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A REMARK ON GELFAND-KIRILLOV DIMENSION

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ABSTRACT. Let A be a finitely generated non-PI Ore domain and Q the quotient division algebra of A. If C is the center of Q, then $\operatorname{GKdim} C \leq \operatorname{GKdim} A - 2$.

Throughout k is a commutative field and $\dim_k$ is the dimension of a k-vector space. Let A be a k-algebra and M a right A-module. The **Gelfand-Kirillov dimension** of M is

$$\operatorname{GKdim} M = \sup_{V, M_0} \overline{\lim}_{n\to\infty} \log_n \dim_k M_0 V^n$$

where the supremum is taken over all finite dimensional subspaces $V \subset A$ and $M_0 \subset M$. If $F \supset k$ is another central subfield of A, we may also consider the Gelfand-Kirillov dimension of M over F which will be denoted by GKdim_F to indicate the change of the field. We refer to [BK], [GK] and [KL] for more details.

Let Z be a central subdomain of A. Then A is localizable over Z and the localization is denoted by A_Z. For any right A-module M, $M \otimes A_Z$ is denoted by M_Z. Let F be the quotient field of Z. The first author [Sm, 2.7] proved the following theorem:

Let A be an almost commutative algebra and Z a central subdomain. Suppose M is a right A-module such that $M_Z \neq 0$. Then

$$\operatorname{GKdim} M \geq \operatorname{GKdim}_F M_Z + \operatorname{GKdim} Z.$$

As a consequence of this, if A is almost commutative but non-PI and Z is a central subalgebra such that every nonzero element in Z is regular in A, then $\operatorname{GKdim} Z \leq \operatorname{GKdim} A - 2$.

It is natural to ask if the above theorem (and hence the consequence) is true for all algebras. In this paper we will precisely prove this.

Theorem 1. *Let A be an algebra and Z a central subdomain. Suppose M is a right A-module such that $M_Z \neq 0$. Then*

$$\operatorname{GKdim} M \geq \operatorname{GKdim}_F M_Z + \operatorname{GKdim} Z.$$

An algebra is called **locally PI** if every finitely generated subalgebra is PI. As a consequence of Theorem 1, we have

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Corollary 2. *Let A be algebra and Z a central subdomain. If A_Z is nonzero, then*

$$\mathrm{GKdim}\, A \geq \mathrm{GKdim}_F\, A_Z + \mathrm{GKdim}\, Z.$$

Furthermore, if A_Z is not locally PI, then

$$\mathrm{GKdim}\, A \geq 2 + \mathrm{GKdim}\, Z.$$

For the second inequality in Corollary 2, Z need not be a domain. Let Z be any central subalgebra of A of finite GKdimension such that A_Z is not locally PI. By the Noether normalization theorem, there is a subalgebra $Z_1 \subset Z$ isomorphic to the polynomial ring on d variables where $d = \mathrm{GKdim}\, Z$. Since $A_Z = (A_{Z_1})_Z$, A_{Z_1} is nonzero and not locally PI. Hence, by Corollary 2, $\mathrm{GKdim}\, Z_1 \leq \mathrm{GKdim}\, A - 2$. Therefore $\mathrm{GKdim}\, Z = \mathrm{GKdim}\, Z_1 \leq \mathrm{GKdim}\, A - 2$.

A stronger version of Corollary 2 also holds. We need another invariant defined by Gelfand and Kirillov. Let A be an algebra. The **Gelfand-Kirillov transcendence degree** of A is

$$\mathrm{Tdeg}\, A = \sup_V \inf_b \mathrm{GKdim}\, k[bV]$$

where V ranges over all finite dimensional subspaces of A and b ranges over the regular elements of A. If A is a commutative domain, then both $\mathrm{GKdim}\, A$ and $\mathrm{Tdeg}\, A$ are equal to the classical transcendence degree of A, denoted by $\mathrm{trdeg}\, A$. If $F \supset k$ is a central field of A, the Gelfand-Kirillov transcendence degree of A over F will be denoted by Tdeg_F to indicate the change of the field.

Theorem 3. *Let A be a semiprime Goldie algebra and Q the classical quotient algebra of A. Let F be a central subfield of Q. Then*

$$\mathrm{Tdeg}\, Q \geq \mathrm{Tdeg}_F\, Q + \mathrm{trdeg}\, F.$$

If moreover A is not locally PI, then

$$\mathrm{GKdim}\, A \geq 2 + \mathrm{GKdim}\, F.$$

The statement in the abstract is an obvious consequence of Theorem 3.

We now give the proofs. For simplicity a **subspace** means a finite dimensional subspace over k and a **subframe** of an algebra means a subspace containing the identity. Our proofs are based on the following easy observation.

Lemma 4. *Let $F \supset k$ be a commutative field and M a right F-module. Let $M_0 \subset M$ and $W \subset F$ be subspaces over k. Then*

$$\dim_k M_0 W \geq (\dim_F M_0 F)(\dim_k W).$$

Proof. Pick a basis of M_0F over F, say $\{x_1, \cdots, x_p\} \subset M_0$. Then $M_0F = \oplus_{i=1}^p x_i F$ and hence $M_0W \supset \oplus_{i=1}^p x_i W$. Therefore $\dim_k M_0W \geq (\dim_F M_0F)(\dim_k W)$. □

Proof of Theorem 1. Since Z is central, by the proof of [KL, 4.2], we have $\mathrm{GKdim}\, M \geq \mathrm{GKdim}\, M_Z$. By [KL, 4.2], $\mathrm{GKdim}\, Z = \mathrm{GKdim}\, F$ where F is the quotient field of Z. Hence it suffices to show $\mathrm{GKdim}\, M_Z \geq \mathrm{GKdim}_F\, M_Z + \mathrm{GKdim}\, F$. Therefore we may assume $Z = F$ is a central field of A, and we need to show that $\mathrm{GKdim}\, M \geq \mathrm{GKdim}_F\, M + \mathrm{GKdim}\, F$. Let d be any number less than $\mathrm{GKdim}\, F$. Then there exists a subframe $S \subset F$ such that $\dim_k S^n \geq n^d$ for all $n \gg 0$. Let e be any number less than $\mathrm{GKdim}_F\, M$. Then there exist a subspace $M_0 \subset M$, and a subframe $V \subset A$

such that $\dim_F M_0F(VF)^n \geq n^e$ for infinitely many n. Since $A \supset F$, we may assume $V \supset S$. Since F is central, $M_0F(VF)^n = M_0V^nF$. By Lemma 4,

$$\dim_k M_0V^{2n} \geq \dim_k M_0V^nS^n \geq (\dim_F M_0V^nF)(\dim_k S^n) \geq n^e n^d = n^{e+d}$$

for infinitely many n. Hence $\operatorname{GKdim} M \geq e + d$. By the choices of e and d, we obtain $\operatorname{GKdim} M \geq \operatorname{GKdim}_F M + \operatorname{GKdim} F$ as desired. □

Proof of Corollary 2. The first inequality follows from Theorem 1.1 by letting $M = A$. If A_Z is not locally PI, then $\operatorname{GKdim}_F A_Z > 1$ by [SSW], and $\operatorname{GKdim}_F A_Z \geq 2$ by [Be]. Hence the second inequality follows. □

As pointed out in [Sm, p. 37] the inequalities in Corollary 2 may be strict even if Z is the maximal central subring. By a result of M. Lorenz [Lo] the same example in [Sm, p. 37] shows also that the inequalities in Theorem 3 may be strict. The proof of Theorem 3 is similar to that of Theorem 1.

Proof of Theorem 3. Since F is commutative, for any $d < \operatorname{trdeg} F (= \operatorname{GKdim} F)$, there is a subframe $S \subset F$ such that $\dim_k S^n \geq n^d$ for all $n \gg 0$. Let e be any number less than $\operatorname{Tdeg}_F Q$. By the proof of [Zh, 3.1] there is a subframe $V \subset A$ such that for every regular element $b \in Q$, $\operatorname{GKdim} F[bVF] > e$. This is equivalent to saying that, for every regular element $b \in Q$, $\dim_F(F + bVF)^n \geq n^e$ for infinitely many n. We may assume $V \supset S$. Since F is central, $\dim_F(k + bV)^n b^n F = \dim_F(F + bVF)^n$. By Lemma 4,

$$\dim_k(k + bV)^n(bS)^n \geq (\dim_F(F + bVF)^n)(\dim_k S^n).$$

Hence

$$\dim_k(k + bV)^{2n} \geq \dim_k(k + bV)^n(bS)^n \geq n^e n^d = n^{e+d}$$

for infinitely many n. This means that $\operatorname{GKdim} k[bV] \geq e + d$ and hence $\operatorname{Tdeg} Q \geq e + d$. By the choices of e and d, $\operatorname{Tdeg} Q \geq \operatorname{Tdeg}_F Q + \operatorname{trdeg} F$.

Now we assume A is not locally PI. Then Q is not locally PI. By [SSW] and [Be], $\operatorname{GKdim}_F Q \geq 2$ and by [Zh, 4.1 and 4.3], $\operatorname{Tdeg}_F Q \geq 2$. Therefore by [Zh, 2.1 and 3.1]

$$\operatorname{GKdim} A \geq \operatorname{Tdeg} A \geq \operatorname{Tdeg} Q \geq \operatorname{Tdeg}_F Q + \operatorname{trdeg} F \geq 2 + \operatorname{GKdim} F.$$

□

If Z is a central subdomian of A, we can similarly prove that $\operatorname{Tdeg} A \geq \operatorname{Tdeg}_F A_Z$ $+ \operatorname{trdeg} Z$ where F is the quotient field of Z.

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