in \mathcal{N}, M \leq N$, the interval (M, N) refers to the set $\{L \in \mathcal{N} : M < L < N\}$. $N - M$ is called an interval projection. We define N_- to be $\vee\{M \in \mathcal{N} : M < N\}$, and N_+ to be $\wedge\{M \in \mathcal{N} : M > N\}$. Since $\mathcal{N}$ is a complete lattice, both N_- and N_+ lie in $\mathcal{N}$. If $N_- \neq N$, we say that N_- is the immediate predecessor of N. If $N_+ \neq N$, we say that N_+ is the immediate successor of N. If $N - N_-$ is nonzero, then it is a minimal projection, or atom, in the core of $\mathcal{N}$, the von Neumann algebra generated by $\mathcal{N}$. A nest $\mathcal{N}$ is said to be continuous if it contains no atoms; it is said to be purely atomic if its atoms span H.

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The nest algebra $\mathcal{A}_\mathcal{N}$ is defined to be the algebra $\{A \in B(H) : (I-N)AN = 0 \text{ for all } N \in \mathcal{N}\}$, where I is the identity operator. This is equivalent to the algebra of all operators that leave invariant each subspace N of $\mathcal{N}$.

We are now ready to formally define the normalizer of a nest algebra.

Definition 1. Given a nest $\mathcal{N}$, the normalizer of $\mathcal{A}_\mathcal{N}$ is the set $\mathcal{S}_\mathcal{N}$ of operators T in $B(H)$ with the property that $T^*\mathcal{A}_\mathcal{N}T \subseteq \mathcal{A}_\mathcal{N}$ and $T\mathcal{A}_\mathcal{N}T^* \subseteq \mathcal{A}_\mathcal{N}$.

Note that $\mathcal{S}_\mathcal{N}$ is a semigroup (but not an algebra).

Theorem 2. *Let $T \in B(H)$. Then $T \in \mathcal{S}_\mathcal{N}$ if and only if there are order homomorphisms $\Phi, \Phi' : \mathcal{N} \to \mathcal{N}$ such that for every $N \in \mathcal{N}$, $TN = \Phi(N)T$ and $T^*N = \Phi'(N)T^*$.*

Proof. Suppose $T \in B(H)$ and there are order homomorphisms $\Phi, \Phi' : \mathcal{N} \to \mathcal{N}$ such that for every $N \in \mathcal{N}$, $TN = \Phi(N)T$ and $T^*N = \Phi'(N)T^*$. Let $A \in \mathcal{A}_\mathcal{N}$. Then for every $N \in \mathcal{N}$, $(T^*ATN)H = (T^*A\Phi(N)T)H \subseteq \big(T^*\Phi(N)\big)H = (NT^*)H \subseteq N$. Thus, $T^*\mathcal{A}_\mathcal{N}T \subseteq \mathcal{A}_\mathcal{N}$. Similarly, $T\mathcal{A}_\mathcal{N}T^* \subseteq \mathcal{A}_\mathcal{N}$, so $T \in \mathcal{S}_\mathcal{N}$.

Assume $T \in \mathcal{S}_\mathcal{N}$. Define $\Phi_T : \mathcal{N} \to \mathcal{N}$ in the following way: $\Phi_T(N) = \wedge\{L \in \mathcal{N} : TN \subseteq L\}$, $N \in \mathcal{N}$. Then Φ_T is an order homomorphism of $\mathcal{N}$ into itself such that $TN = \Phi_T(N)TN$ for all $N \in \mathcal{N}$. To show that $TN = \Phi_T(N)T$, we will show that $\Phi_T(N)T(I-N) = 0$.

Suppose there is $N \in \mathcal{N}$ such that $\Phi_T(N)T(I-N) \neq 0$. Assume first that $\Phi_T(N)$ has no immediate predecessor. Then there is $L \in \mathcal{N}, L < \Phi_T(N)$ such that $LT(I-N) \neq 0$. By the definition of $\Phi_T(N)$, $(\Phi_T(N) - L)TN \neq 0$. Let $x \in N$ be such that $(\Phi_T(N) - L)TNx = y \neq 0$. Since $LT(I-N) \neq 0$, we have that $(I-N)T^*L \neq 0$. Let $w \in L$ be such that $(I-N)T^*Lw = z \neq 0$. Since $w \in L$, $y \in (I - L_-)$, [R, Lemma 3.3] implies that $y^* \otimes w \in \mathcal{A}_\mathcal{N}$, where $(y^* \otimes w)(v) = \langle v, y\rangle w$ for $v \in H$. Since $L, \Phi_T(N)$ are both in $\mathcal{N}$, $A = L(y^* \otimes w)(\Phi_T(N) - L) \in \mathcal{A}_\mathcal{N}$. Thus $T^*AT \in \mathcal{A}_\mathcal{N}$ by hypothesis. But $(I-N)T^*ATNx = ||y||^2 z \neq 0$, a contradiction.

Assume now that $\Phi_T(N)T(I-N) \neq 0$, and that $\Phi_T(N)_-$ exists. Then there is $x \in N$ such that $\big(\Phi_T(N) - \Phi_T(N)_-\big)TNx = y \neq 0, ||y|| = 1$. Now, either $\big(\Phi_T(N)_-\big)T(I-N) \neq 0$ or $\big(\Phi_T(N) - \Phi_T(N)_-\big)T(I-N) \neq 0$. If

$$\big(\Phi_T(N)_-\big)T(I-N) \neq 0,$$

the argument of the previous paragraph (with L replaced by $\Phi_T(N)_-$) will give a contradiction. If

$$\big(\Phi_T(N) - \Phi_T(N)_-\big)T(I-N) \neq 0,$$

then $(I-N)T^*\big(\Phi_T(N) - \Phi_T(N)_-\big) \neq 0$. Letting $z \in \big(\Phi_T(N) - \Phi_T(N)_-\big)$ be such that $(I-N)T^*z \neq 0$, we have by [R, Lemma 3.3] that $y^* \otimes z \in \mathcal{A}_\mathcal{N}$, but $(I-N)T^*(y^* \otimes z)TN \neq 0$, again giving a contradiction. We conclude that for every $N \in \mathcal{N}, \Phi_T(N)T(I-N) = 0$, so that $TN = \Phi_T(N)TN$.

A similar argument with T replaced by T^* establishes the existence of an order homomorphism $\Phi' : \mathcal{N} \to \mathcal{N}$ such that for every $N \in \mathcal{N}$, $T^*N = \Phi'_T(N)T^*$. $\square$

The maps Φ_T, Φ'_T defined above are left continuous in the order topology on $\mathcal{N}$, that is, if $\{N_\lambda\}$ is a net of projections in $\mathcal{N}$, $N_\lambda < N \in \mathcal{N}$, and N_λ converges to N in the order topology, then $\Phi_T(N_\lambda)$ converges to $\Phi_T(N)$, and $\Phi'_T(N_\lambda)$ converges to $\Phi'_T(N)$. This fact can be shown directly, but it is also a consequence of Lemma 3 below.

In [EP], Erdos and Power define a left order continuous order homomorphism $\Theta_{\mathcal{U}} : \mathcal{N} \to \mathcal{N}$ associated with a norm closed $\mathcal{A}_{\mathcal{N}}$ bimodule $\mathcal{U}$ in the following way:

$$\Theta_{\mathcal{U}}(N) = \vee\{ran(XN) : X \in \mathcal{U}\}, \quad N \in \mathcal{N}.$$

For $T \in B(H)$, let (T) denote the strongly closed $\mathcal{A}_{\mathcal{N}}$ bimodule generated by T.

Lemma 3. *For $T \in B(H)$, $\Phi_T = \Theta_{(T)}$.*

Proof. Let $N \in \mathcal{N}$. Since $ran(TN) \subseteq \Theta_{(T)}(N)$, we have $\Phi_T(N) \leq \Theta_{(T)}(N)$. For the other direction, note that if $M \in \mathcal{N}$ is such that $ran(TN) \subseteq M$, then $ran(ATA'N) \subseteq M$ for all $A, A' \in \mathcal{A}_{\mathcal{N}}$. Therefore $ran\big(\sum_{i=1}^n A_iTA_i'N\big) \subseteq M$ for $A_i, A_i' \in \mathcal{A}_{\mathcal{N}}, 1 \leq i \leq n \in \mathbf{N}$. But elements of the form $\sum_{i=1}^n A_iTA_i'$ strongly generate (T), so that $X \in (T)$ implies that $ran(XN) \subseteq M$. This implies that $\Theta_{(T)}(N) \leq M$ for $M \in \mathcal{N}$ such that $ran(TN) \subseteq M$. But $ran(TN) \subseteq \Phi_T(N)$, so that $\Theta_{(T)}(N) \leq \Phi_T(N)$. □

In [EP, Theorem 1.5], it is shown that if $\mathcal{U}$ is a strongly closed $\mathcal{A}_{\mathcal{N}}$ bimodule, then $\mathcal{U} = \{X \in B(H) : XN = \Theta_{\mathcal{U}}(N)XN \text{ for all } N \in \mathcal{N}\}$. Corollary 4 is an immediate consequence of Lemma 3 and this fact.

Corollary 4. *For $T \in \mathcal{S}_{\mathcal{N}}$, $(T) = \{X \in B(H) : XN = \Phi_T(N)X$ for every $N \in \mathcal{N}\}$.*

Since the adjoint operation is not strongly continuous, it is not immediate that $\mathcal{S}_{\mathcal{N}}$ is strongly closed. This fact can be established using Theorem 2.

Proposition 5. *$\mathcal{S}_{\mathcal{N}}$ is strongly closed.*

Proof. Let $\{T_\lambda\}_{\lambda\in\Lambda}$ be a net in $\mathcal{S}_{\mathcal{N}}$ and let $T \in B(H)$ with T_λ converging strongly to $T \in B(H)$. Suppose there is $N \in \mathcal{N}$ with $\Phi_T(N)T(I - N) \neq 0$. Then there is $x \in (I - N)$ and $\Phi_T(N)Tx = y \neq 0$. Since $T_\lambda x \to Tx$, there is $\lambda_0 \in \Lambda$ such that $\lambda \geq \lambda_0$ implies that $\Phi_T(N)T_\lambda x = y_\lambda \neq 0$. But, $\Phi_{T_\lambda}(N)T_\lambda x = 0$ for all $\lambda \in \Lambda$, so $\Phi_{T_\lambda}(N) < \Phi_T(N)$ for all $\lambda \geq \lambda_0$.

If $P \in \mathcal{N}, P < \Phi_T(N)$, there exists a $z \in N$ such that $Tz \in \Phi_T(N), Tz \notin P$. Since $T_\lambda z \to Tz$, there is $\lambda_1 > \lambda_0$ such that $\lambda \geq \lambda_1$ implies $T_\lambda z \notin P$, so that $\Phi_{T_\lambda}(N) > P$. We thus conclude that $\Phi_{T_\lambda}(N) \to \Phi_T(N)$ in the order topology, which is equivalent to convergence in the strong operator topology [D, Theorem 2.13]. Since $\Phi_{T_\lambda}(N)$ is a projection for every λ, the family $\{\Phi_{T_\lambda}(N)\}_{\lambda\in\Lambda}$ is uniformly bounded in norm by 1. [KR, Remark 2.5.10] then implies that

$$\Phi_{T_\lambda}(N)T_\lambda(I - N) \to \Phi_T(N)T(I - N)$$

in the strong operator topology. But $\Phi_{T_\lambda}(N)T_\lambda(I - N) = 0$ for $\lambda \in \Lambda$, forcing $\Phi_T(N)T(I - N)$ to be 0, contradicting the hypothesis.

Thus, $\Phi_T(N)T(I - N) = 0$ so that $TN = \Phi_T(N)T$ for every $N \in \mathcal{N}$. A similar argument shows that $T^*N = \Phi_T'(N)T^*$ for every $N \in \mathcal{N}$, so $T \in \mathcal{S}_{\mathcal{N}}$ and $\mathcal{S}_{\mathcal{N}}$ is strongly closed. □

In general, the condition $TN = \Phi_T(N)T$ for all $N \in \mathcal{N}$ is not sufficient to guarantee that $T \in \mathcal{S}_{\mathcal{N}}$. The proof of Theorem 2 shows that $TN = \Phi_T(N)T$ for all $N \in \mathcal{N}$ is equivalent only to $T^*\mathcal{A}_{\mathcal{N}}T \subseteq \mathcal{A}_{\mathcal{N}}$. We close this section by showing that in the special case when $\mathcal{N}$ is a continuous nest, then $TN = \Phi_T(N)T$ for all $N \in \mathcal{N}$ is sufficient to guarantee that $T \in \mathcal{S}_{\mathcal{N}}$.

Proposition 6. *If $\mathcal{N}$ is a continuous nest, then $T \in \mathcal{S}_{\mathcal{N}}$ if and only if $TN = \Phi_T(N)T$ for every $N \in \mathcal{N}$.*

Proof. Suppose $TN = \Phi_T(N)T$ for all $N \in \mathcal{N}$. Define the order homomorphism $\Psi_T : \mathcal{N} \to \mathcal{N}$ by the following equation:

$$\Psi_T(N) = \vee\{M : \Phi_T(M) \leq N\}, \quad N \in \mathcal{N}.$$

We will show that $T^*N = \Psi_T(N)T^*$, by which Theorem 2 will imply that $T \in \mathcal{S}_{\mathcal{N}}$.

Let $N \in \mathcal{N}$ and note that $T\Psi_T(N) = \Phi_T\big(\Psi_T(N)\big)T$, so that $T^*\Phi_T\big(\Psi_T(N)\big) = \Psi_T(N)T^*$. Since $\Psi_T(N) = \vee\{M : \Phi_T(M) \leq N\}$ and Φ_T is left continuous, we see that $\Phi_T\big(\Psi_T(N)\big) \leq N$. Expand T^*N as follows:

$$(*) \qquad \begin{aligned} T^*N &= T^*\Phi_T\big(\Psi_T(N)\big) + T^*\Big(N - \Phi_T\big(\Psi_T(N)\big)\Big) \\ &= \Psi_T(N)T^* + T^*\Big(N - \Phi_T\big(\Psi_T(N)\big)\Big). \end{aligned}$$

So to show that $T^*N = \Psi_T(N)T^*$, we show that $T^*\Big(N - \Phi_T\big(\Psi_T(N)\big)\Big) = 0$. We first claim that $ran(T^*N) = ran\big(\Psi_T(N)T^*\big)$: Since $T\Psi_T(N) = \Phi_T\big(\Psi_T(N)\big)T$, we have

$$ran(T^*N) \supseteq ran\big(T^*\Phi_T\Psi_T(N)\big) = ran\big(\Psi_T(N)T^*\big).$$

To get containment in the other direction, note that if $M > \Psi_T(N)$, then $\Phi_T(M) \geq N$, so that

$$ran(T^*N) \subseteq ran\big(T^*\Phi_T(M)\big) = ran(MT^*).$$

Since this is true for every $M > \Psi_T(N)$, continuity of $\mathcal{N}$ implies that $ran(T^*N) \subseteq ran\big(\Psi_T(N)T^*\big)$. With the inclusion established both ways, we conclude that $ran(T^*N) = ran\big(\Psi_T(N)T^*\big)$.

Let $x \in N - \Phi_T\big(\Psi_T(N)\big)$, $T^*x = z$. Since $z \in ran(T^*N)$, $z \in ran\big(\Psi_T(N)T^*\big)$, that is, $z \in \Psi_T(N)$. So $\langle Tx, z\rangle = \|z\|^2$, $z \in \Psi_T(N)$, and $x \in N - \Phi_T\big(\Psi_T(N)\big)$. But

$$\Phi_T\big(\Psi_T(N)\big) = \wedge\{M : ran\big(T\Psi_T(N)\big) \subseteq M\},$$

so $\|z\|^2 \neq 0$ implies that $x \in \Phi_T\big(\Psi_T(N)\big)$. Thus, $z = 0$ for arbitrary $x \in N - \Phi_T\big(\Psi_T(N)\big)$, so that $T^*\Big(N - \Phi_T\big(\Psi_T(N)\big)\Big) = 0$ in $(*)$ above, and $T^*N = \Psi_T(N)T^*$. □

Note that in light of the proof of Theorem 2, Proposition 6 can be restated as follows: For $\mathcal{N}$ continuous, $T^*\mathcal{A}_{\mathcal{N}}T \subseteq \mathcal{A}_{\mathcal{N}}$ if and only if $T\mathcal{A}_{\mathcal{N}}T^* \subseteq \mathcal{A}_{\mathcal{N}}$

Recall that the diagonal of $\mathcal{N}$ is the algebra $\mathcal{A}_{\mathcal{N}} \cap \mathcal{A}_{\mathcal{N}}^*$, which is also the commutant $\mathcal{N}'$ of $\mathcal{N}$. Since $\mathcal{N}$ is abelian, $\mathcal{N} \subseteq \mathcal{N}'$ so that $\mathcal{N}'' \subseteq \mathcal{N}'$, i.e., the core of $\mathcal{N}$ is contained in the diagonal of $\mathcal{N}$. If A is an operator in the diagonal, then $A, A^* \in \mathcal{A}_{\mathcal{N}}$ so that $ABA^*, A^*BA \in \mathcal{A}_{\mathcal{N}}$ for all $B \in \mathcal{A}_{\mathcal{N}}$ and we have that the diagonal of $\mathcal{N}$ is contained in $\mathcal{S}_{\mathcal{N}}$.

If $\mathcal{N}$ is continuous, then $\mathcal{S}_{\mathcal{N}}$ contains no compact operators, for if $T \in \mathcal{S}_{\mathcal{N}}$, then T^*T is in the diagonal of $\mathcal{N}$, and the diagonal of $\mathcal{N}$ contains no compacts. If $\mathcal{N}$ is purely atomic, then the finite rank elements of $\mathcal{S}_{\mathcal{N}}$ strongly generate $\mathcal{S}_{\mathcal{N}}$: There is a net $\{J_\lambda\}$ of finite rank projections in the diagonal of $\mathcal{N}$ converging strongly to I and so $\{J_\lambda T\}$ lies in $\mathcal{S}_{\mathcal{N}}$ and converges strongly to T for any $T \in \mathcal{S}_{\mathcal{N}}$. The remainder of this paper culminates in Proposition 11 and Theorem 12, which give decomposition results for finite rank elements of $\mathcal{S}_{\mathcal{N}}$. In all that follows, T is assumed to be a nonzero element of $\mathcal{S}_{\mathcal{N}}$.

Lemma 7. *Let $N_1 \leq N_2 \leq M_1 \leq M_2$ be elements of $\mathcal{N}$, and let $P = N_2 - N_1$, $Q = M_2 - M_1$. Then for $T \in \mathcal{S}_{\mathcal{N}}$, $ran(TP) \perp ran(TQ)$.*

Proof. $QT^*TP = 0$, so

$$ran(TP) \perp \big(ker(QT^*)\big)^{\perp}$$

and $\big(ker(QT^*)\big)^{\perp}$ contains $ran(TQ)$. □

Lemma 8. *Let $T \in \mathcal{S}_{\mathcal{N}}$ have rank $n < \infty$. Then there exist N_λ, N_μ, N_ν in $\mathcal{N}$ such that*

(i) *N_λ is maximal among all $N \in \mathcal{N}$ such that $TN = 0$.*
(ii) *N_μ is minimal among all $N \in \mathcal{N}$ such that $TN = T$.*
(iii) *N_ν is minimal among all $N \in \mathcal{N}$ such that $TN \neq 0$.*

Proof. (i) and (ii) follow easily from the fact that $\mathcal{N}$ is a complete lattice. For (iii), suppose there does not exist N_ν minimal among all N such that $TN \neq 0$. For every neighborhood (J, K) of N_λ, there is $L \in \mathcal{N}$ with $N_\lambda < L < K$ (by assumption, N_λ has no immediate successor). We associate to each neighborhood such a projection L and partially order the neighborhoods by reverse inclusion. This gives a net of projections $\{L_t\}$ in $\mathcal{N}$ converging strongly to N_λ such that $L_t > N_\lambda$ for every t.

We pick t_0 and $x \in L_{t_0}$ such that $TL_{t_0}x = y \neq 0$. We then pick $t_1 > t_0$ such that $TL_{t_1}x \neq y$. Letting $P_1 = L_{t_0} - L_{t_1}$, we have that $TP_1 \neq 0$. Continuing inductively, we obtain $n+1$ pairwise orthogonal interval projections $P_1, P_2, \ldots, P_{n+1}$ such that $TP_i \neq 0$, $i = 1, 2, \ldots, n+1$. But $T\big(\sum_1^{n+1} P_i\big) \in \mathcal{S}_{\mathcal{N}}$, so Lemma 7 implies that $ran(TP_i) \perp ran(TP_j)$ for $i \neq j$. But this implies that $T\big(\sum_1^{n+1} P_i\big)$ has rank at least $n+1$, a contradiction. The result follows. □

In Lemma 7, if P, Q are distinct they are orthogonal, and $PB(H)Q \subseteq \mathcal{A}_{\mathcal{N}}$. If $P = Q$ is a minimal interval projection, then $PB(H)P \subseteq \mathcal{A}_{\mathcal{N}}$. Further, if P, Q are any two minimal interval projections, it is not difficult to see that $PB(H)Q \subseteq \mathcal{S}_{\mathcal{N}}$: Let $S \in PB(H)Q$. Since Q is minimal, $Q = N - N_-$ for some $N \in \mathcal{N}$. If $L \leq N_-$, $S^*\mathcal{A}_{\mathcal{N}}SL = 0 \subseteq L$. If $L > N_-$, then $S^*\mathcal{A}_{\mathcal{N}}SL = S^*\mathcal{A}_{\mathcal{N}}SQ \subseteq Q \subseteq L$. Similarly, $S\mathcal{A}_{\mathcal{N}}S^*L \subseteq L$ for all $L \in \mathcal{N}$.

Lemma 9. *Let $T \in \mathcal{S}_{\mathcal{N}}$ have rank $m < \infty$, and suppose that there are minimal interval projections P, Q such that $T = PTQ$. Then there exist rank one operators $T_1, T_2, \ldots, T_m$ in $\mathcal{S}_{\mathcal{N}}$ such that $T = \sum_{i=1}^m T_i$.*

Proof. Since $T \in PB(H)Q$ is of rank m, there are m rank one operators $T_1, T_2, \ldots,$ T_m in $PB(H)Q$ such that $T = \sum_{i=1}^m T_i$. By the observation immediately preceding this lemma, each of these rank one operators is in $\mathcal{S}_{\mathcal{N}}$. □

Lemma 10. *Let $T \in \mathcal{S}_{\mathcal{N}}$ and suppose that Q is a minimal interval projection and that $TQ \neq 0$. Then there exists a minimal interval projection P such that $TQ = PTQ$.*

Proof. For $QT^* = (TQ)^* \in \mathcal{S}_{\mathcal{N}}$, define N_λ, N_μ, and N_ν as in Lemma 8. Then $N_\mu - N_\nu, N_\nu - N_\lambda$ are orthogonal projections such that $QT^* = QT^*(N_\mu - N_\nu) + QT^*(N_\nu - N_\lambda)$. Now, $TQB(H)QT^* \subseteq \mathcal{A}_{\mathcal{N}}$, so that

$$0 = (N_\mu - N_\nu)TQB(H)QT^*(N_\nu - N_\lambda).$$

This implies that at most one of $(N_\mu - N_\nu)TQ$, $QT^*(N_\nu - N_\lambda)$ is nonzero. By construction, $QT^*(N_\nu - N_\lambda) \neq 0$, so $QT^* = QT^*(N_\nu - N_\lambda)$, or $TQ = (N_\nu - N_\lambda)TQ$. The result follows with $P = N_\nu - N_\lambda$. $\square$

Proposition 11. *Let $T \in \mathcal{S}_\mathcal{N}$ have rank $n < \infty$. Then there exist projections $N_1 < N_2 < \cdots < N_k$, $M_1 < M_2 < \cdots < M_k$, $k \leq n$, in $\mathcal{N}$ such that $T = \sum_{i=1}^k P_i T Q_i$, where $P_i = \big(M_i - (M_i)_-\big)$ and $Q_i = \big(N_i - (N_i)_-\big)$.*

Proof. For the given T, let $N_1 = N_\nu$ as in Lemma 8. Then $(N_1)_- = (N_\nu)_- = N_\lambda$ and $T = T\big(N_1 - (N_1)_-\big) + T(I - N_1)$, with both $T\big(N_1 - (N_1)_-\big)$ and $T(I - N_1)$ in $\mathcal{S}_\mathcal{N}$. By Lemma 10, there is an $M_1 \in \mathcal{N}$ such that

$$T\big(N_1 - (N_1)_-\big) = \big(M_1 - (M_1)_-\big)T\big(N_1 - (N_1)_-\big) \neq 0.$$

By Lemma 7, $\big(M_1 - (M_1)_-\big) \perp ran\big(T(I - N_1)\big)$.

If $T(I - N_1) \neq 0$, we repeat the above construction to obtain an M_2 and an N_2 such that $T = \big(M_1 - (M_1)_-\big)T\big(N_1 - (N_1)_-\big) + \big(M_2 - (M_2)_-\big)T\big(N_2 - (N_2)_-\big) + T(I - N_2)$, where at least the first two terms in this sum are nonzero, $N_1 < N_2$, and $\big(M_2 - (M_2)_-\big), \big(M_1 - (M_1)_-\big), ran\big(T(I - N_2)\big)$ are pairwise orthogonal. Further, $M_1 < M_2$: Let Φ_T be defined as in Theorem 2. By construction, $M_1 \leq \Phi_T(N_1)$. Now,

$$0 = TN_1\big(N_2 - (N_2)_-\big) = \Phi_T(N_1)T\big(N_2 - (N_2)_-\big) = \Phi_T(N_1)\big(M_2 - (M_2)_-\big)T.$$

Since $(M_2 - (M_2)_-\big)T$ is nonzero, we have that

$$\Phi_T(N_1)\big(M_2 - (M_2)_-\big)T \neq \big(M_2 - (M_2)_-\big)T,$$

so that $\Phi_T(N_1) < M_2$.

Repeating the above process k times, we have $N_1 < N_2 < \cdots < N_k$, $M_1 < M_2 < \cdots < M_k$ such that

$$T = \sum_{i=1}^k \big((M_i - M_i)_-\big)T\big((N_i - N_i)_-\big) + T(I - N_k),$$

where the support (range) of any term in the sum is orthogonal to the support (range) of any other term. Since T has rank n, there exists a $k \leq n$ such that $T(I - N_k) = 0$. $\square$

Theorem 12. *Let $T \neq 0$ in $\mathcal{S}_\mathcal{N}$ have rank $n < \infty$. Then there exist n rank one operators $T_1, T_2, \ldots, T_n$ in $\mathcal{S}_\mathcal{N}$ such that $T = \sum_{i=1}^n T_i$.*

Proof. We use Propositon 11 to write T as $\sum_{i=1}^k P_i T Q_i$. By construction, the rank of T is the sum of the ranks of the individual terms, and by Lemma 9 each term of rank m can be written as a sum of m rank one operators in $\mathcal{S}_\mathcal{N}$, for any m. $\square$

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Department of Mathematics, Illinois Wesleyan University, Bloomington, Illinois 61702

E-mail address: `kcoates@sun.iwu.edu`