PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 1, January 1998, Pages 85–95
S 0002-9939(98)04462-1

A RIGID SPACE HOMEOMORPHIC TO HILBERT SPACE

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(Communicated by James West)

Abstract. A rigid space is a topological vector space whose endomorphisms are all simply scalar multiples of the identity map. This is in sharp contrast to the behavior of operators on ℓ_2, and so rigid spaces are, from the viewpoint of functional analysis, fundamentally different from Hilbert space. Nevertheless, we show in this paper that a rigid space can be constructed which is topologically homeomorphic to Hilbert space. We do this by demonstrating that the first complete rigid space can be modified slightly to be an AR-space (absolute retract), and thus by a theorem of Dobrowolski and Torunczyk is homeomorphic to ℓ_2.

Rigid topological vector spaces first appeared in the literature in 1977 with an example by Waelbroeck, in the paper [11]. This first rigid space, however, was not complete, and the existence of a complete rigid space was first confirmed by Kalton and Roberts in [5]. In that paper, the constructed space is not only complete and rigid, but is also quotient-rigid and a subspace of $L_0[0,1]$ (quotient-rigid meaning that every quotient of the space inherits the rigid character). A rigid space which serves as the domain space of a non-trivial compact operator was constructed in [9], illustrating that rigid spaces can have relatively rich topologies. In this paper we demonstrate that rigid spaces can in fact be topologically homeomorphic to Hilbert space, thus illustrating the degree to which the two concepts of topological homeomorphism and topological vector space isomorphism can differ.

To do this, we will first modify slightly the rigid space constructed in [5], and then employ a characterization of ANR-spaces (absolute neighborhood retract spaces) due to the first author, which appeared in its original form in [6] and in a refined form in [7]. It is this second version that we will apply in this paper.

The first section of the paper contains the characterization we will apply, as well as an explanation of the modifications we must make to the first rigid space. We will then show in the second and third sections that the rigid space under consideration is an AR-space.

1. Theorems and a construction

We begin with an explanation of the characterization of ANR spaces to be found in [7].

Received by the editors January 23, 1996.
1991 *Mathematics Subject Classification.* Primary 46A16, 54F65; Secondary 46C05, 54G15.

Let $\{\mathcal{U}_n\}$ be a sequence of open covers of a metric space X. For a given open cover $\mathcal{U}_n$, let

$$\operatorname{mesh}(\mathcal{U}_n) = \sup\{\operatorname{diam}(U) : U \in \mathcal{U}_n\}.$$

We say that $\mathcal{U}_n$ is a *zero sequence* if $\operatorname{mesh}(\mathcal{U}_n) \to 0$ as $n \to \infty$. For a given open cover $\mathcal{U}$, we let $\mathcal{N}(\mathcal{U})$ denote the *nerve* of $\mathcal{U}$. The nerve of an open cover is the simplicial complex

$$\{\sigma : \sigma = \langle U_1, \ldots, U_n\rangle, \quad U_i \in \mathcal{U}, \quad n \in \mathbb{N}\}$$

made up of all $\sigma = \langle U_1, \ldots, U_n\rangle$ for which $\bigcap_{i=1}^n U_i \neq \emptyset$. $\mathcal{N}(\mathcal{U})$ is endowed with the Whitehead (or weak) topology (see [1] or [3] for a discussion). Finally, define $\mathcal{U} = \bigcup_{i=1}^{\infty} \mathcal{U}_n$ and let $\mathcal{K}(\mathcal{U}) = \bigcup_{n=1}^{\infty} \mathcal{N}(\mathcal{U}_n \cup \mathcal{U}_{n+1})$, and for any $\sigma \in \mathcal{K}(\mathcal{U})$ let

$$n(\sigma) = \max\{n \in \mathbb{N} : \sigma \in \mathcal{N}(\mathcal{U}_n \cup \mathcal{U}_{n+1})\}.$$

We use the following version of the ANR-characterization theorem; see [6], [7], [8].

Theorem 1.1. *A metric space X with no isolated points is an ANR if and only if there is a zero sequence $\{\mathcal{U}_n\}$ of open covers of X and a map $g : \mathcal{K}(\mathcal{U}) \to X$ such that $g|\mathcal{U} \to X$ is a selection (i.e. $g(U) \in U$ for every $U \in \mathcal{U}$) and such that for any sequence of simplices $\{\sigma_k\}$ in $\mathcal{K}(\mathcal{U})$ with $n(\sigma_k) \to \infty$ and $g(\sigma_k^0) \to x_0 \in X$ we have $g(\sigma_k) \to x_0 \in X$. Here, σ_k^0 represents the collection of vertices $\{U_1, \ldots, U_n\}$ making up the simplex σ_k, and $g(\sigma_k^0)$ and $g(\sigma_k)$ represent the sets of images of, respectively, the vertices of σ_k and the convex combinations of the vertices of σ_k.*

The goal in this paper is to show that the rigid space to be constructed has this property. This then shows that the rigid space is an AR space, and so by the result of Dobrowolski and Torunczyk [2] the space is homeomorphic to ℓ_2 (Dobrowolski and Torunczyk showed that for a complete, separable infinite-dimensional linear metric space, $X \cong \ell_2 \iff X$ is an AR).

The topology of the rigid space will be generated by an F-norm on the space. Recall that an F-norm is defined as follows.

Definition 1.1. Let X be a vector space. A map $||\cdot|| : X \to [0, \infty)$ is an F-norm if

(1) $||x|| = 0 \iff x = 0$,
(2) $||x + y|| \leq ||x|| + ||y||$,
(3) $||\alpha x|| \leq ||x||$ whenever $|\alpha| \leq 1$, and
(4) $||\alpha x|| \to 0$ whenever $|\alpha| \to 0$.

In addition, the construction in [5] often makes use of quasi-norms, similar to F-norms but with the following characteristics:

(1) $||x|| = 0 \iff x = 0$,
(2) $||x + y|| \leq C(||x|| + ||y||)$, C independent of x and y, and
(3) $||\alpha x|| = |\alpha|\, ||x||$, α a scalar.

The type of "norm" in use at a given point in the construction will be clear from the context.

We now proceed with an overview of the construction of the first rigid space, along with our modifications. For simplicity of terminology we first introduce the following definition.

Definition 1.2. We say that a sequence $\{p_n\}$ is a *1-approaching* sequence if the following conditions hold:

$$1/2 = p_0 < p_1 < ... < p_n < ... < 1,$$

$$\lim_{n\to\infty} p_n = 1.$$

Let $\{p_n\}$ be a 1-approaching sequence. We define the space $\ell(\{p_n\})$ by

$$\ell(\{p_n\}) = \{x \equiv \sum_{n=0}^{\infty} x_n e_n : \|x\| \equiv \sum_{n=0}^{\infty} |x_n|^{p_n} < \infty\}, \tag{1}$$

where e_n denotes the characteristic function of $[n, n+1]$.

The space $\ell(\{p_n\})$ is equipped with the F-norm $\|\cdot\|$ defined in (1).

In [5], the existence of a sequence of finite-dimensional spaces $\{V_n\}_{n=0}^{\infty}$ is demonstrated, each a subspace of L_{p_n} and each with basis $\{v_{n,k}\}_{k=1}^{l(n)}$, with the basis elements possessing certain properties and with $1/2 = p_0 < p_1 < \cdots < 1$. For the purposes of that paper an explicit construction of the basis elements was not necessary, but for our purposes we will find it convenient to be more precise. To that end, let $v_{n,k}$ be the characteristic function of the k^{th} sub-interval of $[0, 1]$ of length $l(n)^{-1}$, where $[0, 1]$ has been sub-divided into $l(n)$ essentially disjoint sub-intervals of equal length. Note that with this specification of the basis elements, the spaces V_n have the properties of Lemma 3.1 of [5].

In [5], each V_n is translated by the map τ_n, which takes functions of $[0, 1]$ to functions of $[n, n+1]$ by $\tau_n f(x) = f(x - n)$. Keeping the notation of [5], we will let $U_n = \tau_n V_n$. Let M be the closed linear span of $\{e_n\}$, and let Y be the closure of $\bigcup U_n$, where closure in both cases is relative to the F-norm defined on the space Z of real-valued functions on $[0, \infty)$ by

$$||f|| = \sum_{n=0}^{\infty} \int_n^{n+1} |f(x)|^{p_n}\, dx.$$

As in [5], let $Z(a, b)$ be the subspace of Z of functions with support on $[a, b]$. Lemma 3.2 of [5] proved the following:

Lemma. *Suppose $f \in Z(0, n)$ with $||f|| = 1$. Then there exists a linear operator $A : Z(n, n+1) \to Z(0, n)$ with $Ae_n = f$ and $||A|| = 1$.*

(In this context, $Z(0, n)$ and $Z(n, n+1)$ are equipped with quasi-norms and, as in Banach spaces, $||A|| = \sup_{||x||\leq 1} ||Ax||$.)

We will have to modify this lemma slightly, so that in addition to the above, $A : U_n \to (U_0 + \cdots + U_{n-1})$. To do this, we have to assume of course that $f \in (U_0 + \cdots + U_{n-1})$, but this will be the case in the application of the lemma. As in the proof of Lemma 3.2 of [5], decompose f so that $f = h_0 + \cdots + h_{n-1}$, where $h_i \in U_i$. For each i, decompose h_i so that $h_i = h_1^i + \cdots + h_{l(i)}^i$, with the support of h_j^i being the j^{th} sub-interval of $[i, i+1]$ (recall that $[i, i+1]$ has been partitioned into $l(i)$ sub-intervals of equal length). Let e_j^n denote the characteristic function of the j^{th} sub-interval of $[n, n+1]$, for each $j = 1, \ldots, l(n)$. For the moment, fix i. By Lemma 2.2 of [5] there exist linear operators F_j^i, each mapping the ${p_n}^{\text{th}}$–integrable functions with support on the j^{th} sub-interval of $[n, n+1]$ to the ${p_i}^{\text{th}}$–integrable functions with support on the j^{th} sub-interval of $[i, i+1]$, for each $j = 1, \ldots, l(i)$. Furthermore,

$$F_j^i e_j^n = h_j^i \text{ and } \left|\left|F_j^i\right|\right| = \left|\left|h_j^i\right|\right|.$$

For $j = l(i)+1, \ldots, l(n)$, let F_j^i be the zero operator, and let $F_i = F_1^i + \cdots + F_{l(n)}^i$. Then $F_i e_n = h_i$ and $F_i : U_n \to U_i$. Also, due to the partitioning of the intervals,

$$\begin{aligned} ||F_i||^{p_i} &= \sup_{||x|| \leq 1} \int |F_1^i x + \cdots + F_{l(n)}^i x|^{p_i} = \sup_{||x|| \leq 1} \left(\int |F_1^i x|^{p_i} + \cdots + \int |F_{l(n)}^i x|^{p_i} \right) \\ &\leq \left|\left|F_1^i\right|\right|^{p_i} + \cdots + \left|\left|F_{l(n)}^i\right|\right|^{p_i} = \left|\left|h_1^i\right|\right|^{p_i} + \cdots + \left|\left|h_{l(i)}^i\right|\right|^{p_i} = ||h_i||^{p_i}. \end{aligned}$$

At this point the remainder of the proof is as in Lemma 3.2 of [5], with the operator A being defined by $A = F_0 + \cdots + F_{n-1}$.

The next step in the construction of the rigid space involves defining an operator S mapping Z to Z. This begins with the selection of a sequence of elements $\{\gamma_k\}_{k=1}^{\infty}$, with $\gamma_k \in U_0 + \cdots + U_{k-1}$. By Lemma 3.2 of [5], operators $A_k : Z(k, k+1) \to Z(0, k)$ with $||A_k|| = 1$ and $A_k e_k = \gamma_k$ can be chosen. Note that by our modification to Lemma 3.2 above, we can assume also that $A_k : U_k \to (U_0 + \cdots + U_{k-1})$. A map $T : Z \to Z$ is then defined by $T = \sum_{k=1}^{\infty} c_k A_k E_k$ (each E_k is the projection map from Z onto $Z(k, k+1)$, and $\{c_k\}$ is a sequence of reals). Our modification to Lemma 3.2 implies that, in addition, T maps Y into Y. Let $\tilde{T}$ denote the restriction of T to the subspace Y. Finally, the map $S : Z \to Z$ is defined by $S = I - T$. We will want to work with the restriction map $\tilde{S} : Y \to Y$ as well, defined by $\tilde{S} = I - \tilde{T}$. In [5] it is shown that $||T|| \leq 1/4$, and that therefore S is invertible. Combining this argument with the fact that we have made $T : Y \to Y$, we obtain $\left|\left|\tilde{T}\right|\right| \leq 1/4$, and therefore $\tilde{S} : Y \to Y$ is also invertible.

Kalton and Roberts then showed in [5] that the sequences $\{l(n)\}$ and $\{p_n\}$ can be chosen so that the quotient space $X = Y/S(M)$, where $M = \ell(\{p_n\})$, is a rigid space.

Remark 1.1. In [5], the sequences $\{p_n\}$ and $\{l(n)\}$ are constructed inductively. In the inductive step, p_n is chosen sufficiently close to 1 so as to obtain the desired behavior, and $l(n)$ then depends on p_n. In our modification of the rigid space, we will want to choose p_n possibly closer to 1 in the inductive step, for reasons given in the proof of Theorem 3.5. Since our choice of p_n is, if anything, larger than the choice of p_n in [5], this has no impact on the construction of the rigid space.

2. The AR-property for Y

In this section we prove the following theorem:

Theorem 2.1. *Y is an AR.*

Proof. We aim to verify the conditions of Theorem 1.1. Let $\{\mathcal{U}_n\}$ be a zero sequence of open covers of Y. Let $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$, $\mathcal{K}(\mathcal{U}) = \bigcup_{n=1}^{\infty} \mathcal{N}(\mathcal{U}_n \cup \mathcal{U}_{n+1})$, and let $f_0 : \mathcal{U} \to Y$ be a selection.

We extend f_0 to a map $f : \mathcal{K}(\mathcal{U}) \to Y$ as follows. For any simplex $\sigma = \langle U_1, \ldots, U_m \rangle \in \mathcal{K}(\mathcal{U})$, $U_j \in \mathcal{U}$ for $j = 1, ..., m$. Since $f_0(U_j) \in Y$, we have

$$f_0(U_j) = \sum_{n=0}^{\infty} \sum_{i=1}^{l(n)} x_{j,i}^n e_i^n, \quad j = 1, ..., m,$$

where e_i^n represents the characteristic function of i^{th} sub-interval of $[n, n+1]$.

For every $x \in \sigma$, with

$$x = \sum_{j=1}^{m} \lambda_j U_j, \quad \lambda_j \geq 0, j = 1, \dots, m \text{ and } \sum_{j=1}^{m} \lambda_j = 1,$$

we define

$$f(x) = \sum_{n=0}^{\infty} \sum_{i=1}^{l(n)} \left| \sum_{j=1}^{m} \lambda_j |x_{j,i}^n|^{p_n} \tau(x_{j,i}^n) \right|^{1/p_n} \tau_i^n e_i^n, \tag{2}$$

where $\tau : \mathbb{R} \to \mathbb{R}$ is the sign function:

$$\tau(t) = \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ -1 & \text{if } t < 0, \end{cases} \tag{3}$$

and

$$\tau_i^n = \tau\left(\sum_{j=1}^{m} \lambda_j |x_{j,i}^n|^{p_n} \tau(x_{j,i}^n) \right).$$

Observe that for every $U \in \mathcal{U}$ we have

$$\begin{aligned} f(U) &= \sum_{n=0}^{\infty} \sum_{i=1}^{l(n)} \left| |x_i^n|^{p_n} \tau(x_i^n) \right|^{1/p_n} \tau(x_i^n) e_i^n \\ &= \sum_{n=0}^{\infty} \sum_{i=1}^{l(n)} x_i^n e_i^n = f_0(U). \end{aligned}$$

Therefore $f|\mathcal{U} = f_0$.

Now assume that $\{\sigma_k\}$ is a sequence of simplices in $\mathcal{K}(\mathcal{U})$ with $n(\sigma_k) \to \infty$, such that $f(\sigma_k^0) \to x_0 \in Y$ as $k \to \infty$. We have to show that $f(\sigma_k) \to x_0$ as $k \to \infty$.

Since $x_0 \in Y$, we have

$$x_0 = \sum_{n=0}^{\infty} \sum_{i=1}^{l(n)} x_i^n e_i^n.$$

Let $\sigma_k = \langle U_1^k, \dots, U_{m(k)}^k \rangle$. Then we have

$$f(U_j^k) = \sum_{n=0}^{\infty} \sum_{i=1}^{l(n)} x_{j,i}^n(k) e_i^n, \quad j = 1, \dots, m(k).$$

For every $x_k \in \sigma_k$, with

$$x_k = \sum_{j=1}^{m(k)} \lambda_j(k) U_j^k, \quad \lambda_j(k) \geq 0, j = 1, \dots, m(k) \text{ and } \sum_{j=1}^{m(k)} \lambda_j(k) = 1,$$

we have

$$f(x_k) = \sum_{n=0}^{\infty} \sum_{i=1}^{l(n)} \left| \sum_{j=1}^{m(k)} \lambda_j(k) |x_{j,i}^n(k)|^{p_n} \tau(x_{j,i}^n(k)) \right|^{1/p_n} \tau_i^n(k) e_i^n, \text{ see (2)},$$

where

$$\tau_i^n(k) = \tau\left(\sum_{j=1}^{m(k)} \lambda_j(k)|x_{j,i}^n(k)|^{p_n}\tau(x_{j,i}^n(k))\right), \text{ see (3).}$$

Since $f(\sigma_k^0) \to x_0$, we have

$$\max\{\|f(U_j^k) - x_0\|, j = 1, ..., m(k)\} \to 0 \text{ as } k \to \infty.$$

Therefore

$$\max\left\{\sum_{n=0}^{\infty}\sum_{i=1}^{l(n)} l(n)^{-1}|x_{j,i}^n(k) - x_i^n|^{p_n}, j = 1, ..., m(k)\right\} \to 0 \tag{4}$$

as $k \to \infty$.

Now given $\epsilon > 0$, take $k_0 \in \mathbb{N}$ such that

$$\max\left\{\sum_{n=0}^{\infty}\sum_{i=1}^{l(n)} l(n)^{-1}|x_{j,i}^n(k) - x_i^n|^{p_n}, j = 1, ..., m(k)\right\} < \epsilon \tag{5}$$

whenever $k > k_0$.

Take $N \in \mathbb{N}$ so that

$$\sum_{n=N+1}^{\infty}\sum_{i=1}^{l(n)} l(n)^{-1}|x_i^n|^{p_n} < \epsilon.$$

Then for $k > k_0$ and $j = 1, ..., m(k)$ we get

$$\begin{aligned}\sum_{n=N+1}^{\infty}\sum_{i=1}^{l(n)} l(n)^{-1}|x_{j,i}^n(k)|^{p_n} &\le \sum_{n=N+1}^{\infty}\sum_{i=1}^{l(n)} l(n)^{-1}|x_{j,i}^n(k) - x_i^n|^{p_n} \\ &\quad + \sum_{n=N+1}^{\infty}\sum_{i=1}^{l(n)} l(n)^{-1}|x_i^n|^{p_n} \\ &< \epsilon + \epsilon = 2\epsilon.\end{aligned} \tag{6}$$

Therefore

(7)

$$\begin{aligned}\sum_{n=N+1}^{\infty}\sum_{i=1}^{l(n)} l(n)^{-1}\sum_{j=1}^{m(k)}\lambda_j(k)|x_{j,i}^n(k)|^{p_n} &= \sum_{j=1}^{m(k)}\lambda_j(k)\left(\sum_{n=N+1}^{\infty}\sum_{i=1}^{l(n)} l(n)^{-1}|x_{j,i}^n(k)|^{p_n}\right) \\ &< \sum_{j=1}^{m(k)}\lambda_j(k)(2\epsilon) = 2\epsilon.\end{aligned}$$

Denote

$$A_i^n(k) = \left|\left|\sum_{j=1}^{m(k)}\lambda_j(k)|x_{j,i}^n(k)|^{p_n}\tau(x_{j,i}^n(k))\right|^{1/p_n}\tau_i^n(k) - x_i^n\right|^{p_n}. \tag{8}$$

We claim that there exists a $\delta_i^n > 0$ such that

$$A_i^n(k) < 2^{-n}\epsilon \text{ whenever } |x_{j,i}^n(k) - x_i^n| < (\delta_i^n)^{1/p_n}. \tag{9}$$

To prove the claim we consider two cases:

Case 1. $x_i^n = 0$. Take $\delta_i^n = 2^{-n}\epsilon$. Then we get

$$\begin{aligned} A_i^n(k) &= \left|\sum_{j=1}^{m(k)} \lambda_j(k)|x_{j,i}^n(k)|^{p_n}\tau(x_{j,i}^n(k))\right| \\ &< \sum_{j=1}^{m(k)} \lambda_j(k)2^{-n}\epsilon = 2^{-n}\epsilon. \end{aligned}$$

Case 2. $x_i^n \neq 0$. We may assume that $x_i^n > 0$ (the case $x_i^n < 0$ is similar). Take $\delta_i^n < \min\{x_i^n, 2^{-n}\epsilon\}$. Then the inequality

$$|x_{j,i}^n(k) - x_i^n| < (\delta_i^n)^{1/p_n}$$

implies that

$$x_i^n - (\delta_i^n)^{1/p_n} < x_{j,i}^n(k) < x_i^n + (\delta_i^n)^{1/p_n}.$$

Since $\sum_{j=1}^{m(k)} \lambda_j(k) = 1$, it follows that

$$\left|\left|\sum_{j=1}^{m(k)} \lambda_j(k)|x_{j,i}^n(k)|^{p_n}\tau(x_{j,i}^n(k))\right|^{1/p_n} \tau_i^n(k) - x_i^n\right|^{p_n} < \delta_i^n < 2^{-n}\epsilon.$$

The claim is proved.

From (4) we get

$$|x_{j,i}^n(k) - x_i^n| \to 0 \text{ as } k \to \infty$$

for every $n = 0, 1, ...$ and $i = 1, ..., l(n)$.

For every $n = 0, ..., N$, take $K(n) \in \mathbb{N}$ such that

$$\max\ \left\{|x_{j,i}^n(k) - x_i^n|^{p_n}, j = 1, ..., m(k)\right\} < (\delta_i^n)^{1/p_n} \tag{10}$$

whenever $k \geq K(n)$ and $i = 1, ..., l(n)$. Denote

$$K = \ \max\ \{k_0, K(0), ..., K(N)\}.$$

Then from (8), (9) and (10) we have

$$A_i^n(k) = \left|\left|\sum_{j=1}^{m(k)} \lambda_j(k)|x_{j,i}^n(k)|^{p_n}\tau(x_{j,i}^n(k))\right|^{1/p_n} \tau_i^n(k) - x_i^n\right|^{p_n} < 2^{-n}\epsilon$$

whenever $k > K$.

Consequently for $k > K$, we get

$$\begin{aligned} \sum_{n=0}^{N}\sum_{i=1}^{l(n)} l(n)^{-1}A_i^n(k) &< \sum_{n=0}^{N}\sum_{i=1}^{l(n)} l(n)^{-1}2^{-n}\epsilon \\ &< \sum_{n=0}^{\infty} 2^{-n}\epsilon < 2\epsilon \end{aligned} \tag{11}$$

and

$$\begin{aligned}\sum_{n=N+1}^{\infty}\sum_{i=1}^{l(n)} l(n)^{-1}A_i^n(k) &\le \sum_{n=N+1}^{\infty}\sum_{i=1}^{l(n)} l(n)^{-1}\sum_{j=1}^{m(k)}\lambda_j(k)|x_{j,i}^n(k)|^{p_n}\\ &\quad+\sum_{n=N+1}^{\infty}\sum_{i=1}^{l(n)} l(n)^{-1}\sum_{j=1}^{m(k)}\lambda_j(k)|x_i^n|^{p_n}\\ &< 2\epsilon+\epsilon = 3\epsilon.\end{aligned}\tag{12}$$

Therefore from (11) and (12) we obtain

$$\begin{aligned}\|f(x_k)-x_0\| &= \sum_{n=0}^{\infty}\sum_{i=1}^{l(n)} l(n)^{-1}\left|\left|\sum_{j=1}^{m}\lambda_j|x_{j,i}^n(k)|^{p_n}\tau(x_{j,i}^n)(k)\right|^{1/p_n}\tau_i^n - x_i^n\right|^{p_n}\\ &= \sum_{n=0}^{N}\sum_{i=1}^{l(n)} l(n)^{-1}A_i^n(k)+\sum_{n=N+1}^{\infty}\sum_{i=1}^{l(n)} l(n)^{-1}A_i^n(k)\\ &< 2\epsilon+3\epsilon = 5\epsilon\end{aligned}$$

whenever $k > K$.

Consequently $f(\sigma_k) \to x_0$ as $k \to \infty$. Hence Y is an ANR by Theorem 1.1. Since Y is contractible, Y is an AR. □

Remark 2.1. Observe that if $l(n) = 1$ for every $n \in \mathbb{N}$, then $Y = M$. Therefore from Theorem 2.1 we obtain that M is also an AR. In the next section we shall show that for a certain choice of 1-approaching sequence $\{p_n\}$, the space $M = \ell(\{p_n\})$ is locally convex.

3. Proof of the main result

We will begin with a lemma concerning 1-approaching sequences.

Lemma 3.1. *There exists a 1-approaching sequence $\{p_n\}$ such that for any $n \in \mathbb{N}$ and for every $x_1, \dots, x_n \ge 0$, with $\sum_{i=0}^{n} x_i \le 1$,*

$$x_0^{p_0} + ... + x_n^{p_n} + (1 - x_0 - ... - x_n)^{p_{n+1}} < 3. \tag{13}$$

Proof. We will use the fact that the function

$$f(x_0, ..., x_n) = x_0^{p_0} + ... + x_n^{p_n} + 1 - x_0 - ... - x_n,$$

where $x_1, \dots, x_n \ge 0$ and $\sum_{i=0}^{n} x_i \le 1$, attains the maximum

$$f_{\max} = p_0^{\frac{p_0}{1-p_0}} + ... + p_n^{\frac{p_n}{1-p_n}} + 1 - p_0^{\frac{1}{1-p_0}} - ... - p_n^{\frac{1}{1-p_n}}$$

at $x_i = p_i^{\frac{1}{1-p_i}}$, $i = 0, ..., n$. This may be easily confirmed by the use of Gundelfinger's Rule (see, for instance, p. 219 of [10]).

Now, to prove the lemma, first choose $p_0 = 1/2$. Assume that p_i has been chosen up to n, with

$$(1 + 2^{-i})^{-1} \le p_i < 1, i = 0, ..., n, \tag{14}$$

such that condition (13) holds. Consider the function

$$f(x_0, ..., x_n) = x_0^{p_0} + ... + x_n^{p_n} + 1 - x_0 - ... - x_n,$$

where $x_i \geq 0$, $i = 0, ..., n$, and $\sum_{i=0}^n x_i \leq 1$. By the above fact and by (14) we have

$$\begin{aligned} f(x_0, ..., x_n) &\leq p_0^{\frac{p_0}{1-p_0}} + ... + p_n{}^{\frac{p_n}{1-p_n}} + 1 - p_0^{\frac{1}{1-p_0}} - ... - p_n{}^{\frac{1}{1-p_n}} \\ &= 1 + \frac{1}{4} + \sum_{i=1}^n p_i^{\frac{1}{1-p_i}}(p_i^{-1} - 1) \\ &< \frac{5}{4} + \sum_{i=1}^n (p_i^{-1} - 1) \leq \frac{5}{4} + \sum_{i=1}^n 2^{-i} < \frac{9}{4} \end{aligned}$$

for any $x_i \geq 0$, $i = 0, ..., n$, with $\sum_{i=0}^n x_i \leq 1$.

Therefore we can choose p_{n+1}, with

$$(1 + 2^{-n-1})^{-1} \leq p_{n+1} < 1$$

such that

$$x_0^{p_0} + ... + x_n^{p_n} + (1 - x_0 - ... - x_n)^{p_{n+1}} < 3$$

for any $x_i \geq 0$, $i = 0, ..., n$, with $\sum_{i=0}^n x_i \leq 1$. □

We will use Lemma 3.1 to show that the space $M = \ell(\{p_n\})$ can be assumed to be locally convex.

Theorem 3.2. *There exists a 1-approaching sequence $\{p_n\}$ such that $\ell(\{p_n\})$ is a locally convex space.*

Proof. Let $x^0, ..., x^k \in \ell(\{p_n\})$, $x^i = \sum_{n=0}^\infty x_n^i e_n$, with

$$\|x^i\| = \sum_{n=0}^\infty |x_n^i|^{p_n} < 1 \text{ for every } i = 0, ..., k.$$

Let $\alpha_i \geq 0, i = 0, ..., k$, with $\sum_{i=0}^k \alpha_i = 1$. We will show that $\| \sum_{i=0}^k \alpha_i x^i \| < 3$.

Observe that

$$\sum_{i=1}^k \alpha_i x^i = \sum_{n=0}^\infty (\sum_{i=1}^k \alpha_i x_n^i) e_n = \sum_{n=0}^\infty \lambda_n e_n,$$

where $\lambda_n = \sum_{i=1}^k \alpha_i x_n^i$. Then we get

$$\begin{aligned} \sum_{n=0}^\infty |\lambda_n| = \sum_{n=0}^\infty |\sum_{i=1}^k \alpha_i x_n^i| &\leq \sum_{i=1}^k \alpha_i \sum_{n=0}^\infty |x_n^i| \\ &\leq \sum_{i=1}^k \alpha_i \sum_{n=0}^\infty |x_n^i|^{p_n} < \sum_{i=1}^k \alpha_i = 1. \end{aligned}$$

Therefore from Lemma 3.1 we get

$$\| \sum_{i=1}^k \alpha_i x^i \| = \sum_{n=0}^\infty |\lambda_n|^{p_n} < 3.$$

This uniform bound on convex combinations of elements from within the unit ball implies the existence of a local base at 0 of convex sets, and so $\ell(\{p_n\})$ is locally convex. □

Remark 3.1. Observe that by the proof of Theorem 3.2 we get the following stronger result: there exists a 1-approaching sequence $\{p_n^0\}$ such that for any 1-approaching sequence $\{p_n\}$, with $p_n \geq p_n^0$, the resulting space $M = \ell(\{p_n\})$ is locally convex.

Remark 3.2. It is natural to ask whether $\ell(\{p_n\})$ is locally convex for any 1-approaching sequence $\{p_n\}$. The answer to this question is no, as we shall see by the following example.

Example. Let $\{t_n\}$ be any 1-approaching sequence. Take a sequence $\{m(n)\}$ of natural numbers such that $m(n)^{1-t_n} > n$ for every $n \in \mathbb{N}$. Now let $\{p_k\}$ be any 1-approaching sequence for which $p_{m(n)} = t_n$. Then the resulting space $\ell(\{p_k\})$ is not locally convex.

Proof. We have

$$\begin{aligned}\|\sum_{i=1}^{m(n)} \frac{1}{m(n)} e_i\| &= \sum_{i=1}^{m(n)} |\frac{1}{m(n)}|^{p_i} > \sum_{i=1}^{m(n)} |\frac{1}{m(n)}|^{t_n} \\ &= m(n)^{1-t_n} > n \to \infty\end{aligned}$$

as $n \to \infty$. Therefore $\ell(\{p_k\})$ is not locally convex. □

Let X be a linear metric space, let M denote a closed linear subspace of X, let $E = X/M$ denote the quotient space and let $\pi : X \to E$ denote the quotient map. We say that a map $g : E \to X$ is a *selection* if $g(x) \in \pi^{-1}(x)$ for every $x \in E$. The proof of our result uses the following theorem of Michael, see [1], Proposition 7-1, p.87.

Theorem 3.3. *Let M be a locally convex closed linear subspace of a complete linear metric space X. Then there exists a continuous selection $g : E \to X$.*

From Theorem 3.3 we get

Theorem 3.4. *Let M be a locally convex closed linear subspace of a complete linear metric space X. If X is an AR, then X/M is an AR.*

Proof. Let $f_0 : A \to X/M$ be a continuous map from a closed subset A of a metric space Z into X/M. Since M is locally convex, by Theorem 3.3 there exists a selection $g : X/M \to X$. Since X is an AR, there exists a continuous map $h : Z \to X$ such that $h|A = g \circ f_0$. Then $f = \pi \circ h$, where $\pi : X \to X/M$ denotes the quotient map, is an extension of f_0.

Consequently, X/M is an AR, and the theorem is proved. □

Now we are able to prove our main result in this paper.

Theorem 3.5. *There exists a 1-approaching sequence $\{p_n\}$ such that the resulting rigid space $X = Y/S(M)$ constructed in Section 1 is an AR, and therefore is homeomorphic to ℓ_2.*

Proof. First observe that by Remark 1.1 we can assume that the 1-approaching sequence $\{p_n\}$ used in the construction of the rigid space is such that the space M is locally convex, by first identifying a sequence $\{\tilde{p}_n\}$ which will satisfy Lemma 3.1 and then specifying that in the inductive step in the creation of the rigid space, p_n is chosen to be at least as large as $\tilde{p}_n$.

Now, since S is an isomorphism, $S(M)$ is locally convex. By Theorem 2.1 Y is an AR, and so $Y/S(M)$ is an AR by Theorem 3.4. □

References

1. C. Bessaga and A. Pelczynski, *Selected Topics in Infinite-Dimensional Topology*, PWN – Polish Scientific Publishers, Warsaw, 1975. MR **57:**17657
2. T. Dobrowolski and H. Torunczyk, *On Metric Linear Spaces Homeomorphic to ℓ_2 and Compact Convex Sets Homeomorphic to Q*, Bull. Acad. Polon. Sci. Ser. Sci. Math. **27** (1979), 883–887. MR **82j:**57010
3. J. Dugundji, *Topology*, Allyn and Bacon, Inc., Boston, 1966. MR **33:**1824
4. N.J. Kalton, N.T. Peck and J.W. Roberts, *An F-Space Sampler*, Cambridge University Press, Cambridge, 1984. MR **87c:**46002
5. N.J. Kalton and J.W. Roberts, *A Rigid Subspace of L_0*, Trans. Amer. Math. Soc. **266** (1981), 645–654. MR **82j:**46039
6. Nguyen To Nhu, *Investigating the ANR-property of Metric Spaces*, Fund. Math. **124** (1984), 243–254; **141** (1992), 297. MR **86d:**54018; MR **93k:**54042
7. Nguyen To Nhu and K. Sakai, *The Compact Neighborhood Extension Property and Local Equi-connectedness*, Proc. Amer. Math. Soc. **121** (1994), 259–265. MR **94g:**54009
8. Nguyen To Nhu and Ta Khac Cu, *Probability Measure Functors Preserving the ANR-property of Metric Spaces*, Proc. Amer. Math. Soc. **106** (1989), 493–501. MR **89m:**60013
9. P. Sisson, *A Rigid Space Admitting Compact Operators*, Studia Math. **112** (1995), 213–228. MR **96d:**46003
10. A. Taylor and W. Mann, *Advanced Calculus, Third Edition*, John Wiley and Sons, Inc., New York, 1983. MR **83m:**26001
11. L. Waelbroeck, *A Rigid Topological Vector Space*, Studia Math. **59** (1977), 227–234. MR **55:**3730

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