PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 2, February 1998, Pages 545–552
S 0002-9939(98)04059-3

# A WEIGHTED POINCARÉ INEQUALITY WITH A DOUBLING WEIGHT

RITVA HURRI-SYRJÄNEN

(Communicated by Palle E. T. Jorgensen)

Abstract. We show that unbounded John domains (and even a larger class of domains than John domains) satisfy the weighted Poincaré inequality

$$\inf_{a\in\mathbb{R}} \|u(x)-a\|_{L^q(D,w_1)} \le C\|\nabla u(x)\|_{L^p(D,w_2)}$$

whenever $u$ is a Lipschitz function on $D$, $w_1$ is a doubling weight, and weights satisfy certain cube conditions, and $C = C(D,p,q,w_1,w_2)$.

## 1. Introduction

In this note we generalize results considering weighted Poincaré inequalities. My work was stimulated by a paper of Chua [C]. If $D$ is a bounded John domain and if there exists a constant $C_1 < \infty$ such that the inequality

$$(1.1)\qquad |Q|^{\frac{1}{n}-1}\left(\int_Q w_1(x)\,dx\right)^{1/q}\left(\int_Q w_2(x)^{-1/p-1}\,dx\right)^{p-1/p} \le C_1$$

holds for all Whitney cubes of $D$, then

$$\|u(x)-u_{D,w_1}\|_{L^q(D,w_1)} \le C_2\|\nabla u(x)\|_{L^p(D,w_2)}$$

whenever $w_1$ is a doubling weight and $1 < p < q < \infty$. If $p = q$, instead of (1.1) we need to require that

$$(1.1*)\qquad |Q|^{1/n}\left(\frac{1}{|Q|}\int_Q w_1^r(x)\,dx\right)^{1/pr}\left(\frac{1}{|Q|}\int_Q w_2^{-r/p-1}\right)^{p-1/pr} \le C_1$$

holds for any $r > 1$. These results are implicitly in Chua's paper.

We study the following generalized inequality:

$$(1.2)\qquad \inf_{a\in\mathbb{R}} \|u(x)-a\|_{L^q(D,w_1)} \le C\|\nabla u(x)\|_{L^p(D,w_2)}\ ,$$

where $u$ is a Lipschitz function and $1 < p \le q < \infty$. If $D$ satisfies (1.2), we write $D \in \mathcal{P}(q,p)$ with $w_1$ and $w_2$ and $C = \mathcal{K}_{q,p}(D,w_1,w_2)$. We show that unbounded John domains $D$ satisfy (1.2) whenever (1.1) holds for $1 < p < q < \infty$ (respectively (1.1$*$) for $p = q$) and $w_1$ is a doubling weight, Theorem 1.8. We also show that there is a larger class than John domains which satisfy (1.1) for $1 < p < q < \infty$ (respectively (1.1$*$) for $p = q$) under above conditions, Theorem 1.3. In this manner

Received by the editors January 5, 1996 and, in revised form, August 22, 1996.
1991 *Mathematics Subject Classification.* Primary 46Exx, 26Dxx.

a weighted result for so-called rooms and corridors domains is obtained, Example 4.1.

Our main theorems are Theorems 1.3 and 1.8.

**1.3 Theorem.** *Let a domain $\mathcal{G}$ be the union of domains $D_i \in \mathcal{P}(q,p)$ with a doubling weight $w_1$ and a weight $w_2$ such that*

$$\mathcal{K}_{q,p}(D_i, w_1, w_2) \leq C_0 < \infty \ , \qquad i = 1, 2, \dots \ . \tag{1.4}$$

*Suppose that each domain $D_i$ lies in a cube $Q_i$ with the following three properties. There are constants $C_i$, $i = 1, 2, 3$, such that*

$$\sum_{j=1}^{\infty} \chi_{Q_j}(x) \leq C_1 \chi_{\bigcup_{j=1}^{\infty} Q_j}(x) \tag{1.5}$$

*for all $x \in R^n$,*

$$Q_i \subset C_2 Q_j \ , \tag{1.6}$$

*where $j = 1, 2, \dots, i$, and*

$$\mathcal{K}_{q,p}(D_i, w_1, w_2)^q w_1(Q_i) \leq C_3 \min\{w_1(D_i \cap D_{i-1}), w_1(D_i \cap D_{i+1})\} \ . \tag{1.7}$$

*Then $\mathcal{G}$ is a $(q,p)$-Poincaré domain with $w_1$ and $w_2$.*

**1.8 Theorem.** *Let $w_1$ be a doubling weight. Suppose that $D$ is an unbounded John domain. Suppose that there exists a constant $C < \infty$ such that the inequality* (1.1) *for $1 < p < q < \infty$ (respectively* (1.1∗) *for $p = q$) holds for all cubes $Q \subset D$. Then $D$ is a weighted $(q,p)$-Poincaré domain with $w_1$ and $w_2$.*

The proofs for Theorem 1.3 and Theorem 1.8 are given in §3. An example and corollaries in §4 reveal that we have generalized results of Chao et al. [CWZ], Evans and Harris [EH], and Hurri [H].

## 2. Preliminaries

*Notation.* Throughout this paper we let $D$ be a domain of euclidean $n$-space $R^n$, $n \geq 2$. We suppose that $1 \leq p \leq q < \infty$ unless otherwise stated.

The diameter of a set $A$ is written as $\operatorname{dia}(A)$. We write $tQ$ for the cube with the same center as $Q$ and dilated by a factor $t > 1$.

We let $C(*, \cdots, *)$ denote a constant which depends only on the quantities appearing in the parentheses.

A *weight* (function) is a nonnegative measurable function on $R^n$. A weight $w$ is a *doubling weight* (that is, $w$ satisfies a doubling condition) if there exists a constant $t < \infty$ such that

$$\int_{2Q} w(x)\, dx \leq t \int_Q w(x)\, dx \text{ for all cubes } Q \subset R^n \ .$$

$A_p$-weights are doubling weights.

The average of a function $u$ over a domain $D$ with finite Lebesgue measure $|D|$ is

$$u_D = \frac{1}{|D|} \int_D u(x)\, dx \ ,$$

and with an arbitrary weight $w$

$$u_{D,w} = \frac{1}{\int_D w(x)\,dx} \int_D u(x)w(x)\,dx$$

whenever

$$0 < \int_D w(x)\,dx < \infty \ .$$

We write

$$\|u\|_{L^p(D,w)} = \left(\int_D |u(x)|^p w(x)\,dx\right)^{1/p} ,$$

where $u$ is a Lipschitz function on $D$. The distributional gradient is written as $\nabla u = (\partial_1 u, \dots, \partial_n u)$.

We recall a lemma due to Strömberg and Wheeden.

**2.1 Lemma (**[StW, Lemma 2.3]**,** [C, Lemma 2.5]**).** *Let $\{Q_\alpha\}_{\alpha\in\mathcal{I}}$ be an arbitrary family of cubes in $R^n$. If $\{a_\alpha\}_{\alpha\in\mathcal{I}}$ is a family of nonnegative real numbers and $w$ is a doubling weight, then for $1 \le p < \infty$ and $N \ge 1$ we have*

$$\Big\| \sum_\alpha a_\alpha \chi_{NQ_\alpha} \Big\|_{L^p(R^n,w)} \le C(n,p,N,w) \Big\| \sum_\alpha a_\alpha \chi_{Q_\alpha} \Big\|_{L^p(R^n,w)} .$$

The Hölder inequality and the Minkowski inequality yield the following useful lemma.

**2.2 Lemma.** *Let $D$ be a domain and $A \subset D$ be a set such that $\int_A w(x)\,dx < \infty$. Then for each $a \in R$*

$$\|u - u_{A,w}\|_{L^p(D,w)} \le 2 \left(\frac{\int_D w(x)\,dx}{\int_A w(x)\,dx}\right)^{1/p} \|u - a\|_{L^p(D,w)}$$

*where $u$ is a Lipschitz function on $D$.*

**$(q,p)$-Poincaré domains with weights $w_1$ and $w_2$.** Let $D \subset R^n$ be a domain and let $1 \le p \le q < \infty$. Let $w_1$ and $w_2$ be weight functions. If there exists a constant $\mathcal{K} = \mathcal{K}(D,p,q,w_1,w_2) < \infty$ such that the inequality

$$\inf_{a\in R} \|u - a\|_{L^q(D,w_1)} \le \mathcal{K} \|\nabla u\|_{L^p(D,w_2)} \tag{2.3}$$

holds for all Lipschitz functions $u$, then $D$ is a $(q,p)$-Poincaré domain with weights $w_1$ and $w_2$. We write $D \in \mathcal{P}(q,p)$ with $w_1$ and $w_2$.

**John domains.** Let $E$ be a closed arc with endpoints $a$ and $b$. The subarc between $x$ and $y$ is denoted by $E[x,y]$. For $x$ in $E \setminus \{a,b\}$ write

$$q(x) = \min\{\mathrm{dia}(E[a,x]), \mathrm{dia}(E[b,x])\} \ .$$

Let $c \ge 1$. A domain $D$ in $R^n$ is a $c$-John domain, if each pair of distinct points $a$ and $b$ in $D$ can be joined by an arc $E$ such that

$$cigE(a,b) = \bigcup \left\{ B\Big(x, \frac{q(x)}{c}\Big) \;\Big|\; x \in E \setminus \{a,b\} \right\} \subset D \ .$$

Balls, convex domains, domains with smooth boundaries are John domains as well as a snowflake domain.

Bojarski proved that a bounded $b$-John domain satisfies the $(q,p)$-Poincaré inequality with $w_1 = w_2 = 1$ [B, Chapter 6]. Unbounded John domains are $(\frac{np}{n-p}, p)$-Poincaré domains with $w_1 = w_2 = 1$ [H-S, Corollary 4.6].

We recall the following lemma due to Väisälä.

**2.4 Lemma** ([V, Theorem 4.6]). *Let $D$ be an unbounded $b$-John domain. Then there are $b_0$-John domains $D_i$ such that $D_i \subset \bar{D}_i \subset D_{i+1}$, $i = 1, 2, \ldots$, and $D = \bigcup_{i=1}^{\infty} D_i$.*

Sawyer and Wheeden have given several sufficient and necessary conditions for weights $w_1$ and $w_2$ so that a cube is a weighted $(q,p)$-Poincaré domain. We write down here one of their theorems which was refined by Chua for a doubling weight $w_1$.

**2.5 Lemma** ([C, Theorem 2.14]). *Suppose that $w_1$ is a doubling weight. Then for all Lipschitz functions $u$ and a cube $Q_0$*

$$\|u - u_{Q_0,w_1}\|_{L^q(Q_0,w_1)} \le \mathcal{K}_{q,p} \|\nabla u\|_{L^p(Q_0,w_2)}$$

*where*

$$\mathcal{K}_{p,q} = \mathcal{K}_{q,p}(w_1) \sup_{Q \subset Q_0} |Q|^{\frac{1}{n}-1} \left( \int_Q w_1(x)\, dx \right)^{1/q} \left( \int_Q w_2(x)^{-1/p-1}\, dx \right)^{p-1/p}$$

*whenever $p < q$.*

*If $p = q$, then*

$$\|u - u_{Q_0,w_1}\|_{L^p(Q_0,w_1)} \le \mathcal{K}_p \|\nabla u\|_{L^p(Q_0,w_2)}$$

*with*

$$\mathcal{K}_p = \mathcal{K}_p(r, w_1) \sup_{Q \subset Q_0} |Q|^{1/n} \left( \fint_Q w_1(x)^r\, dx \right)^{1/pr} \left( \fint_Q w_2(x)^{-r/(p-1)}\, dx \right)^{(p-1)/pr}$$

*for any $r > 1$.*

*Proof.* Let $f = u - u_{Q_0,w_1}$ in [C, Theorem 2.14]. □

Chua considered Boman's chain condition domains [C]. An especial case of his theorem [C, Theorem 1.5] is the following lemma.

**2.6 Lemma.** *Let $D$ be a bounded $b_0$-John domain and let $w_1$ be a doubling weight. Suppose that there exists a constant $C < \infty$ such that for all cubes $Q$ in $D$ (1.1) holds whenever $1 < p < q < \infty$ (respectively, (1.1$*$) holds for $p = q$).*

*Then*

$$\|u - u_{D,w_1}\|_{L^q(D,w_1)} \le \mathcal{K}_{p,q}(b_0, C, w_1) \|\nabla u\|_{L^p(D,w_2)} .$$

*Proof.* Lemma 2.5 and [C, Theorem 1.5]. □

*Proof for Theorem 1.3.* By the given decomposition of $\mathcal{G}$

$$\begin{aligned} \int_{\mathcal{G}} |u(y) - u_{D_1,w_1}|^q w_1(y)\, dy &\le \sum_{i=1}^{\infty} \int_{D_i} |u(y) - u_{D_1,w_1}|^q w_1(y)\, dy \\ &\le 2^{q-1} \left( \sum_{i=1}^{\infty} \int_{D_i} |u(y) - u_{D_i,w_1}|^q w_1(y)\, dy + \sum_{i=1}^{\infty} \int_{D_i} |u_{D_i,w_1} - u_{D_1,w_1}|^q w_1(y)\, dy \right). \end{aligned}$$

Since $D_i \in \mathcal{P}(q,p)$ with weights $w_1$ and $w_2$ and $\mathcal{K}_{q,p}(D_i, w_1, w_2) \le C_0$, we obtain

$$\begin{aligned}
&\sum_{i=1}^{\infty} \int_{D_i} |u(y) - u_{D_i,w_1}|^q w_1(y)\, dy \\
&\quad \le \sum_{i=1}^{\infty} \left( \mathcal{K}_{q,p}(D_i, w_1, w_2) \left( \int_{D_i} |\nabla u(y)|^p w_2(y)\, dy \right)^{1/p} \right)^q \\
&\quad \le C_0^q \left( \int_{\mathcal{G}} |\nabla u(y)|^p w_2(y)\, dy \right)^{q/p} .
\end{aligned}$$

To estimate the second sum we use the triangle inequality and the weighted $(q,p)$-Poincaré inequality in $D_i$. First,

$$\begin{aligned}
&|u_{D_j,w_1} - u_{D_{j+1},w_1}|^q \\
&\quad = \frac{1}{w_1(D_j \cap D_{j+1})} \int_{D_j \cap D_{j+1}} |u_{D_j,w_1} - u_{D_{j+1},w_1}|^q w_1(x)\, dx \\
&\quad \le \frac{2^{q-1}}{w_1(D_j \cap D_{j+1})} \sum_{h=j}^{j+1} \mathcal{K}_{q,p}^q (D_h, w_1, w_2) \left( \int_D |\nabla u(x)|^p w_2(x)\, dx \right)^{q/p} .
\end{aligned}$$

Hence, the triangle inequality, the condition (1.7), and the engulfing property,

$$D_i \subset Q_i \subset C_2 Q_j, \qquad j = 1, 2, \dots, i,$$

yield that

$$\begin{aligned}
&\sum_{i=1}^{\infty} \int_{D_i} |u_{D_i,w_1} - u_{D_1,w_1}|^q w_1(x)\, dx \\
&\quad \le \sum_{i=1}^{\infty} \int_{D_i} \left( \sum_{j=1}^{i-1} |u_{D_j,w_1} - u_{D_{j+1},w_1}| \chi_{D_i}(x) \right)^q w_1(x)\, dx \\
&\quad \le C_4 \int_{R^n} \left( \sum_{j=1}^{\infty} \frac{1}{w_1(Q_j)^{1/q}} \left( \int_{D_j} |\nabla u(y)|^p w_2(y)\, dy\, \chi_{C_2 Q_j}(x) \right)^{1/p} \right)^q w_1(x)\, dx.
\end{aligned}$$

Lemma 2.1 implies

$$\begin{aligned}
&\sum_{i=1}^{\infty} \int_{D_i} |u_{D_j,w_1} - u_{D_{j+1},w_1}|^q w_1(x) dx \\
&\quad \le C_5 \int_{R^n} \left( \sum_{j=1}^{\infty} \frac{1}{w_1(Q_j)^{1/q}} \left( \int_{D_j} |\nabla u(y)|^p w_2(y)\, dy\, \chi_{Q_j}(x) \right)^{1/p} \right)^q w_1(x)\, dx \\
&\quad \le C_6 \sum_{j=1}^{\infty} \frac{1}{w_1(Q_j)} \left( \int_{D_j} |\nabla u(y)|^p w_2(y)\, dy \right)^{q/p} \int_{R^n} \chi_{Q_j}(x) w_1(x)\, dx \\
&\quad \le C_7 \left( \int_{\mathcal{G}} |\nabla u(y)|^p w_2(y)\, dy \right)^{q/p} .
\end{aligned}$$

□

*Proof for Theorem 1.8.* Since $D$ is an unbounded John domain, there are $b_0$-John domains $D_i \subset \bar{D}_i \subset D_{i+1}$ such that $D = \bigcup_{i=1}^{\infty} D_i$ by Lemma 2.4.

We set

$$u_i = \frac{1}{\int_{D_i} w_1(x)\,dx} \int_{D_i} u(x) w_1(x)\,dx \ .$$

We will use $D_1$ to obtain for $|u_i|$ an upper bound which does not depend on $i$. The triangle inequality yields

$$\begin{aligned} |u_i| &= \left(\int_{D_1} w_1(x)\,dx\right)^{-1} \int_{D_1} |u_i| w_1(x)\,dx \\ &\le \left(\int_{D_1} w_1(x)\,dx\right)^{-1} \left(\int_{D_1} |u(x) - u_i| w_1(x)\,dx + \int_{D_1} |u(x)| w_1(x)\,dx\right) \end{aligned}$$

where we may assume that

$$0 < \int_{D_1} w(x)\,dx < \infty \text{ and } \int_{D_1} |u(x)| w_1(x)\,dx < \infty \ .$$

By Lemma 2.6

$$\begin{aligned} \int_{D_i} |u(x) - u_i| w_1(x)\,dx &\le \left(\int_{D_1} w_1(x)\,dx\right)^{1-\frac{1}{q}} \|u - u_i\|_{L^q(D_i, w_1)} \\ &\le \left(\int_{D_1} w_1(x)\,dx\right)^{1-\frac{1}{q}} \|u - u_i\|_{L^q(D_i, w_1)} \\ &\le \left(\int_{D_1} w_1(x)\,dx\right)^{1-\frac{1}{q}} \mathcal{K}_{q,p}(b_0, C, w_1) \|\nabla u\|_{L^p(D, w_2)} \ . \end{aligned}$$

Thus $(u_i)$ is a bounded sequence and hence there exists a convergent subsequence $(u_{ij})$ and $b \in R$ such that $\lim_{j\to\infty} u_{ij} = b$.

Since

$$\lim_{j\to\infty} \chi_{D_j}(x) |u(x) - u_j|^q = \chi_D(x) |u(x) - b|^q,$$

Fatou's lemma and Lemma 2.6 yield that

$$\begin{aligned} \int_D |u(x) - b|^q w_1(x)\,dx &= \int_D \lim_{j\to\infty} \chi_{D_j}(x) |u(x) - u_j|^q w_1(x)\,dx \\ &\le \underline{\lim}_{j\to\infty} \int_D \chi_{D_j}(x) |u(x) - u_j|^q w_1(x)\,dx \\ &\le \underline{\lim}_{j\to\infty} \left(\left(\mathcal{K}_{q,p}(b_0, C, w_1) \int_{D_j} |\nabla u(x)|^p w_2(x)\,dx\right)^{1/p}\right)^q \\ &\le \underline{\lim}_{j\to\infty} \left(\mathcal{K}_{q,p}(b_0, C, w_1) \int_D |\nabla u(x)|^p w_2(x)\,dx\right)^{q/p} \\ &= \left(\mathcal{K}_{q,p}(D, w_1) \int_D |\nabla u(x)|^p w_2(x)\,dx\right)^{q/p} \ . \end{aligned}$$

□

## 4. Further remarks

Example 4.1 considers the rooms and corridors domain.

**4.1 Example.** Let $G = \bigcup_{i=1}^{\infty} D_i$ be a domain where the sets $D_i$, $i = 1, 2, \ldots$, are defined as follows: Let $(h_i)$ and $(\delta_{2i})$ be sequences such that $h_i = M^{-i}$, $M > 1$, and $\delta_{2i} = bM^{-2ai}$, $b > 0$, $a > 1$. We set $\eta_{2i} = M^{-2(i+1)}$, $i = 1, 2, \ldots$, and $\sum_{i=1}^{k} h_i = d_k$. We define

$$D_{2i-1} = (d_{2i-1} - h_{2i-1}, d_{2i-2}) \cdot (-\frac{1}{2}h_{2i-1}, \frac{1}{2}h_{2i+1})^{n-1},$$

$$D_{2i} = (d_{2i-1} - \frac{1}{2}\eta_{2i}, d_{2i-1} + h_{2i} + \frac{1}{2}\eta_{2i}) \cdot (-\frac{1}{2}\delta_{2i}, \frac{1}{2}\delta_{2i})^{n-1}.$$

By [H] $G$ is an ordinary Poincaré domain, if and only if $p \geq (n-1)(a-1)$.

Adjoin the cubes $Q_i$ to the sets $D_i$ as follows: $Q_{2i-1} = D_{2i-1}$ and

$$Q_{2i} = (d_{2i-1} - \frac{1}{2}\eta_{2i}, d_{2i-1} + h_{2i} + \frac{1}{2}\eta_{2i}) \cdot (-\frac{1}{2}(h_{2i} + \eta_{2i}), \frac{1}{2}(h_{2i} + \eta_{2i}))^{n-1}.$$

We choose $w_1(x) = 1$ and $w_2(x) = d(x, \partial G)^{\alpha p}$. To check (1.7) we need the fact that the weighted Poincaré constant in this case is $|D|^{q + \frac{1-\alpha}{n} - \frac{1}{p}} c^n$ for a $c$-John domain by [H-S2]. Using this we obtain that weighted Poincaré inequality (1.3) holds whenever

$$n(2 - a - \frac{q}{p} + p(1-a)) + q(1-\alpha) - 1 + a \geq 0\,.$$

This generalizes the results of [EH] and [H] to the weighted case.

The proof for Theorem 1.8 has the following interesting corollaries.

**4.2 Corollary.** *Let $w_1$ be a doubling weight. Suppose that $D$ is an unbounded John domain. Suppose that there exists a constant $C < \infty$ such that for all cubes $Q \subset D$ the inequality* (1.1) *holds whenever $1 < p < q < \infty$ (respectively,* (1.1$*$) *for $p = q$). Then there is a constant $\mathcal{K}_{q,p}(w_1) < \infty$ such that the inequality*

$$\|u\|_{L^q(D,w_1)} \leq \mathcal{K}_{q,p}(D, w_1) \|\nabla u\|_{L^p(D,w_2)}$$

*holds for all Lipschitz functions $u$.*

*Proof.* Note that

$$u_j = \frac{1}{\int_{D_j} w_1(x)\,dx} \int_{D_j} u(x) w_1(x)\,dx \to 0$$

when we assume that $\int_D u(x) w_1(x)\,dx < \infty$. This follows from the proof for Theorem 1.8. □

An especial case of Corollary 4.2 is the following result.

**4.3 Corollary.** *If $D$ is an unbounded John domain then there exists a constant $C = C(n, p) < \infty$ such that*

$$\|u\|_{L^{np/n-p}(D)} \leq C \|\nabla u\|_{L^p(D)} \tag{4.4}$$

*holds for all Lipschitz functions $u$.*

Corollary 4.3 generalizes the result of Chen et al. [CWZ] who showed that (4.4) holds for domains with a cone condition.

## References


[B] Bojarski, B., *Remarks on Sobolev imbedding inequalities*, Complex Analysis (Joensuu 1987), Lecture Notes in Math., Springer-Verlag, Berlin and Heidelberg **1351** (1988), 52–68. MR **90b:**46068

[C] Chua, S.-K., *Weighted Sobolev inequalities on domains satisfying the chain condition*, Proc. Amer. Math. Soc. **117** (1993), 449–457. MR **93d:**46050

[CWZ] Chen, Z.Q., R.J. Williams and Z. Zhao, *A Sobolev inequality and Neumann heat kernel estimate for unbounded domains*, Math. Research Letters **1** (1994), 177–184. MR **90d:**70034

[EH] Evans, W.D. and D.J. Harris, *Sobolev embeddings for generalized ridged domains*, Proc. London Math. Soc. **54** (3) (1987), 141–175. MR **88b:**46056

[H] Hurri, R., *Poincaré domains in* $R^n$, Ann. Acad. Sci. Fenn. Ser. IA, Dissertations **71** (1988), 1–41. MR **90a:**30074

[H-S] Hurri-Syrjānen, R., *Unbounded Poincaré domains*, Ann. Acad. Sci. Fenn. Ser. IA **17** (1992), 409–423. MR **93k:**46022

[H-S2] Hurri-Syrjānen, R., *An improved Poincaré inequality*, Proc. Amer. Math. Soc. **120** (1994), 213–222. MR **94b:**46047

[SW] Sawyer, E. and R.L. Wheeden, *Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces*, Amer. J. Math. **114** (1992), 813–874. MR **94i:**42024

[StW] Strömberg, J.-O. and R.L. Wheeden, *Fractional integrals on weighted* $H^p$ *and* $L^p$ *spaces*, Trans. Amer. Math. Soc. **287** (1985), 293–321. MR **86f:**42016

[V] Vāisālā, J., *Exhaustions of John domains*, Ann. Acad. Sci. Fenn. Ser. IA **19** (1994), 47–54. MR **94i:**30024



Department of Mathematics, University of Texas, Austin, Texas 78712

*E-mail address*: `syrjanen@math.utexas.edu`

*Current address*: Department of Mathematics, P.O. Box 4, FIN-00014 University of Helsinki, Finland

*E-mail address*: `hurrisyr@helsinki.fi`