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UNIQUE CONTINUATION ON THE BOUNDARY FOR DINI DOMAINS

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ABSTRACT. We show that the normal derivative of a harmonic function which vanishes on an open subset of the boundary of a Dini domain cannot vanish on a subset of positive surface measure.

1. INTRODUCTION

In [L], the following unique continuation question was raised: If u is a harmonic function in a Lipschitz domain Ω which vanishes on an open subset Γ of the boundary $\partial\Omega$, and if u is not identically zero, does it follow that the surface measure of the set $\{x \in \Gamma : \nabla u = 0\}$ is zero? In [L], it was shown that the answer is affirmative if Ω is a $C^{1,1}$ domain. In [AEK], it was proven that the answer is affirmative also for convex domains. Further, in [AE], it was proven that the fact holds for Dini domains, and thus, in particular, for $C^{1,\alpha}$ domains for all $\alpha \in (0,1]$. The approach in [AE] consists of a local change of variables around points on the boundary, which transforms the harmonic operator to a more general elliptic one, while the boundary becomes (in a certain sense) locally convex with respect to this operator.

The purpose of the present paper is to present a short and elementary proof of the Adolfsson-Escauriaza result. By [AEK], it is sufficient to prove that u satisfies a uniform doubling type condition on Γ. This is done by studying the logarithmic convexity of averages of u^2 over the balls centered at points $x_0 \in \Gamma$, rather than spherical shell averages as in [L], [AEK], and [AE], and take into account the fact that a Dini domain is locally star-shaped with respect to suitably chosen points close to x_0.

2. THE MAIN RESULTS

In this section, we recall the definition of a Dini domain and state the main results.

Definition 2.1. An open connected domain $\Omega \subseteq \mathbb{R}^d$ is a Dini domain if for each point $x_0 \in \partial\Omega$, there exists a local coordinate system $(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$, $R_0 > 0$, and a function $\phi\colon \mathbb{R}^{d-1} \to \mathbb{R}$ such that

(i) $B_{R_0}(x_0) \cap \Omega = \{(x', x_d) \in B_{R_0}(x_0) : x_d < \phi(x')\}$,

(ii) $B_{R_0}(x_0) \cap \partial\Omega = \{(x', x_d) \in B_{R_0}(x_0) : x_d = \phi(x')\}$,

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(iii) $|\nabla\phi(x_1') - \nabla\phi(x_2')| \le \psi(|x_1' - x_2'|)$ for all $(x_1', \phi(x_1')), (x_2', \phi(x_2')) \in B_{R_0}(x_0)$ where ψ satisfies

$$\int_0^1 \frac{\psi(r)}{r}\, dr < \infty. \tag{2.1}$$

Theorem 2.2. *Let Ω be a Dini domain, and let Γ be an open subset of the boundary $\partial\Omega$. If $u \in C(\Gamma \cup \Omega)$ is harmonic in Ω, and if u vanishes on Γ, then, for every $x_0 \in \Gamma$,*

$$\sup_{x_0\in\Gamma} \sup_{r\in(0,1)} \frac{\int_{B_{2r}(x_0)\cap\Omega} u^2\, dx}{\int_{B_r(x_0)\cap\Omega} u^2\, dx} < \infty.$$

The following corollary of Theorem 2.2 is our main result.

Theorem 2.3. *Let Ω be a Dini domain, and let Γ be an open subset of the boundary $\partial\Omega$. If $u \in C(\Gamma \cup \Omega)$ is harmonic in Ω, and if u vanishes on Γ, then the surface measure of $\{x \in \Gamma : |\nabla u(x)| = 0\}$ is zero.*

Proof of Theorem 2.3. It is proven in [AEK] (or cf. [L]) that Theorem 2.2 implies Theorem 2.3. □

Theorem 2.2 is proven in the next section.

3. Proof of Theorem 2.2

Let $\Omega \subseteq \mathbb{R}^d$ be a Lipschitz domain, and let u be harmonic in Ω.

For any $x_0 \in \overline{\Omega}$, denote

$$H_{x_0}(r) = \int_{B_r(x_0)\cap\Omega} u^2\, dx.$$

Lemma 3.1. *Let $B_{R_0}(x_0) \cap \Omega$ be star-shaped with respect to some $x_0 \in \overline{\Omega}$, and assume that u vanishes continuously on $B_{R_0}(x_0) \cap \partial\Omega$. If $0 < r_1 < r_2 < r_3 < R_0$, then*

$$\frac{\log \dfrac{H_{x_0}(r_2)}{H_{x_0}(r_1)}}{\log \dfrac{r_2}{r_1}} \le \frac{\log \dfrac{H_{x_0}(r_3)}{H_{x_0}(r_2)}}{\log \dfrac{r_3}{r_2}}.$$

The interior version of the above lemma was obtained in [K].

Proof of Lemma 3.1. Assume $x_0 = 0$; also, denote $H(r) = H_0(r)$ and $B_r = B_r(0)$. Fix an arbitrary $r \in (0, R_0)$. Since $B_{R_0} \cap \Omega$ is star-shaped with respect to 0, and since u vanishes on $B_R \cap \partial\Omega$,

$$\begin{aligned} H'(r) &= \int_{(\partial B_r)\cap\Omega} u^2\, d\sigma = \int_{\partial(B_r\cap\Omega)} u^2\, d\sigma = \frac{1}{r}\int_{B_r\cap\Omega} \operatorname{div}(u^2 x) dx \\ &= \frac{d}{r} H(r) + \frac{2}{r}\int_{B_r\cap\Omega} u\, x\cdot\nabla u\, dx. \end{aligned} \tag{3.1}$$

Denote

$$\begin{aligned} I(r) &= 2\int_{B_r\cap\Omega} u\, x\cdot\nabla u\, dx = -\int_{B_r\cap\Omega} u\,\nabla u\cdot\nabla(r^2 - |x|^2)\, dx \\ &= \int_{B_r\cap\Omega} |\nabla u|^2 (r^2 - |x|^2)\, dx, \end{aligned}$$

where we again used that u vanishes on $B_{R_0} \cap \partial\Omega$ and that u is harmonic. Differentiating the last expression, we get

$$I'(r) = 2r \int_{B_r\cap\Omega} |\nabla u|^2 \, dx + \int_{(\partial B_r)\cap\Omega} |\nabla u|^2 (r^2 - |x|^2) \, d\sigma = 2r \int_{B_r\cap\Omega} |\nabla u|^2 \, dx.$$

Note that, using integration by parts,

$$\begin{aligned}
\frac{1}{r} \int_{B_r\cap\Omega} & x \cdot \nabla(|\nabla u|^2)(r^2 - |x|^2) \, dx \\
&= -\frac{d}{r} I(r) + \frac{2}{r} \int_{B_r\cap\Omega} |x|^2 |\nabla u|^2 \, dx + \frac{1}{r} \int_{B_r\cap\partial\Omega} |\nabla u|^2 (r^2 - |x|^2) \, x \cdot n \, d\sigma \\
&= -\frac{d+2}{r} I(r) + \frac{2}{r} \int_{B_r\cap\Omega} |\nabla u|^2 (r^2 - |x|^2) \, dx \\
&\qquad + \frac{2}{r} \int_{B_r\cap\Omega} |x|^2 |\nabla u|^2 \, dx + \frac{1}{r} \int_{B_r\cap\partial\Omega} |\nabla u|^2 (r^2 - |x|^2) \, x \cdot n \, d\sigma \\
&= -\frac{d+2}{r} I(r) + 2r \int_{B_r\cap\Omega} |\nabla u|^2 \, dx + \frac{1}{r} \int_{B_r\cap\partial\Omega} |\nabla u|^2 (r^2 - |x|^2) \, x \cdot n \, d\sigma,
\end{aligned}$$

where n denotes the outward unit normal on $\partial\Omega$. Therefore,

$$\begin{aligned}
I'(r) = \frac{d+2}{r} I(r) &+ \frac{2}{r} \int_{B_r\cap\Omega} x_k \, (\partial_k \partial_j u)(\partial_j u)(r^2 - |x|^2) \, dx \\
&- \frac{1}{r} \int_{B_r\cap\partial\Omega} |\nabla u|^2 (r^2 - |x|^2) \, x \cdot n \, d\sigma,
\end{aligned}$$

whence, integrating by parts (with respect to j) the second expression on the right-hand side,

$$\begin{aligned}
I'(r) = \frac{d+2}{r} I(r) - \frac{2}{r} \int_{B_r\cap\Omega} |\nabla u|^2 (r^2 - |x|^2) \, dx &+ \frac{4}{r} \int_{B_r\cap\Omega} (x \cdot \nabla u)^2 \, dx \\
&+ \frac{2}{r} \int_{B_r\cap\partial\Omega} (x \cdot \nabla u)(n \cdot \nabla u)(r^2 - |x|^2) \, d\sigma \\
&- \frac{1}{r} \int_{B_r\cap\partial\Omega} |\nabla u|^2 (r^2 - |x|^2) x \cdot n \, d\sigma.
\end{aligned}$$

The sum of the last two terms is nonnegative since on $B_R \cap \partial\Omega$ the tangential derivative of u vanishes and $x \cdot n \geq 0$ almost everywhere on $B_r \cap \partial\Omega$. We get

$$I'(r) \geq \frac{d}{r} I(r) + \frac{4}{r} \int_{B_r\cap\Omega} (x \cdot \nabla u)^2 \, dx.$$

Denoting by

$$N(r) = \frac{I(r)}{H(r)}$$

the frequency function, we obtain by differentiation

$$
\begin{aligned}
N'(r) &= \frac{1}{H(r)^2}\big(I'(r)H(r) - I(r)H'(r)\big)\\
&\geq \frac{4}{rH(r)^2}\left(\int_{B_r\cap\Omega}(x\cdot\nabla u)^2\,dx\int_{B_r\cap\Omega}u^2\,dx - \left(\int_{B_r\cap\Omega}u\,x\cdot\nabla u\,dx\right)^2\right) \geq 0,
\end{aligned}
$$

and we conclude that $N(r)$ is an increasing function of $r \in (0, R_0)$. Now, (3.1) implies

$$
\log\frac{H(r_2)}{H(r_1)} = d\log\frac{r_2}{r_1} + \int_{r_1}^{r_2}\frac{N(r)}{r}\,dr \leq (d + N(r_2))\log\frac{r_2}{r_1}. \tag{3.2}
$$

Similarly,

$$
\log\frac{H(r_3)}{H(r_2)} = d\log\frac{r_3}{r_2} + \int_{r_2}^{r_3}\frac{N(r)}{r}\,dr \geq (d + N(r_2))\log\frac{r_3}{r_2}. \tag{3.3}
$$

The asserted inequality is then obtained by combining (3.2) and (3.3). □

Consider the following setting. Let $\phi\colon \mathbb{R}^{d-1} \to \mathbb{R}$ be continuously differentiable. Denote

$$
\Omega = B_2 \cap \{(x', x_d) \in \mathbb{R}^{d-1}\times\mathbb{R} : \phi(x') < x_d\},
$$

where $B_2 = B_2(0)$. For every $r \in (0,2)$, let

$$
\Lambda(r) = \sup|n(x_2) - n(x_1)|
$$

where the supremum is taken over all $x_1, x_2 \in B_2 \cap \partial\Omega$ such that $|x_2 - x_1| \leq r$, and where $n(x)$ denotes the outward unit normal at $x \in B_2 \cap \partial\Omega$. Note that Λ is non-decreasing. Assume also $\phi(0) = 0$ (i.e., $0 \in \partial\Omega$).

Throughout this section, we assume that $u \in C(\overline{\Omega})$ is harmonic in Ω and that it vanishes on $B_2 \cap \partial\Omega$. Theorem 2.2 will be an immediate consequence of the following lemma.

Lemma 3.2. *Assume*

$$
\int_0^2 \frac{\Lambda(r)}{r}\,dr < \infty, \tag{3.4}
$$

$\Lambda(r) \neq 0$ *for* $r \in (0,2)$*, and* $\Lambda(2) \leq 1/32$*. Then*

$$
C_1 = \prod_{m=1}^{\infty}\frac{\log\dfrac{2+16\Lambda(2^{-m})}{1-16\Lambda(2^{-m})}}{\log\dfrac{4-16\Lambda(2^{-m})}{2+16\Lambda(2^{-m})}} < \infty, \tag{3.5}
$$

and for all $x_0 \in B_1 \cap \partial\Omega$*, we have*

$$
H_{x_0}(2r) \leq C_2 H_{x_0}(r)\ , \qquad r \in \left(0, \frac{1}{2}\right),
$$

where $C_2 = \big(H_{x_0}(1)/H_{x_0}(1/2)\big)^{2C_1}$.

Proof of Lemma 3.2. Note that the function

$$f(x)=\frac{\log\dfrac{2+x}{1-x}}{\log\dfrac{4-x}{2+x}}$$

is increasing and continuously differentiable on $(-2,1)$. Since also $f(0)=1$, we have

$$1\le f(x)\le 1+e^{Cx}\ ,\qquad x\in[0,1/2],$$

for some constant $C>0$. Hence, the product in (3.5) converges provided

$$\sum_{m=1}^{\infty}\Lambda(2^{-m})<\infty.$$

However, the last fact follows directly from (3.4) taking into account that Λ is non-decreasing.

Without loss of generality, we may assume $x_0=0$ and $n(0)=(0,1)$.

Fix any $r\in(0,1)$. Let $(x_1',\phi(x_1')),(x_2',\phi(x_2'))\in B_1$ be arbitrary points. It is easy to check that $|\nabla\phi(x_j')|\le 3\Lambda(r)/2$ for $j=1,2$; the mean value theorem implies

$$|\phi(x_2')-\phi(x_1')|\le\frac{3}{2}\Lambda(r)|x_2'-x_1'|.$$

Denoting $a=4\Lambda(r)r$ and $y_0=(0,-a)$, we claim that $B_{r-a}(y_0)\cap\Omega$ is star-shaped with respect to x_0. Indeed, if $(x_1',\phi(x_1')),(x_2',\phi(x_2'))\in B_1$ are arbitrary points for which $x_1'\ne x_2'$ and $x_2'\ne 0$, then

$$\frac{|\phi(x_2')+a|}{|x_2'|}\ge\frac{a-|\phi(x_2')|}{|x_2'|}\ge\frac{4\Lambda(r)r}{|x_2'|}-\frac{3}{2}\Lambda(r)\ge\frac{5}{3}\Lambda(r)>\frac{3}{2}\Lambda(r)\ge\frac{|\phi(x_2')-\phi(x_1')|}{|x_2'-x_1'|},$$

and thus the points $(0,-a)$, $(x_1',\phi(x_1'))$, and $(x_2',\phi(x_2'))$ are not collinear.

Next, we claim

$$\log\frac{H(r/2)}{H(r/4)}\le\frac{\log\dfrac{2+16\Lambda(r)}{1-16\Lambda(r)}}{\log\dfrac{4-16\Lambda(r)}{2+16\Lambda(r)}}\log\frac{H(r)}{H(r/2)}.\tag{3.7}$$

For this purpose, let $r_1=r/4-a$, $r_2=r/2+a$, and $r_3=r-a$. Note that, since $\Lambda(r)<1/16$, we have $0<r_1<r_2<r_3<0$. Lemma 3.1 then implies

$$\log\frac{H_{y_0}(r_2)}{H_{y_0}(r_1)}\le\frac{\log\dfrac{r/2+a}{r/4-a}}{\log\dfrac{r-a}{r/2+a}}\log\frac{H_{y_0}(r_3)}{H_{y_0}(r_2)}.$$

Now, noting that $B_{r_1}(y_0)\subseteq B_{r/4}$, $B_{r_2}(y_0)\supseteq B_{r/2}$, and $B_{r_3}(y_0)\subseteq B_r$, we get (3.7).

By using (3.7) for $r=2^{-m}$ with $m=0,1,\dots$ and then iterating the obtained inequalities, we get

$$H(2^{-m})\le\left(\frac{H(1)}{H(1/2)}\right)^{C_1}H(2^{-m-1})\ ,\qquad m=1,2,\dots\ ,$$

and Lemma 3.2 follows. □

Proof of Theorem 2.2. Note that, in the setting of Lemma 3.2,

$$\sup_{x_0 \in B_1 \cap \partial\Omega} \frac{H_{x_0}(1)}{H_{x_0}(1/2)} < \infty,$$

and Theorem 2.2 follows from Lemma 3.2. □

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