MICROLOCAL ANALYSIS OF ULTRADISTRIBUTIONS

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Abstract. The ultradistributional wave front sets of an ultradistribution $u$ are characterized by the behaviour of $K \ast u$ on the boundary of the tube domain $DR^n$, where $K$ is the kernel analysed by Hörmander.

1. Introduction

We follow the results and ideas of Chapter 8 in [2] and characterize the ultradistributional wave front sets of type $WF^*$ and $WF_*$ for a tempered ultradistribution $u$ by using its convolution with the kernel $K$ (cf. [2]). We give the assertions which are analogous to the corresponding ones for distributions.

The paper is organized as follows. In Section 2 we recall the definitions of ultradistributional wave front sets $WF^*$ and $WF_*$. These notions are introduced by Eida [1] and Komatsu [4] and called singular spectrums $SS^*$ and $SS_*$. In Section 3 we recall the definition of tempered ultradistribution spaces $S'*$ (cf. [5], [7]) and a theorem from [8] which is needed for later use. In Section 4 we give the microlocal analysis of a $u \in S'^*$ (Theorems 2, 3 and Corollary 1). Necessary and sufficient conditions for an analytic function in the tube domain $DR^n$ which determine its boundary value in $S'^*$ are given in Theorem 2.

2. Notation and notions

By $M_p$, $p \in \mathbb{N}_0$, we denote a sequence of positive numbers with $M_0 = 1$. We refer to [3] and [6] for the meaning of conditions (M.1), (M.2)', (M.2), (M.3)' and (M.3). We also use the following condition ([6]):

$$(M.1)' \quad M_{p-1}M_{p+1}^* \leq M_p^*, \quad p \in \mathbb{N}, \quad M_0^* = 1, \quad M_p^* = M_p/p!, \quad p \in \mathbb{N}.$$

Let $M_p$ satisfy (M.1) and (M.3)'. The associated function $\tilde{M}(\rho)$ and the growth function $\tilde{M}(\rho)$ related to $M_p$ are defined by

$$M(\rho) = \sup_{p \in \mathbb{N}_0} \ln \frac{\rho^p}{M_p}, \quad \tilde{M}(\rho) = \sup_{p \in \mathbb{N}_0} \ln \frac{\rho^p}{M_p^*}, \quad \rho > 0.$$

Note, for a given $L > 0$ there is $L_1 > 0$ such that ([8])

$$M(L|\xi|) - |\eta||\xi| \leq \tilde{M}(L_1/|\eta|), \quad \xi, \eta \in \mathbb{R}^n. \tag{1}$$
We denote by $\Omega$ an open set in $\mathbb{R}^n$, and $K \subset \subset \Omega$ means that $K$ is a compact subset of $\Omega$. Recall,

$$
\|\varphi\|_{K,h,M_p} = \sup_{x \in K, \alpha \in \mathbb{N}^n_0} \frac{|\varphi^{(\alpha)}(x)|}{h^{|\alpha|}M_{|\alpha|}}, \varphi \in C^\infty(\Omega).
$$

The symbol $\ast$ is used for both $(M_p)$ and $(M_p)$. We refer to [3] for the definitions of $\mathcal{D}'(\Omega)$, $\mathcal{D}'_K(\Omega)$ and $\mathcal{D}''(\Omega)$. Throughout the paper we will assume that (M.1), (M.2)$'$ and (M.3)$'$ hold. Eida [1] and Komatsu [4] have defined $SS_*$- and $SS^*$-singular support of a hyperfunction $f$. We will recall the definitions related to ultradistributions and call them wave front sets.

Let $N_p$ be a sequence of positive numbers which satisfies (M.1), (M.2)$'$, (M.3)$'$ and $N_0 = 1$. Then ([3]):

$$(N_p) \leq (M_p) \quad \text{(resp., } \{N_p\} \leq \{M_p\})$$

if there are constants $L > 0$ and $C > 0$ (resp., for every $\epsilon > 0$ there is $C_\epsilon > 0$) such that

$$N_p \leq C L^p M_p \quad \text{(resp., } N_p \leq C \epsilon^p M_p), \quad p \in \mathbb{N}_0.$$ 

Let $f \in \mathcal{D}'$, where $\ast = N_p \leq \ast = M_p$. Then $(x, \omega) \in S^*\Omega = \Omega \times S^{n-1}$ is not in $WF_f$ (resp., not in $WF^* f$) if there exist a neighborhood $U \subset \Omega$ of $x$ and a conic neighborhood $\Gamma$ of $\omega$ of the form $\Gamma = \{\xi \neq 0; |\xi|^{-1} - |\omega| < \eta\}$ such that for every $\phi \in \mathcal{D}'(U)$ the following hold:

In the $(M_p)$ case, for every $\epsilon > 0$ there is $C_\epsilon > 0$ such that $|\widehat{\phi f}(\xi)| \leq C_\epsilon e^{-M(\epsilon |\xi|)}$, $\xi \in \Gamma$ (resp., there are $k > 0$ and $C > 0$ such that $|\widehat{\phi f}(\xi)| \leq C e^{M(k |\xi|)}$, $\xi \in \Gamma$).

In the $(M_p)$ case, there exist $k > 0$ and $C > 0$ such that $|\widehat{\phi f}(\xi)| \leq C e^{-M(k |\xi|)}$, $\xi \in \Gamma$ (resp., for every $\epsilon > 0$, there is $C_\epsilon > 0$ such that $|\widehat{\phi f}(\xi)| \leq C_\epsilon e^{M(\epsilon |\xi|)}$, $\xi \in \Gamma$).

Note, the notion $WF_f(M_p)$ is equal to Hörmander’s notion $WF_L$.

The definition of the singular spectrum $SSf$, where $f \in \mathcal{B}(\Omega)$, is given by Sato (cf. [10]). For an $f \in \mathcal{D}'(\Omega)$, $(x, \omega) \in S^*\Omega$ is not in $SSf$ if this point is not in $SS\{f\}$, where $\{f\}$ denotes the corresponding hyperfunction. This notion is equal to Hörmander’s $WF_A f$, the analytic wave front set of $f$ ([2], Definition 9.3.2, Theorem 9.6.3) and we will use this notation.

We also use the definition according to which $(x, \xi) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$ is an element of the corresponding singular spectrum defined above if this holds for $(x, \xi/|\xi|)$. Let $DR^n = \{z \in \mathbb{C}^n; |\Im z| < 1\}$ and $S^*\mathbb{R}^n = \partial DR^n$. Recall ([4]), $O^*|_{DR^n}$ (resp., $O^*|_{DR^n(U)}$) is a sheaf over C$^0$ of holomorphic functions in $DR^n$ which satisfy the following growth condition near $S^*\mathbb{R}^n$:

Let $U$ be an open set in $\mathbb{C}^n$. Then a function $F(z)$ is in $O^*|_{DR^c(U)}$ (resp., $O^*|_{DR^c(U)}$) if $F$ is holomorphic in $DR^n \cap U$ such that for every compact set $K \subset \subset U$,

- in the $(M_p)$ case, there are $C > 0$ and $k > 0$,
- in the $(M_p)$ case, for every $k > 0$ there is $C > 0$ such that

$$|F(z)| \leq C e^{M(k |z|)}, \quad z = x + \sqrt{-1}y \in K \cap DR^n$$

(resp., for every compact set $K \subset \subset U$),

- in the $(M_p)$ case, for every ultradifferential operator $P(\partial)$ of class $(M_p)$, in the $(M_p)$ case, for every ultradifferential operator $P(\partial)$ of class $(M_p)$

$$P(\partial_z)F \quad \text{is bounded in } K \cap DR^n.$$
3. Tempered ultradistributions

We will recall the definitions and the basic structural properties of tempered ultradistributions (cf. [5] and [7]). Let $m > 0$. The space of smooth functions $\varphi$ on $\mathbb{R}^n$ which satisfy

$$\sigma_{m,2}(\varphi) = \left( \sum_{\alpha, \beta \in \mathbb{N}_0^n} \int_{\mathbb{R}^n} \frac{|m|^{\alpha + \beta}}{M_{|\alpha|} M_{|\beta|}} (1 + |x|^2)^{\beta/2} \varphi^{(\alpha)}(x)^2 \, dx \right)^{1/2} < \infty,$$

equipped with the topology induced by the norm $\sigma_{m,2}$, is denoted by $\mathcal{S}^{M_p,m}_2$. The strong duals of $\mathcal{S}^{M_p}$ is $\text{proj} \lim_{m \to \infty} \mathcal{S}^{M_p,m}_2$ and $\mathcal{S}^{M_p} = \text{ind} \lim_{m \to 0} \mathcal{S}^{M_p,m}_2$ are called spaces of tempered ultradistributions of Beurling and Roumieau type.

For every fixed $p \in [1, \infty]$, the family of norms $\{\sigma_{m,p}; \ m > 0\}$ is equivalent to the family of norms $\{\sigma_{m,p}; \ m > 0\}$ where instead of the $L^2$ norm we put the $L^p$ norm. In fact, in the sequel we will use the family of norms

$$s_h(\phi) = \sup\{ \frac{\int |^{\alpha|}}{M_{|\alpha|}} | \phi^{(\alpha)}(x) | e^{M(h|x|)} ; \alpha \in \mathbb{N}_0^n, x \in \mathbb{R}^n \}, \ h > 0,$$

which is equivalent to $\{\sigma_{h,2}; \ h > 0\}$.

$\mathcal{S}^{M_p}$ and $\mathcal{S}^{M_p}_2$ are $(FS)$- and $(LS)$-spaces respectively. If (M.2) holds, they are $(FN)$- and $(LN)$-spaces respectively and $\mathcal{D}^* \hookrightarrow \mathcal{S}^* \hookrightarrow \mathcal{E}^*$, $\mathcal{S}^* \hookrightarrow \mathcal{S}$, where "$A \hookrightarrow B$" means that $A$ is dense in $B$ and the inclusion mapping is continuous.

An $f \in \mathcal{D}^*$ is in $\mathcal{S}^*$ if and only if there exists a family $F_{\alpha, \beta}$, $\alpha, \beta \in \mathbb{N}_0^n$, in $L^2(\mathbb{R}^n)$ such that

$$f = \sum_{\alpha, \beta \in \mathbb{N}_0^n} ((1 + |x|^2)^{\beta/2} F_{\alpha, \beta})^{(\alpha)} \quad \text{in} \quad \mathcal{S}^*,$$

and in the case $\mathcal{S}^{M_p}$, there exists $k > 0$ (resp., in the case $\mathcal{S}^{M_p}_2$), for every $k > 0$ such that

$$\left( \sum_{\alpha, \beta \in \mathbb{N}_0^n} \int_{\mathbb{R}^n} \frac|M_{|\alpha|} M_{|\beta|}}{k^{\alpha + \beta}} F_{\alpha, \beta}(x)^2 \right)^{1/2} < \infty.$$

The Fourier transformation is an isomorphism of $\mathcal{S}^*$ onto itself.

We recall the next theorem for later use.

Theorem 1 ([8]). Assume (M.1)*, (M.2) and (M.3)* hold for $\dag = N_p$ $(N_0 = 1)$. Let $\Gamma$ be an open convex cone in $\mathbb{R}^n$ and $F$ be an analytic function in $Z = \{ z \in \mathbb{C}^n; \ \text{Im} \ z \in \Gamma, \ |\text{Im} \ z| < \gamma \}$ for some $\gamma > 0$. Moreover, assume

$$|F(x + \sqrt{-1}y)| \leq C_{a,b} e^{N(a|x|) + \tilde{N}(x)}, \ x + \sqrt{-1}y \in Z,$$

in the $(N_p)$ case for some $a > 0$, $b > 0$, and $C_{a,b} > 0$, and in the $(N_p)$ case for every $a > 0$, $b > 0$ there exists $C_{a,b} > 0$. Then

$$F(x + \sqrt{-1}y) \stackrel{S^*}{} \rightarrow F(x + \sqrt{-1}0), \ y \to 0, \ y \in \Gamma.$$
Moreover, there exists
\[ F(x + \sqrt{-1}t), \varphi(x) = \int_{\mathbb{R}^n} F(x + \sqrt{-1}Y)\Phi(x + \sqrt{-1}Y)dx \]
\[ 2\sqrt{-1} \sum_{i=1}^{n} Y_i \int_{0}^{1} \int_{\mathbb{R}^n} \frac{\partial}{\partial \xi_i} \Phi(x + \sqrt{-1}tY)F(x + \sqrt{-1}Yt)dtdx. \]
Moreover, there exists \( C > 0 \) such that
\[ |\langle F(x + \sqrt{-1}t), \varphi(x) \rangle| \leq C_{S_{\mathbb{R}}} \varphi, \varphi \in S', \]
where \( \Phi(z) \) is the almost analytic extension of \( \varphi \) (cf. [6] or [8]).

4. Microlocal analysis of ultradistributions

As in [2], put
\[ I(\xi) = \int_{|\omega| = 1} e^{-i(\omega, \xi)}d\omega, \xi \in \mathbb{R}^n, \quad K(z) = (2\pi)^{-n} \int e^{i\sqrt{-1}(z, \xi)} I(\xi) d\xi, z \in DR^n. \]
We call \( K \) Hörmander’s kernel. Recall ([2], Lemma 8.4.10) that \( K(z) \) is an analytic function in \( \Omega = \{ z \in \mathbb{C}^n; \langle z, z \rangle \notin (-\infty, -1] \} \). The properties of this function given in the quoted lemma imply
\[ |K(x + \sqrt{-1}y)| \leq \frac{Cn!e^{-c|z|}}{(1 - |y|)^n}, \quad z \in DR^n, \]
for some \( C > 0 \) and \( c > 0 \). By using the Cauchy formula on the contour
\[ \Gamma_{t + \sqrt{-1}y} = \Gamma_{t_1 + \sqrt{-1}y_1} \times \cdots \times \Gamma_{t_n + \sqrt{-1}y_n}, \]
where \( \Gamma_{t_j + \sqrt{-1}y_j} \) is the boundary of
\[ \{(\tau_j, u_j); \tau_j - \tau_j < 1, |y_j - u_j| < (1 - |y|)/2\}, \]
it follows that for some \( C > 0 \) and \( c > 0 \)
\[ \frac{\partial^{\alpha}}{\partial t^{\alpha}} K(t + \sqrt{-1}y) < C \frac{\alpha n!}{(1 - |y|)^{n+|\alpha|}} e^{-c|z|}, \quad t \in \mathbb{R}^n, \quad |y| < 1, \quad \alpha \in \mathbb{N}^n. \]
This implies that \( K(\cdot + \sqrt{-1}y) \in S' \) for every fixed \( y, |y| < 1 \).

The next theorem is proved in [2] for tempered distributions.

**Theorem 2.** Let \( \mathbb{N} = N_p \) and \( \mathbb{N} = M_p \) satisfy (M.1)\,*, (M.2) and (M.3)' and let \( N_p \leq M_p \). Let \( u \in S' \) and \( U(z) = \langle u, K(z) \rangle = \langle u(t), K(x + \sqrt{-1}y) \rangle, z \in DR^n \).
\( a) \) Then \( U \) is analytic in \( DR^n \) and
\[ \text{in the} \ \{N_p\} \ \text{case, for some} \ a > 0, \ b > 0 \ \text{there is} \ C_{a,b} > 0, \ \text{in the}\ \{N_p\} \ \text{case, for every} \ a > 0, \ b > 0 \ \text{there is} \ C_{a,b} > 0, \ \text{such that} \]
\[ |U(z)| \leq C_{a,b} e^{N(a|z|) + N(b(1-|y|))}, \quad z = x + \sqrt{-1}y \in DR^n. \]
For every \( \omega \in S^{n-1} \) there exists the limit
\[ \lim_{y \to 0} \langle U(x + \sqrt{-1}y), \phi(x) \rangle = \langle U(x + \sqrt{-1}\omega), \phi(x) \rangle, \ \phi \in S', \]
and the mapping \( S^{n-1} \ni \omega \mapsto U(\cdot + \sqrt{-1}\omega) \in S^{\dagger} \) is continuous. Moreover,
\[ \langle u, \phi \rangle = \int_{\mathbb{R}^n} \langle U(\cdot + \sqrt{-1}\omega), \phi \rangle d\omega, \ \phi \in S'. \]
b) Conversely, if $U$ is given satisfying (4), then (5) defines an ultradistribution in $S'$. 

c1) $q = (x_0, \omega_0) \notin WF^* u$ if and only if $U$ is $O_+$ in a neighborhood of $x_0 - \sqrt{-1} \omega_0$.

c2) $q \notin WF_u$ if and only if $U$ is $O$, in a neighborhood of $x_0 - \sqrt{-1} \omega_0$.

c3) $q \notin WF_A u$ if and only if $U$ is analytic at $x_0 - \sqrt{-1} \omega_0$ (i.e. in a neighborhood of this point).

Proof. a) We will give only the proof in the $(M_p)$ case. There exist $h > 0, C_h > 0$ and suitable constants which do not depend on $h$, such that

$$ |\langle u(t), K(z-t) \rangle| \leq C_h \sup_{\alpha \in \mathbb{N}_0^N} \| \frac{h^{\alpha}}{2\pi} e^{\frac{\alpha}{2} N(\langle h | t \rangle)} \|_{\mathcal{N}_h} $$

$$ \leq C C_h \sup_{\alpha \in \mathbb{N}_0^N} \frac{\hbar \alpha!}{\mathcal{N}_h(1-|y|^{\alpha}+h^{\alpha} N(\langle h | t \rangle))} $$

$$ \leq C_1 C_h e^{N(h|x|)} \sup_{\alpha \in \mathbb{N}_0^N} \frac{\hbar \alpha!}{(1-|y|^{\alpha}+h^{\alpha} N(\langle h | t \rangle)} $$

$$ \leq C_1 C_h e^{N(h|x|)+N(\hbar^{-\alpha})}, \ z \in DR^n. $$

This proves (4). Since the Fourier transformation of $U(\cdot + \sqrt{-1} y)$, $|y| < 1$, is $e^{-\langle y, \xi \rangle} \hat{u}(\xi)/I(\xi), \xi \in \mathbb{R}^n$, and

$$ I(\xi) = (2\pi)^{(n-1)/2} \frac{e^{\frac{|\xi|}{2}}}{|\xi|^{(n-1)/2}} (1 + O(1/|\xi|), |\xi| \to \infty $$

(2), p. 287), it follows that $y \mapsto U(\cdot + \sqrt{-1} y)$, $|y| \leq 1$, is a continuous function $\{ y; |y| \leq 1 \} \to S'(N_0)$. Let us prove this. For any $\phi \in S^1, \xi \in \mathbb{R}^n, \alpha \in \mathbb{N}_0^n$ and $|y| \leq 1$ we have

$$ |(e^{-\langle y, \xi \rangle}(1/I(\xi)) \phi(\xi))^{(\alpha)}| $$

$$ = |\sum_{i \leq \alpha} \binom{\alpha}{i} (y)^{\alpha-i} e^{-\langle y, \xi \rangle} \sum_{p \leq i} \binom{i}{p} \phi^{(i-p)}(\xi)(1/I(\xi))^{(p)}| $$

Since $|(I(\xi))^{(\beta)}| \leq I(\xi), \xi \in \mathbb{R}^n, \beta \in \mathbb{N}_0^n$, we conclude

$$ |(1/I(\xi))^{(p)}| \leq 2^{|p|} |I(\xi), \xi \in \mathbb{R}^n. $$

Now by (6),

$$ |(e^{-\langle y, \xi \rangle}(1/I(\xi)) \phi(\xi))^{(\alpha)}| \leq C_3 |\alpha| |(1 + |\xi|^{(n-1)/2} \sup_{|i| \leq \alpha} |\phi^{(i)}(\xi)| $$

which implies:

$$ \langle u, \phi \rangle = \lim_{r \to 1} \left( \int_{|\omega|=1} \frac{e^{-r \omega \cdot \xi} \hat{u}(\xi)}{I(\xi)} d\omega, \hat{\phi}(\xi) \right) $$

$$ = \lim_{r \to 1} \int_{|\omega|=1} \left( \frac{e^{-r \omega \cdot \xi} \hat{u}(\xi)}{I(\xi)} \hat{\phi}(\xi) \right) d\omega = \int_{|\omega|=1} \langle U(x + \sqrt{-1} \omega), \phi(x) \rangle d\omega, $$

where $y = r \omega, \omega \in S^{n-1}$. 


b) Conversely, if \(U\) is analytic in \(D\mathbb{R}^n\) and satisfies (4), then Theorem 1, more precisely (2) and (3), imply that

\[
\lim_{z \in D\mathbb{R}^n} U_y = U_\omega, \quad |(\omega_1, \ldots, \omega_n)| = 1,
\]

exists in \(S^{(Np)}\), where \(U_y = U(\cdot + \sqrt{-1}y), |y| < 1\) and

\[
\langle U_\omega, \phi \rangle = \int U(x) \Phi(x - \sqrt{-1}\omega)dx
\]

\[
-2\sqrt{-1} \sum \omega_i \int_0^1 \int_{\mathbb{R}^n} U(x + \sqrt{-1}(1 - t)\omega) \frac{\partial \Phi}{\partial x_1} (x + \sqrt{-1}(1 - t)\omega) dt dx.
\]

In fact, we have to put \(U(z + \sqrt{-1}\omega) = F(z), Y = \omega\), and the conclusion follows as in Theorem 1.

Moreover, it follows that \(S^{n-1} \ni \omega \mapsto U_\omega \in S^{(Np)}\) is continuous, which implies that by

\[
\phi \mapsto \langle u, \phi \rangle = \int \langle U_\omega(x), \phi(x) \rangle d\omega, \phi \in S^{(M_p)},
\]

an element from \(S^{(Np)}\) is defined. One can easily show that \(U = u * K\).

c) We will prove the “if” parts.

Suppose \(q \notin WF^u\). Let \(\varphi \in D^*(U)\), where \(U = \{x; |x - x_0| < 2r\}\) and \(0 \leq \varphi \leq 1, \varphi \equiv 1\) on \(\{x; |x - x_0| < r\}\).

Note that

\[
|\varphi u(\xi)| < C_1 e^{M(L_1|\xi|)}, \quad \xi \in \mathbb{R}^n,
\]

for some \(L_1 > 0\) and some \(C_1 > 0\) in the \((M_p)\) case (resp., for every \(L_1 > 0\) there is \(C_1 > 0\) in the \(\{M_p\}\) case). Put \(U = \varphi u * K + (1 - \varphi)u * K = U_1 + U_2\). This decomposition is also used in parts \(b)\) and \(c)\). Let \(x \in \{x; |x - x_0| < r/2\}\). Then,

\[
(1 - \varphi(t))K(x + \sqrt{-1}y - t) = 0, \quad |t - x_0| < r, \quad x + \sqrt{-1}y \in \mathbb{C}^n.
\]

If \(|t - x_0| \geq r\), then \(|x - t| \geq r/2\) and for \(|y| < 1 + r/2,\)

\[
t \mapsto K(x + \sqrt{-1}y - t)
\]

is a function which belongs to \(S^*\).

This implies that

\[
U_2(z) = \langle u(t), (1 - \varphi(t)), K(z - t) \rangle, \quad z \in D\mathbb{R}^n,
\]

is analytic in

\[
\{x + \sqrt{-1}y; |y| < 1 + r/2, |x - x_0| < r/2\},
\]

which is a neighborhood of \(x_0 - \sqrt{-1}\omega_0\). So in the proofs of \(c1)\), \(c2)\) and \(c3)\) below we have to consider \(U_1\).

c1) Let \(\gamma\) be an open convex cone which contains \(\omega_0, \epsilon > 0\) and \(|y + \omega_0| < \epsilon, |y| < 1\). We shall use the following inequalities:

\[
M(L|\xi|) - (\xi, y) - |\xi| \leq M(\bar{L}/(1 - |y|)), \quad \xi \in \gamma,
\]

which holds for some \(\bar{L} > 0\) (see (1)), and

\[
-\langle y, \xi \rangle - |\xi| \leq -\epsilon|\xi|, \quad \xi \notin \gamma,
\]

which holds if \(\epsilon\) is small enough.
Let $|y + \omega_0| < \epsilon$, $|y| < 1$, $|x - x_0| < r$. Then $(z = x + \sqrt{-1}y)$,
\[ |U_1(z)| = |\mathcal{F}(\varphi * K(z, \cdot))(\xi)| \]

\[ \leq \int_{\mathbb{R}^n} \frac{e^{-\langle y, \xi \rangle}}{I(\xi)} \hat{\varphi}(\xi) d\xi = \int_{\xi \in \gamma} + \int_{\xi \notin \gamma}. \]

By using (7) and (8) for the first integral and (7) and (9) for the second integral, we obtain that $U_1$ is $\mathcal{O}^*$ in some neighborhood of $x_0 - \sqrt{-1}\omega_0$.

(2) Let $q \notin WF_u$. By using
\[ |P(\partial_z)(u \varphi * K)(z)| \leq \int_{\mathbb{R}^n} \frac{e^{-\langle y, \xi \rangle}}{I(\xi)} P(\sqrt{-1}I\xi)\hat{u}\varphi(\xi) d\xi, \]

in a similar way we obtain that for every ultradifferential operator $P(\partial_z)$ of the corresponding class, $P(\partial_z)U$ is bounded in a neighborhood of $x_0 - \sqrt{-1}\omega_0$.

(3) If $(x_0, \omega_0) \notin WF_A u$, then $(x_0, \omega_0) \notin WF_A u$. By [2], Definition 9.3.2, it is equivalent to the analyticity of $u \varphi * K$ at $x_0 - \sqrt{-1}\omega_0$.

The converse parts in c1), c2) and c3) follow from the next lemma (see [2], Lemma 8.4, for tempered distributions).

Lemma 1. Let $d\mu$ be a measure on $S^{n-1}$ and $\Gamma$ an open convex cone in $\mathbb{R}^n$ such that
\[ (y, \omega) < 0 \quad \text{if} \quad 0 \neq y \in \Gamma, \, \omega \in \text{supp } d\mu. \]

Let $U$ be analytic in $\text{DR}^n$, and satisfy (4) in $\text{DR}^n$. Then
\[ F(z) = \int_{S^{n-1}} U(z + \sqrt{-1}\omega) d\mu(\omega), \quad \text{Im } z \in \Gamma, |\text{Im } z| < \gamma, \]

is analytic and
\[ |F(z)| \leq C_{a,b} e^{N(a|z|) + S(b/(1-|y|))}, \quad \text{Im } z \in \Gamma, \quad |\text{Im } z| < \gamma, \]

for some $a, b$ and some $C_{a,b} > 0$ in the $(N_p)$ case, and for every $a, b$ there is $C_{a,b} > 0$ in the $(N_p)$ case.

For every measure $d\mu$ on $S^{n-1}$ and
\[ U_\mu = \int_{S^{n-1}} U(\cdot + \sqrt{-1}\omega) d\mu(\omega), \]

there holds:
\[ WF^* U_\mu \subset \{ (x, \omega); -\omega \in \text{supp } d\mu \} \quad \text{and } U \text{ is not } \mathcal{O}^* \text{ at } x - \sqrt{-1}\omega. \]
\[ WF_* U_\mu \subset \{ (x, \omega); -\omega \in \text{supp } d\mu \} \quad \text{and } U \text{ is not } \mathcal{O}_* \text{ at } x - \sqrt{-1}\omega. \]
\[ WF_A U_\mu \subset \{ (x, \omega); -\omega \in \text{supp } d\mu \} \quad \text{and } U \text{ is not analytic at } x - \sqrt{-1}\omega. \]

Proof. Let $\omega \in \text{supp } d\mu$ and $\text{Im } z \in \Gamma$. Then for some $C > 0$,
\[ 1 - |\text{Im}(z + \sqrt{-1}\omega)| > \frac{|\text{Im } z|}{2} \quad \text{if } |\text{Im } z| < C \quad ([5], \text{Lemma 8.4.12}). \]

This implies that we may use Theorem 1 for $F$. Thus, $F(\cdot + \sqrt{-1}\Gamma_0) \subset S^d$.

Note
\[ WF^* U_\mu \subset WF_* U_\mu \subset WF_A U_\mu \subset \mathbb{R}^n \times \Gamma^0, \]

where $\Gamma^0 = \{ \xi; (x, \xi) \geq 0 \text{ for every } x \in \Gamma \}$ is the dual cone.

We shall prove only the estimate for $WF^* U_\mu$ since the other parts can be proved similarly. Denote $\text{sing supp }^* u = \pi_1 WF^* u$. There holds
\[ \text{sing supp }^* U_\mu \subset \{ x; U \text{ is not } \mathcal{O}^* \text{ at } x + \sqrt{-1}\omega \text{ for some } \omega \in \text{supp } d\mu \}. \]
Decompose
\[
d\mu = \sum_{j=1}^{r} d\mu_j, \quad \text{supp } d\mu_j \subset \text{supp } d\mu \cap \Gamma_j,
\]
where \( \Gamma_j \) are open convex cones such that \( \text{int } \Gamma_j \neq \emptyset \). By replacing \( d\mu \) and \( \Gamma \) from the first part of the lemma by \( d\mu_j \) and \(- \text{int}(\Gamma_j)\) we obtain
\[
WF^*U_{\mu} \subset \bigcup_{j=1}^{r} \{ (x, \xi) : -\xi \in \Gamma_j, U \text{ is not } \mathcal{O}^* \text{ at } x - \sqrt{-1} \omega \text{ for some } \omega \in \Gamma_j \cap S^{n-1} \}.
\]

If \( -\frac{\xi}{|\xi|} \notin \text{supp } d\mu \) or \( U \) is not \( \mathcal{O}^* \) at \( x - \sqrt{-1} \omega \) for some \( \omega \in \Gamma_j \cap S^{n-1} \), there exists the decomposition of \( d\mu \) such that \( -\xi \notin \Gamma_j \), for \( j = 1, \ldots, r \) or \( -\xi \notin \Gamma_j \) for \( j = 2, \ldots, r \) and \( U \) is \( \mathcal{O}^* \) at \( x + \sqrt{-1} \omega \) for every \( \omega \in \Gamma_j \cap S^{n-1} \). In both cases \( (x, \xi) \notin WF^*U_{\mu} \).

Immediate consequences of this theorem are the next corollary and theorem (see [2], Corollary 8.4.13 and Theorem 8.4.15 for tempered distributions).

**Corollary 1.** Let \( G_j \) be closed subsets of \( S^{n-1} \) such that \( \bigcup_{j=1}^{r} G_j = S^{n-1} \). Any \( u \in S^f(\mathbb{R}^n) \) can be written \( u = \sum_{j=1}^{r} u_j, \quad u_j \in S^f(\mathbb{R}^n) \) \((f \leq *)\) with
a) \( WF^*u_j \subset WF^*u \cap \mathbb{R}^n \times G_j, \quad j = 1, \ldots, r \).

b) \( WF_Au_j \subset WF_Au \cap \mathbb{R}^n \times G_j, \quad j = 1, \ldots, r \).

c) \( WF^*u_j \subset WF^*u_j \cap \mathbb{R}^n \times (G_j \cap G_k) \).

If \( u \in \mathcal{E}^f \), then \( u_j, \quad j = 1, \ldots, r \), have compact supports as well. If \( u = \sum u_j \) is another such decomposition, then
\[
u_j = u_j + \sum_{k} u_{jk}, \quad u_{jk} \in S^f, \quad u_j = -u_{jk} \quad \text{and } WF^*u_{jk} \subset (WF^*u) \cap \mathbb{R}^n \times (G_j \cap G_k).
\]
The same holds for \( WF_Au_{jk} \).

**Theorem 3.** Let \( \Gamma \) be an open convex cone in \( \mathbb{R}^n \), \( u \in D^f(\Omega), \quad \Omega \subset \mathbb{R}^n \), and \( WF_Au \subset \Omega \times \Gamma^0 \). If \( \Omega_1 \subset \subset \Omega \) and \( \Gamma_1 \) is an open convex cone with closure \( \subset \Gamma \cup \{0\} \), then there is an \( F \) analytic in \( \{x + \sqrt{-1}y; x \in \Omega_1, y \in \Gamma_1, |y| < \gamma \} \) such that for some \( k > 0 \) and \( C > 0 \) in the \( (M_p) \) case (resp., for every \( k > 0 \) there is \( C > 0 \) in the \( (M_p) \) case),
\[
|F(x + \sqrt{-1}y)| \leq Ce^{\gamma(k/|y|)}, \quad x \in \Omega_1, \quad y \in \Gamma_1, \quad |y| < \gamma;
\]
and \( F(\cdot + \sqrt{-1}0) - u|_{\Omega_1} \in \mathcal{E}^f(\Omega_1) \).

**REFERENCES**


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