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CORRECTED OUTER FUNCTIONS

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ABSTRACT. Given $0 < p < 2$ and a strictly positive continuous function φ on the unit circle, we construct a bounded analytic function g such that $|g^*| = \varphi$ a.e., and g is in the Besov space A^1_p on the unit disc.

1. INTRODUCTION

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{T} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ be the unit disc and the unit circle. Denote by m_2 and m_1 the corresponding Lebesgue measures, $m_2(\mathbb{D}) = 1$, $m_1(\mathbb{T}) = 1$. $H(\mathbb{D})$ is the space of all analytic functions $f : \mathbb{D} \to \mathbb{C}$.

Put $H^\infty := \{f \in H(\mathbb{D}) : f \text{ is bounded}\}$, $A(\mathbb{D}) = H(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ (the disc-algebra). Let $0 < p \leq 2$; then define

$$\begin{aligned} A^1_p(\mathbb{D}) = \{f \in H(\mathbb{D}) : \|f\|^p_{A^1_p(\mathbb{D})} &= \|f'\|^p_{A_p(\mathbb{D})} \\ &= \int_{\mathbb{D}} |f'(z)|^p (1-|z|)^{p-1}\, dm_2(z) < \infty\}. \end{aligned}$$

Since $\|\cdot\|_{A^1_p(\mathbb{D})}$ is the Besov (quasi) norm, we say that $A^1_p(\mathbb{D})$ is the analytic Besov space. Recall some inclusions between $A^1_p(\mathbb{D})$ and other classical spaces of analytic functions. Let $\ell^p_A = \{f \in H(\mathbb{D}) : \{\hat{f}(n)\}_{n\geq 0} \in \ell^p\}$ and H^p be the Hardy class; then $\ell^p_A \subset A^1_p(\mathbb{D}) \subset H^p$, $0 < p \leq 2$. In particular, $A^1_2(\mathbb{D}) = H^2$.

The aim of the present paper is to prove the following result (we use the symbol g^* to denote the boundary values of $g \in H^\infty$).

Theorem. *Let $0 < p < 2$ and $\varphi \in C(\mathbb{T})$, $\varphi > 0$. Then there exists a function $g \in H^\infty \cap A^1_p(\mathbb{D})$ such that $|g^*| = \varphi$ m_1-almost everywhere.*

Remark. This theorem holds also for some non-continuous functions φ and for non-negative φ with some zeros. Moreover, these results are true in the unit ball of $\mathbb{C}^n$, $n \geq 2$. We will not discuss these generalizations in the present paper.

If $p = 1$, then the result under question was obtained in [2]. Note that the theorem is interesting for small $p > 0$, since $H^\infty \cap A^1_p(\mathbb{D}) \subset H^\infty \cap A^1_q(\mathbb{D})$ if $0 < p < q$.

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Indeed, suppose that $f \in H^\infty$; then, by Cauchy's inequality, $|f'(z)|(1-|z|) \le \mathrm{const}$, $z \in \mathbb{D}$, and therefore

$$\int_{\mathbb{D}} |f'(z)|^{(q-p)+p}(1-|z|)^{(q-p)+p-1} dm_2(z) \le \mathrm{const} \int_{\mathbb{D}} |f'(z)|^p (1-|z|)^{p-1} dm_2(z).$$

To prove the theorem, we apply, as in [2], the approximation construction of A. B. Aleksandrov in $L^p(\mathbb{T})$, $0 < p < 1$ (see [1]). Recall that in [1] this construction yields a solution of the inner function problem in the unit ball of $\mathbb{C}^n$.

Comments.

1. The point of the theorem is the restriction $g \in A^1_p(\mathbb{D})$. Indeed, given a bounded modulus $\varphi \ge 0$, $\log\varphi \in L^1(\mathbb{T})$, the classical *outer* (in sense of Beurling) function is defined by the formula

$$O_\varphi(z) = \exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log\varphi(\zeta)\, dm_1(\zeta)\right), \quad z \in \mathbb{D}.$$

Recall that O_φ satisfies the equality under consideration $|O^*_\varphi| = \varphi$ m_1-a.e. (for further details about the inner-outer factorization see [3]). Therefore, it is important to note that there exists $\varphi \in C(\mathbb{T})$, $\varphi > 0$, such that $O_\varphi \notin A^1_p(\mathbb{D})$ for all $0 < p < 2$ (this has been known for a long time, at least for $p = 1$, see [5] and [6]). For example, the following argument gives the proof:

If $f \in A^1_p(\mathbb{D})$, $1 \le p < 2$, then $\{\hat{f}(2^n)\}_{n\ge 0} \in \ell^p$. On the other hand, given a sequence $\{x_n\}_{n\ge0} \in \ell^2$, there exists $g \in A(\mathbb{D})$ such that $\hat{g}(2^n) = x_n$, $n \in \mathbb{Z}_+$. Hence, if we take $\{x_n\} \in \ell^2 \setminus \ell^p$ for all $0 < p < 2$, we obtain a function $g \in A(\mathbb{D}) \setminus A^1_p(\mathbb{D})$ for all $0 < p < 2$. To finish the argument, put $h = g + 2\|g\|_\infty$; then $|h| > 0$ and h is outer.

2. The theorem has an interpretation in terms of the inner-outer factorization. Indeed, let $0 < q < 2$, $\varphi \in C(\mathbb{T})$, $\varphi > 0$, and $O_\varphi \notin \bigcup_{0<p<2} A^1_p(\mathbb{D})$. Then there exists an inner function I_φ such that $O_\varphi I_\varphi \in A^1_q(\mathbb{D})$. In other words, the outer function O_φ is *corrected* by I_φ.

Given a space $\mathcal{E} \subset H(\mathbb{D})$, recall one notion which is important for investigation of z-invariant subspaces of $\mathcal{E}$.

Definition. Let I be an inner function. We say that I divides $\mathcal{E}$ if

$$IF \in \mathcal{E} \Longrightarrow F \in \mathcal{E} \quad \text{for all } F \in H^q,\ q > 0.$$

Let φ be as above; then the theorem says, in particular, that I_φ does not divide $H^\infty \cap A^1_p(\mathbb{D})$. We refer the reader to the paper [8] for other results on division and non-division by inner functions in the spaces $H^\infty \cap A^1_p(\mathbb{D})$, $0 < p < 2$.

3. Let $0 < p$, $q < \infty$. Then define

$$A^1_{pq}(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \int_{\mathbb{D}} |f'(z)|^p (1-|z|)^{q-1}\, dm_2(z) < \infty \right\}.$$

It is necessary to explain why we consider the case $q = p$ only.

Indeed, let $q > p$ and $f \in H^\infty$; then

$$\int_{\mathbb{D}} |f'(z)|^p (1-|z|)^{q-1}\, dm_2(z) \le \mathrm{const} \int_{\mathbb{D}} (1-|z|)^{q-p-1}\, dm_2(z) < \infty,$$

in other words, $H^\infty \subset A^1_{pq}(\mathbb{D})$, and the theorem is trivial in this case.

On the other hand, let $p > q > p-1$, $q \ge 1$ and $f \in A^1_{pq}(\mathbb{D})$; then (see, e.g, [4, p.67])

$$\int_{-\pi}^{\pi} |t|^{q-p-1} \int_{-\pi}^{\pi} |f(e^{i(x+t)}) - f(e^{ix})|^p \, dx \, dt < \infty.$$

Since there exists a modulus $\varphi \in C(\mathbb{T})$, $\varphi > 0$, such that

$$\int_{-\pi}^{\pi} |t|^{q-p-1} \int_{-\pi}^{\pi} |\varphi(e^{i(x+t)}) - \varphi(e^{ix})|^p \, dx \, dt = \infty,$$

the theorem is not valid for all $q < p$.

2. Auxiliary results

Lemma 2.1 (see, e.g., [7, p.17], where the proof is given even in the ball of $\mathbb{C}^n$). *Let $w \in \mathbb{D}$, $a > 0$, $b > -1$. Then*

$$\int_{\mathbb{T}} \frac{dm_1(\zeta)}{|1-\zeta\overline{w}|^{1+a}} \le \frac{\mathrm{const}(a)}{(1-|w|)^a},$$
$$\int_{\mathbb{D}} \frac{(1-|z|)^b \, dm_2(z)}{|1-z\overline{w}|^{2+a+b}} \le \frac{\mathrm{const}(a,b)}{(1-|w|)^a}.$$

Lemma 2.2. *Let $d \in \mathbb{N}$, $0 < p < 1$, $pd \ge 2$, and*

$$h(t,z) = \frac{i}{(2+t-tz)^d}, \quad t \ge 2, \ z \in \overline{\mathbb{D}}.$$

Then there exist constants $\alpha = \alpha(p,d) \in (0,1)$ and $M_0 = M_0(p,d) \ge 4$ such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left| \mathbb{1}_{[-\Delta,\Delta]}(\theta) - \mathrm{Re}\, h(M_0\Delta^{-1}, e^{i\theta})\right|^p \, d\theta < \alpha \cdot \Delta/\pi, \tag{2.1}$$

$$\|h(M_0\Delta^{-1},\cdot)\|^p_{L^p(\mathbb{T})} < \Delta/\pi, \tag{2.2}$$

$$\|h'(M_0\Delta^{-1},\cdot)\|^p_{A_p(\mathbb{D})} < \Delta/\pi \tag{2.3}$$

for all $\Delta \in (0, \pi/4)$.

Proof. 1. To prove (2.1), we estimate the value of

$$\mathrm{Re}\, h(t, e^{i\theta}) = \frac{-\mathrm{Im}\,(2 + t(1-\cos\theta) + it\sin\theta)^d}{|2+t(1-\cos\theta) - it\sin\theta|^{2d}}.$$

If $|t\theta| < 1$, then $|2 + t(1-\cos\theta) + it\sin\theta| \le 4$. On the other hand, there exists $\varepsilon = \varepsilon(d) \in (0,1)$ such that

$$2|\mathrm{Im}\,(2+t(1-\cos\theta) + it\sin\theta)^d| \ge 2^{d-1}|t\theta| \quad \text{for all } |t\theta| < \varepsilon, \ t \ge 2$$

(we killed the higher degrees of $|t\theta|$). Put $t = M\Delta^{-1}$, $M \ge 2$; then

$$\frac{1}{2\pi}\int_{-\Delta}^{\Delta} |\mathrm{Re}\, h(t,e^{i\theta})|^2 \, d\theta > \frac{1}{2\pi}\int_{-\varepsilon/t}^{\varepsilon/t} |\mathrm{Re}\, h(t,e^{i\theta})|^2 \, d\theta \ge C(d)t^{-1} = C(d)\frac{\Delta}{M}.$$

Let $0 < p < 1$ and $x \in [-1,1]$; then

$$\frac{1}{2}((1+x)^p + (1-x)^p) \le 1 - \frac{p(1-p)}{2}x^2.$$

Note that $\operatorname{Re} h(t, e^{-i\theta}) = -\operatorname{Re} h(t, e^{i\theta})$; thus

$$\frac{1}{2\pi}\int_{-\Delta}^{\Delta} |1 - \operatorname{Re} h(t, e^{i\theta})|^p \, d\theta < \Delta\left(\frac{1}{\pi} - \frac{C_1(p,d)}{M}\right), \tag{2.4}$$

where $C_1(p,d) > 0$.

Now we estimate the complementary integral.

If $|\theta| \in [\Delta, \pi/2]$, then $|h(t, e^{i\theta})| \le |t \sin\theta|^{-d} \le C(d)|t\theta|^{-d}$. On the other hand, if $|\theta| \in [\pi/2, \pi]$, then $|h(t, e^{i\theta})| \le |t|^{-d} \le C(d)|t\theta|^{-d}$. For $t = M\Delta^{-1}$ we obtain

$$\begin{aligned} \frac{1}{2\pi}\int_{[-\pi,\pi]\setminus[-\Delta,\Delta]} |\operatorname{Re} h(t, e^{i\theta})|^p \, d\theta &\le C(d)\int_{\Delta}^{\infty} (t\theta)^{-pd}\, d\theta \\ &\le \Delta \frac{M^{1-pd} C_2(p,d)}{M}. \end{aligned} \tag{2.5}$$

If M_1 is so large that $C_1(p,d) > M_1^{1-pd} C_2(p,d)$ and $M_0 \ge M_1$, then (2.1) follows from (2.4) and (2.5).

2. Lemma 2.1, with $a = pd - 1 > 0$, yields

$$\begin{aligned} \|h(t,\zeta)\|^p_{L^p(\mathbb{T})} &= \int_{\mathbb{T}} \frac{dm_1(\zeta)}{|2 + t - t\zeta|^{pd}} \\ &= \int_{\mathbb{T}} \frac{(t+2)^{-pd}}{|1 - t(t+2)^{-1}\zeta|^{pd}}\, dm_1(\zeta) \\ &\le \operatorname{const}(p,d)\frac{(t+2)^{-pd}}{(1 - t(t+2)^{-1})^{pd-1}} \\ &\le \operatorname{const}_2(p,d) t^{-1}. \end{aligned}$$

Let $t^{-1} = \Delta M_2^{-1}$ and $M_2 > \pi \cdot \operatorname{const}_2(p,d)$; then (2.2) holds if $M_0 \ge M_2$.

3. We have

$$|h'(t,z)| = \frac{td}{|2 + t - tz|^{d+1}}.$$

Hence, by Lemma 2.1, with $a = pd - 1 > 0$ and $b = p - 1 > -1$, we obtain

$$\begin{aligned} \|h'(t,z)\|^p_{A_p(\mathbb{D})} &= \int_{\mathbb{D}} \frac{t^p d^p (t+2)^{-pd-p}(1 - |z|)^{p-1}}{|1 - t(t+2)^{-1} z|^{pd+p}}\, dm_2(z) \\ &\le \operatorname{const}(p,d)\frac{(t+2)^{-pd}}{(1 - t(t+2)^{-1})^{pd-1}} \\ &\le \operatorname{const}_3(p,d) t^{-1}. \end{aligned}$$

Again, let $t^{-1} = \Delta M_3^{-1}$ and $M_3 > \pi \cdot \operatorname{const}_3(p,d)$; then (2.3) holds if $M_0 \ge M_3$.

To finish the proof, define $M_0 = \max\{4, M_1, M_2, M_3\}$. □

3. Elementary functions

Given $R > 0$, define $\mathbb{D}(R) = \{z \in \mathbb{C} : |z| < R\}$.

Lemma 3.1. *Suppose that $p \in (0,1)$. Then there exists a constant $\beta = \beta(p) \in (0,1)$ with the following property: Let $r \in (0, 1/4)$ and $Q = (e^{-3ri}, e^{3ri}) \subset \mathbb{T}$. Let $\varkappa \in (0,1)$, $R \in (0,1)$. Then there exists a function $f \in A(\mathbb{D})$ such that*

$$\operatorname{Re} f < 1 \text{ on } Q, \quad \text{and} \quad \operatorname{Re} f < \varkappa \text{ on } \mathbb{T}\setminus Q, \tag{3.1}$$

$$\|\mathbb{1}_Q - \operatorname{Re} f\|^p_{L^p(\mathbb{T})} < \beta m_1(Q), \tag{3.2}$$

$$\|f\|^p_{L^p(\mathbb{T})} < m_1(Q), \tag{3.3}$$

$$\|f'\|^p_{A_p(\mathbb{D})} < m_1(Q), \tag{3.4}$$

$$|f(z)| < \varkappa \qquad \textit{if} \;\; z \in \mathbb{D}\,(R). \tag{3.5}$$

Remark. We will use this lemma when $m_1(Q)$ and $\varkappa$ are small and R is close to 1.

Proof. Let $0 < \delta < \varkappa \min\{r, (1-R)^2\}$ and $r\delta^{-1} = N \in \mathbb{N}$ (note that $N\varkappa > 1$). Define $\zeta_1 = e^{-ri}$, $\zeta_{j+1} = e^{2\delta i}\zeta_j$, $1 \le j \le N-1$; then $\{\zeta_j\}_{j=1}^N \subset [e^{-ri}, e^{ri}]$. Take $d = d(p) \in \mathbb{N}$ such that $pd \ge 2$ (in particular, $d \ge 2$) and let α and M_0 be those given by Lemma 2.2. We claim that the function

$$f(z) := \sum_{j=1}^N h_j(z) := \sum_{j=1}^N \frac{i}{(2 + M_0\delta^{-1}(1 - z\overline{\zeta}_j))^d}$$

satisfies the conditions of the lemma.

1. Let $\zeta \in \mathbb{T}$; then

$$|2 + M_0\delta^{-1}(1-\zeta\overline{\zeta}_j)| \ge \min(2, M_0\delta^{-1}|1-\zeta\overline{\zeta}_j|) = \min(2, M_0\delta^{-1}|\zeta - \zeta_j|).$$

Now, assume that $\zeta \in Q$ and $\min\{j : \arg(\zeta) \le \arg(\zeta_j)\} = k$; then $|\zeta_{k+l} - \zeta| \ge l\delta$, $l = 1, 2, \dots, N-k$. Therefore

$$\sum_{j=k}^N |h_j(\zeta)| \le 2^{-d} + \sum_{l=1}^\infty (lM_0)^{-d}.$$

Analogously,

$$\sum_{j=1}^{k-1} |h_j(\zeta)| \le 2^{-d} + \sum_{l=1}^\infty (lM_0)^{-d}.$$

Hence, if $\zeta \in Q$, then

$$\frac{1}{2}|f(\zeta)| \le 2^{-d} + \sum_{l=1}^\infty (lM_0)^{-d}.$$

Since $M_0 \ge 4$, the first part of (3.1) holds.

If $\zeta \in \mathbb{T} \setminus Q$, then $|1 - \zeta\overline{\zeta}_j| \ge r = N\delta$ for all $1 \le j \le N$. Hence

$$|f(\zeta)| \le \sum_{j=1}^N |h_j(\zeta)| \le N \cdot N^{-d} < \varkappa^{d-1} \le \varkappa.$$

2. The estimate (2.1) from Lemma 2.2, for $\Delta = \delta$, gives

$$\begin{aligned}\|\mathbb{1}_Q - \operatorname{Re} f\|^p_{L^p(\mathbb{T})} &\le m_1(Q) - N\delta/\pi + \sum_{j=1}^N \|\mathbb{1}_{[\zeta_j e^{-\delta i}, \zeta_j e^{\delta i}]} - \operatorname{Re} h_j\|^p_{L^p(\mathbb{T})} \\ &< m_1(Q) - (N\delta - \alpha(p)N\delta)/\pi = m_1(Q) - (1-\alpha(p)) \cdot r/\pi \\ &< \beta(p) m_1(Q).\end{aligned}$$

3. Since $\|h_j\|^p_{L^p(\mathbb{T})} < \delta/\pi$ (see (2.2)), we obtain

$$\|f\|^p_{L^p(\mathbb{T})} \le \sum_{j=1}^N \|h_j\|^p_{L^p(\mathbb{T})} < N\delta/\pi = r/\pi < m_1(Q).$$

4. The property (2.3) yields

$$\|f'\|^p_{A_p(\mathbb{D})} \le \sum_{j=1}^N \|h'_j\|^p_{A_p(\mathbb{D})} < N\delta/\pi < m_1(Q).$$

5. If $|z| < R$, then $|1 - z\overline{\zeta}_j|^2 \ge (1-R)^2$ for all $1 \le j \le N$. Thus, as above,

$$|f(z)| \le N \cdot \delta^d (1-R)^{-d} \le N\delta^{d/2}\varkappa^{d/2} < \varkappa.$$

The proof is complete. □

4. Approximation construction

Lemma 4.1. *Let $0 < p < 1$. Then there exists a constant $\gamma = \gamma(p) \in (0,1)$ with the following property: Suppose that $\psi \in C(\mathbb{T})$, $\psi > 0$, $R \in [0,1)$, $\varepsilon > 0$; then there exists a function $F \in A(\mathbb{D})$ such that*

$$\operatorname{Re} F < \psi \quad \text{on} \quad \mathbb{T}, \tag{4.1}$$

$$\|\psi - \operatorname{Re} F\|^p_{L^p(\mathbb{T})} < \gamma \|\psi\|^p_{L^p(\mathbb{T})}, \tag{4.2}$$

$$\|F\|^p_{L^p(\mathbb{T})} < \|\psi\|^p_{L^p(\mathbb{T})}, \tag{4.3}$$

$$\|F'\|^p_{A_p(\mathbb{D})} < \|\psi\|^p_{L^p(\mathbb{T})}, \tag{4.4}$$

$$|F| < \varepsilon \quad \text{on} \quad \mathbb{D}(R). \tag{4.5}$$

Proof. Take a linear combination of characteristic functions $h := \sum_{j=1}^J c_j \mathbb{1}_{Q_j}$ (the arcs Q_j are mutually disjoint and small enough, $c_j > 0$) such that

$$2\|\psi - h\|^p_{L^p(\mathbb{T})} < (1 - \beta(p))\|\psi\|^p_{L^p(\mathbb{T})}, \tag{4.6}$$

$$\psi - h \ge \eta \quad \text{for some} \quad \eta > 0. \tag{4.7}$$

Put $c_0 = \max\{c_j : 1 \le j \le J\}$ and $\varkappa = \min\{\varepsilon, \eta\}/(2c_0 J)$. Given the arcs Q_j, $\varkappa$ and R, Lemma 3.1 provides the functions f_j.

We claim that the function $F := \sum_{j=1}^J c_j f_j$ satisfies the conditions of the lemma with $\gamma = (1+\beta)/2$.

Since $2c_0 J\varkappa \le \eta$, (3.1) and (4.7) yield the inequality $\psi - \operatorname{Re} F \ge \eta/2$, so (4.1) holds. By (3.2) and (4.6), we have (4.2). Indeed,

$$\begin{aligned}
\|\psi - \operatorname{Re} F\|^p_{L^p(\mathbb{T})} &\le \|\psi - h\|^p_{L^p(\mathbb{T})} + \sum_{j=1}^J c_j^p \|\mathbb{1}_{Q_j} - \operatorname{Re} f_j\|^p_{L^p(\mathbb{T})} \\
&< \frac{1}{2}(1-\beta)\|\psi\|^p_{L^p(\mathbb{T})} + \beta\|h\|^p_{L^p(\mathbb{T})} \\
&\le \gamma \|\psi\|^p_{L^p(\mathbb{T})}.
\end{aligned}$$

The property (3.3) provides the estimate

$$\|F\|^p_{L^p(\mathbb{T})} \le \sum_{j=1}^J c_j^p \|f_j\|^p_{L^p(\mathbb{T})} < \sum_{j=1}^J c_j^p m_1(Q_j) \le \|\psi\|^p_{L^p(\mathbb{T})}.$$

The same argument shows that (3.4) implies (4.4).

At last, (3.5) ⇒ (4.5), since $2c_0 J\varkappa < \varepsilon$. □

5. Proof of the theorem

Proof. Given a strictly positive continuous modulus φ, put $\psi_0 = \log\varphi$. Without loss of generality, we suppose that $\psi_0 > 0$ and $0 < p < 1$.

Put $\psi = \psi_0$ and $R = R_0 := 0$; then Lemma 4.1 yields a function $F \in A(\mathbb{D})$. Define $F_1 := F$.

Suppose, as induction hypothesis, that $m \in \mathbb{N}$, $\{F_k\}_{k=1}^m \subset A(\mathbb{D})$ and $\{R_k\}_{k=0}^{m-1} \subset [0, 1)$. Assume also that

$$\operatorname{Re}\left(\sum_{k=1}^m F_k\right) < \psi_0 \quad \text{on} \quad \mathbb{T}, \tag{5.1}$$

$$\left\|\psi_0 - \operatorname{Re}\left(\sum_{k=1}^m F_k\right)\right\|_{L^p(\mathbb{T})}^p < \gamma^m \|\psi_0\|_{L^p(\mathbb{T})}^p, \tag{5.2}$$

$$\|F_m\|_{L^p(\mathbb{T})}^p < \gamma^{m-1}\|\psi_0\|_{L^p(\mathbb{T})}^p, \tag{5.3}$$

$$\|F_m'\|_{A_p(\mathbb{D})}^p < \gamma^{m-1}\|\psi_0\|_{L^p(\mathbb{T})}^p, \tag{5.4}$$

$$\left\|\sum_{k=1}^{m-1} F_k'\right\|_{A_p(\mathbb{D}\setminus\mathbb{D}(R_{m-1}))}^p < \gamma^{m-1}, \tag{5.5}$$

$$\left\|(\exp(F_m) - 1)\left(\sum_{k=1}^{m-1} F_k'\right)\right\|_{A_p(\mathbb{D}(R_{m-1}))}^p < \gamma^{m-1}. \tag{5.6}$$

Remark (base of induction). If $m = 1$, then (5.1–5.4) are (4.1–4.4), and the estimates (5.5), (5.6) are trivial.

Step $m + 1$. Define R_m such that (5.5) holds for $\sum_{k=1}^m$ and γ^m. Take $\varepsilon_m > 0$ such that

$$\left\|(\exp(\varepsilon_m) - 1)\left(\sum_{k=1}^{m} F_k'\right)\right\|_{A_p(\mathbb{D}(R_m))}^p < \gamma^m. \tag{5.7}$$

Given $p \in (0, 1)$ and $\psi = \psi_0 - \operatorname{Re}\left(\sum_{k=1}^m F_k\right) > 0$, $R = R_m$, $\varepsilon = \varepsilon_m$, Lemma 4.1 provides the function F_{m+1}.

Note that (4.1–4.4) $\Rightarrow$ (5.1–5.4) and (5.7) $\Rightarrow$ (5.6) for $m+1$. Now the induction construction proceeds.

Recall that $\sum_{m=1}^\infty \gamma^m < \infty$; therefore, by (5.3), the series $\sum_{k=1}^\infty F_k$ converges in H^p, so define

$$g = \exp\left(\sum_{k=1}^\infty F_k\right).$$

We claim that g satisfies the conditions of the theorem.

Put $\Phi := \max(\varphi) < +\infty$. Then (5.1) implies the estimate $|g| \le \Phi$ on the disc $\mathbb{D}$, and therefore $g \in H^\infty$.

On the other hand, (5.2) yields the equality $|g^*| = \varphi$ m_1-a.e.

So we have to prove the property $g \in A^1_p(\mathbb{D})$ only. Introduce extra notations

$$G_m = \left(\sum_{k=1}^m F'_k\right) \exp\left(\sum_{k=1}^m F_k\right), \quad x_m = \|G_{m+1} - G_m\|^p_{A_p(\mathbb{D})}.$$

It is sufficient to show that $\{x_m\}_{m=1}^\infty \in \ell^1$. Fix an integer $m \in \mathbb{N}$. Then

$$x_m \le X_m + Y_m := \left\| F'_{m+1} \exp\left(\sum_{k=1}^{m+1} F_k\right)\right\|^p_{A_p(\mathbb{D})} + \left\| \left[\sum_{k=1}^m F'_k\right] \left[\exp\left(\sum_{k=1}^{m+1} F_k\right) - \exp\left(\sum_{k=1}^m F_k\right)\right]\right\|^p_{A_p(\mathbb{D})}.$$

X). The properties (5.1) and (5.4) imply the estimate

$$X_m \le \gamma^m \Phi \|\psi_0\|^p_{L^p(\mathbb{T})}.$$

Y). By (5.1), (5.5) and (5.6), we obtain

$$Y_m \le 2\Phi \left\|\sum_{k=1}^m F'_k\right\|^p_{A_p(\mathbb{D}\setminus\mathbb{D}(R_m))} + \Phi \left\| (\exp(F_{m+1}) - 1) \left(\sum_{k=1}^m F'_k\right)\right\|^p_{A_p(\mathbb{D}(R_m))} \le \gamma^m \cdot 3\Phi.$$

Since $\sum_{m=1}^\infty \gamma^m < \infty$, the items X) and Y) yield $\sum_{m=1}^\infty x_m < +\infty$.
The proof is finished. □

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