A REMARK ON GELFAND-KIRILLOV DIMENSION

S. PAUL SMITH AND JAMES J. ZHANG

Abstract. Let $A$ be a finitely generated non-PI Ore domain and $Q$ the quotient division algebra of $A$. If $C$ is the center of $Q$, then $\text{GKdim } C \leq \text{GKdim } A - 2$.

Throughout $k$ is a commutative field and $\dim_k$ is the dimension of a $k$-vector space. Let $A$ be a $k$-algebra and $M$ a right $A$-module. The Gelfand-Kirillov dimension of $M$ is

$$\text{GKdim } M = \sup_{V,M_0} \lim_{n \to \infty} \log_n \dim_k M_0 V^n$$

where the supremum is taken over all finite dimensional subspaces $V \subset A$ and $M_0 \subset M$. If $F \supset k$ is another central subfield of $A$, we may also consider the Gelfand-Kirillov dimension of $M$ over $F$ which will be denoted by $\text{GKdim}_F$ to indicate the change of the field. We refer to [BK], [GK] and [KL] for more details.

Let $Z$ be a central subdomain of $A$. Then $A$ is localizable over $Z$ and the localization is denoted by $A_Z$. For any right $A$-module $M$, $M \otimes A_Z$ is denoted by $M_Z$. Let $F$ be the quotient field of $Z$. The first author [Sm, 2.7] proved the following theorem:

Let $A$ be an almost commutative algebra and $Z$ a central subdomain. Suppose $M$ is a right $A$-module such that $M_Z \neq 0$. Then

$$\text{GKdim } M \geq \text{GKdim}_F M_Z + \text{GKdim } Z.$$

As a consequence of this, if $A$ is almost commutative but non-PI and $Z$ is a central subalgebra such that every nonzero element in $Z$ is regular in $A$, then $\text{GKdim } Z \leq \text{GKdim } A - 2$.

It is natural to ask if the above theorem (and hence the consequence) is true for all algebras. In this paper we will precisely prove this.

**Theorem 1.** Let $A$ be an algebra and $Z$ a central subdomain. Suppose $M$ is a right $A$-module such that $M_Z \neq 0$. Then

$$\text{GKdim } M \geq \text{GKdim}_F M_Z + \text{GKdim } Z.$$

An algebra is called **locally PI** if every finitely generated subalgebra is PI. As a consequence of Theorem 1, we have

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Corollary 2. Let $A$ be algebra and $Z$ a central subdomain. If $A_Z$ is nonzero, then
$$\text{GKdim } A \geq \text{GKdim}_F A_Z + \text{GKdim } Z.$$ 
Furthermore, if $A_Z$ is not locally PI, then
$$\text{GKdim } A \geq 2 + \text{GKdim } Z.$$

For the second inequality in Corollary 2, $Z$ need not be a domain. Let $Z$ be any central subalgebra of $A$ of finite GKdimension such that $A_Z$ is not locally PI. By the Noether normalization theorem, there is a subalgebra $Z_1 \subset Z$ isomorphic to the polynomial ring on $d$ variables where $d = \text{GKdim } Z$. Since $A_Z = (A_{Z_1})_Z$, $A_{Z_1}$ is nonzero and not locally PI. Hence, by Corollary 2, $\text{GKdim } Z_1 \leq \text{GKdim } A - 2$. Therefore $\text{GKdim } Z = \text{GKdim } Z_1 \leq \text{GKdim } A - 2$.

A stronger version of Corollary 2 also holds. We need another invariant defined by Gelfand and Kirillov. Let $A$ be an algebra. The **Gelfand-Kirillov transcendence degree** of $A$ is
$$T\text{deg } A = \sup \inf V \cdot b \text{GKdim } k[bV]$$
where $V$ ranges over all finite dimensional subspaces of $A$ and $b$ ranges over the regular elements of $A$. If $A$ is a commutative domain, then both GKdim $A$ and $T\text{deg } A$ are equal to the classical transcendence degree of $A$, denoted by $\text{trdeg } A$. If $F \supset k$ is a central field of $A$, the Gelfand-Kirillov transcendence degree of $A$ over $F$ will be denoted by $T\text{deg}_F$ to indicate the change of the field.

**Theorem 3.** Let $A$ be a semiprime Goldie algebra and $Q$ the classical quotient algebra of $A$. Let $F$ be a central subfield of $Q$. Then
$$T\text{deg } Q \geq T\text{deg}_F Q + \text{trdeg } F.$$ 
If moreover $A$ is not locally PI, then
$$\text{GKdim } A \geq 2 + \text{GKdim } F.$$

The statement in the abstract is an obvious consequence of Theorem 3.

We now give the proofs. For simplicity a **subspace** means a finite dimensional subspace over $k$ and a **subframe** of an algebra means a subspace containing the identity. Our proofs are based on the following easy observation.

**Lemma 4.** Let $F \supset k$ be a commutative field and $M$ a right $F$-module. Let $M_0 \subset M$ and $W \subset F$ be subspaces over $k$. Then
$$\dim_k M_0 W \geq (\dim_F M_0 F)(\dim_k W).$$

**Proof.** Pick a basis of $M_0 F$ over $F$, say $\{x_1, \cdots, x_p\} \subset M_0$. Then $M_0 F = \oplus_{i=1}^p x_i F$ and hence $M_0 W \supset \oplus_{i=1}^p x_i W$. Therefore $\dim_k M_0 W \geq (\dim_F M_0 F)(\dim_k W)$. ∎

**Proof of Theorem 1.** Since $Z$ is central, by the proof of [KL, 4.2], we have $\text{GKdim } M \geq \text{GKdim } M_Z$. By [KL, 4.2], $\text{GKdim } Z = \text{GKdim } F$ where $F$ is the quotient field of $Z$. Hence it suffices to show $\text{GKdim } M_Z \geq \text{GKdim}_F M_Z + \text{GKdim } F$. Therefore we may assume $Z = F$ is a central field of $A$, and we need to show that $\text{GKdim } M \geq \text{GKdim}_F M + \text{GKdim } F$. Let $d$ be any number less than $\text{GKdim } F$. Then there exists a subframe $S \subset F$ such that $\dim_k S^n \geq n^d$ for all $n \gg 0$. Let $e$ be any number less than $\text{GKdim}_F M$. Then there exist a subspace $M_0 \subset M$, and a subframe $V \subset A$
such that \( \dim_F M_0 F(VF)^n \geq n^e \) for infinitely many \( n \). Since \( A \supseteq F \), we may assume \( V \supset S \). Since \( F \) is central, \( M_0 F(VF)^n = M_0 V^n F \). By Lemma 4,
\[
\dim_k M_0 V^{2n} \geq \dim_k M_0 V^n S^n \geq (\dim_F M_0 V^n F)(\dim_k S^n) \geq n^e n^d = n^{e+d}
\]
for infinitely many \( n \). Hence \( \text{GKdim} M \geq e + d \). By the choices of \( e \) and \( d \), we obtain \( \text{GKdim} M \geq \text{GKdim}_F M + \text{GKdim} F \) as desired.

\textbf{Proof of Corollary 2}. The first inequality follows from Theorem 1.1 by letting \( M = A \). If \( A_Z \) is not locally PI, then \( \text{GKdim}_F A_Z > 1 \) by [SSW], and \( \text{GKdim}_F A_Z \geq 2 \) by [Be]. Hence the second inequality follows.

As pointed out in [Sm, p. 37] the inequalities in Corollary 2 may be strict even if \( Z \) is the maximal central subring. By a result of M. Lorenz [Lo] the same example in [Sm, p. 37] shows also that the inequalities in Theorem 3 may be strict.

The proof of Theorem 3 is similar to that of Theorem 1.

\textbf{Proof of Theorem 3}. Since \( F \) is commutative, for any \( d < \text{trdeg}_F (= \text{GKdim} F) \), there is a subframe \( S \subset F \) such that \( \dim_k S^n \geq n^d \) for all \( n \gg 0 \). Let \( e \) be any number less than \( \text{Tdeg}_F Q \). By the proof of [Zh, 3.1] there is a subframe \( V \subset A \) such that for every regular element \( b \in Q \), \( \text{GKdim} F[bVF] > e \). This is equivalent to saying that, for every regular element \( b \in Q \), \( \dim_F (F + bVF)^n \geq n^e \) for infinitely many \( n \). We may assume \( V \supset S \). Since \( F \) is central, \( \dim_F (k + bV)^n bV F = \dim_F (F + bVF)^n \). By Lemma 4,
\[
\dim_k (k + bV)^n (bS)^n \geq (\dim_F (F + bVF)^n)(\dim_k S^n).
\]
Hence
\[
\dim_k (k + bV)^{2n} \geq \dim_k (k + bV)^n (bS)^n \geq n^e n^d = n^{e+d}
\]
for infinitely many \( n \). This means that \( \text{GKdim} k[bV] \geq e + d \) and hence \( \text{Tdeg} Q \geq e + d \). By the choices of \( e \) and \( d \), \( \text{Tdeg} Q \geq \text{Tdeg}_F Q + \text{trdeg} F \).

Now we assume \( A \) is not locally PI. Then \( Q \) is not locally PI. By [SSW] and [Be], \( \text{GKdim}_F Q \geq 2 \) and by [Zh, 4.1 and 4.3], \( \text{Tdeg}_F Q \geq 2 \). Therefore by [Zh, 2.1 and 3.1]
\[
\text{GKdim} A \geq \text{Tdeg} A \geq \text{Tdeg} Q \geq \text{Tdeg}_F Q + \text{trdeg} F \geq 2 + \text{GKdim} F.
\]

If \( Z \) is a central subdomain of \( A \), we can similarly prove that \( \text{Tdeg} A \geq \text{Tdeg}_F A_Z + \text{trdeg} Z \) where \( F \) is the quotient field of \( Z \).

\textbf{References}


Department of Mathematics, Box 354350, University of Washington, Seattle, Washington 98195

E-mail address: smith@math.washington.edu

E-mail address: zhang@math.washington.edu