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POSITIVE DEFINITENESS AND COMMUTATIVITY OF OPERATORS

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ABSTRACT. It is shown that an n–tuple of bounded linear operators on a complex Hilbert space, which is positive definite in the sense of Halmos, must be commutative. Some generalizations of this result to the case of pairs of unbounded operators are obtained.

INTRODUCTION

The Halmos–Bram characterization of bounded subnormal operators [7], [1], [3] has been generalized to the case of commutative families of bounded linear operators by Ito [8]. The Ito theorem says that such a family is subnormal if and only if it is positive definite in the sense of Halmos (H–positive definite for short). This result has further been generalized to the case of families of unbounded operators by the author and Szafraniec [16], and independently by Jorgensen [9]. Contrary to [16], the families of operators taken into consideration in [9] have not been assumed to be commutative; commutativity follows from H–positive definiteness due to [9, Theorem 4.2]:

(J) If a pair of linear operators with common invariant domain is H–positive definite, then it is commutative.

However, the Jorgensen proof of [9, Theorem 4.2] is not correct because the equality (3) on page 518 holds if and only if the operators in question commute (this has been noticed by J. Niechwiej). In other words (J) can only be considered as a conjecture.

In this paper we show that the conjecture (J) is true provided one of the following two conditions holds: 1^0 both components of the pair are bounded (see Theorem 3.2), 2^0 the first component of the pair is either symmetric or unitary and the other one is arbitrary (see Proposition 4.1); (J) is also true for some pairs of operators whose first component is unbounded and the other one bounded (see Theorem 3.3). The crucial role in proofs of these results is played by Theorem 2.1 which describes in an algebraic way H–positive definiteness. Theorem 2.1 can eventually be used to construct a counterexample to the conjecture (J).

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There is yet another characterization of bounded subnormal operators due to Sz.–Nagy [21]. It says that a bounded linear operator is subnormal if and only if it is positive definite in the sense of Sz.–Nagy (S–positive definite for short). It turns out that the conjecture (J) is true provided the expression "H–positive definite" is replaced by "S–positive definite" (see Proposition 5.1).

1. Preliminaries

All linear spaces taken into consideration in this paper are assumed to be complex. Let $\mathcal{D}$ and $\mathcal{E}$ be inner product spaces. Denote by $\boldsymbol{L}(\mathcal{D},\mathcal{E})$ the linear space of all linear operators with domain equal to $\mathcal{D}$ and range contained in $\mathcal{E}$. The space $\boldsymbol{L}(\mathcal{D}) := \boldsymbol{L}(\mathcal{D},\mathcal{D})$ is an algebra with the identity operator $I_{\mathcal{D}}$ as a unit. The algebra of all bounded operators $A \in \boldsymbol{L}(\mathcal{D})$ is denoted by $\boldsymbol{B}(\mathcal{D})$. $\boldsymbol{L}^{\#}(\mathcal{D})$ stands for the $*$–algebra of all operators $A \in \boldsymbol{L}(\mathcal{D})$ for which there exists $A^{\#} \in \boldsymbol{L}(\mathcal{D})$ such that $\langle Af, g\rangle = \langle f, A^{\#}g\rangle$ for $f, g \in \mathcal{D}$; the involution is given by the mapping $A \longmapsto A^{\#}$. If $n \geq 1$, then $\boldsymbol{L}(\mathcal{D})^n$ stands for the n–fold Cartesian product of $\boldsymbol{L}(\mathcal{D})$ by itself. $\boldsymbol{B}(\mathcal{D})^n$ is defined similarly. Given a set Ω, we denote by $\mathcal{F}(\Omega,\mathcal{D})$ the linear space of all functions $f : \Omega \longrightarrow \mathcal{D}$ with finite supports $\{\alpha \in \Omega : f(\alpha) \neq 0\}$.

If $\mathcal{H}$ is a Hilbert space completion of $\mathcal{D}$ and $A \in \boldsymbol{L}(\mathcal{D})$, then we can consider A as a densely defined operator in $\mathcal{H}$ and, consequently, we can talk about its closure $\overline{A}$ as well as of its adjoint A^* within $\mathcal{H}$. One can check that such A belongs to $\boldsymbol{L}^{\#}(\mathcal{D})$ if and only if $A^*(\mathcal{D}) \subseteq \mathcal{D}$; if this is the case, then $A^{\#} = A^*|_{\mathcal{D}}$. In the sequel we will use the notation $\overline{A}$ and A^* without any explicit specification of a Hilbert space completion of $\mathcal{D}$.

Given an n–tuple $\boldsymbol{A} = (A_1,\dots,A_n) \in \boldsymbol{L}(\mathcal{D})^n$, we write $\boldsymbol{A}^\alpha = A_1^{\alpha_1}\cdots A_n^{\alpha_n}$ for $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{Z}_+^n$, where $\mathbb{Z}_+ = \{0,1,2,\dots\}$. In case $A_1,\dots,A_n$ commute, $\boldsymbol{A}$ is said to be *commutative*. We say that $\boldsymbol{A}$ is *H–positive definite* if

$$\sum_{\alpha,\beta\in\mathbb{Z}_+^n} \langle \boldsymbol{A}^\alpha f(\beta), \boldsymbol{A}^\beta f(\alpha)\rangle \geq 0, \quad f \in \mathcal{F}(\mathbb{Z}_+^n,\mathcal{D}).$$

$\boldsymbol{A}$ is said to be *subnormal* if there exist a Hilbert space $\mathcal{K} \supseteq \mathcal{D}$ and an n–tuple $\boldsymbol{N} = (N_1,\dots,N_n)$ of commuting normal operators[1] in $\mathcal{K}$ such that $A_j \subseteq N_j$ for $j = 1,\dots n$; such $\boldsymbol{N}$ is called a *commutative normal extension* of $\boldsymbol{A}$ (note that if $\boldsymbol{A} \in \boldsymbol{B}(\mathcal{D})^n$ is subnormal, then there always exists a commutative normal extension $\boldsymbol{N} \in \boldsymbol{B}(\mathcal{K})^n$). It is well–known that if $\boldsymbol{A}$ is subnormal, then $\boldsymbol{A}$ is commutative and H–positive definite. The converse implication is not true in general (cf. [2], [12], [15]). However if $\boldsymbol{A}$ has sufficiently many analytic or quasianalytic vectors, then the converse implication holds (cf. [16], [9]); in particular this is the case for $\boldsymbol{A} \in \boldsymbol{B}(\mathcal{D})^n$ (cf. [8]).

Theorem 1.1. *If $\boldsymbol{A} \in \boldsymbol{B}(\mathcal{D})^n$ $(n \geq 2)$, then $\boldsymbol{A}$ is subnormal if and only if it is commutative and H–positive definite.*

Denote by $\boldsymbol{S}(\mathcal{D})$ the linear space of all sesquilinear forms over $\mathcal{D}$. Let Ω be an arbitrary set. A kernel $\Phi : \Omega\times\Omega \longrightarrow \boldsymbol{S}(\mathcal{D})$ is said to be *positive definite* if

$$\sum_{\alpha,\beta\in\Omega} \Phi(\alpha,\beta)(f(\beta), f(\alpha)) \geq 0, \quad f \in \mathcal{F}(\Omega,\mathcal{D}).$$

Recall now a version of the Kolmogorov–Aronszajn factorization theorem (cf. [6], [11, KMKA Lemma], [14, Theorem 1.1]).

[1] Recall that two normal operators commute if their spectral measures do.

Theorem 1.2. *If a kernel $\Phi : \Omega \times \Omega \longrightarrow \boldsymbol{S}(\mathcal{D})$ is positive definite, then there exist an inner product space $\mathcal{E}$ and a function $\Psi : \Omega \longrightarrow \boldsymbol{L}(\mathcal{D}, \mathcal{E})$ such that*

- $\Phi(\alpha,\beta)(f,g) = \langle \Psi(\beta)f, \Psi(\alpha)g\rangle$ *for $\alpha, \beta \in \Omega$ and $f, g \in \mathcal{D}$,*
- *$\mathcal{E}$ is the linear span of $\{\Psi(\alpha)f : \alpha \in \Omega, f \in \mathcal{D}\}$.*

Let $(\mathcal{G}, +)$ be a commutative $*$–semigroup with the neutral element 0. A function $\Phi : \mathcal{G} \longrightarrow \boldsymbol{S}(\mathcal{D})$ is said to be *$*$–positive definite* if the kernel $\mathcal{G} \times \mathcal{G} \ni (\alpha, \beta) \longmapsto \Phi(\alpha^* + \beta) \in \boldsymbol{S}(\mathcal{D})$ is positive definite. In the sequel we need the following dilation theorem (cf. [19, Proposition] and also [10, Theorem 4.7]).

Theorem 1.3. *If $\Phi : \mathcal{G} \longrightarrow \boldsymbol{S}(\mathcal{D})$ is $*$–positive definite and $\Phi(0)(f, g) = \langle f, g\rangle$ for $f, g \in \mathcal{D}$, then there exist an inner product space $\mathcal{E} \supseteq \mathcal{D}$ and an involution preserving semigroup homomorphism $\Pi : \mathcal{G} \longrightarrow \boldsymbol{L}^{\#}(\mathcal{E})$ such that*

- $\Phi(\alpha)(f, g) = \langle \Pi(\alpha)f, g\rangle$ *for $f, g \in \mathcal{D}$ and $\alpha \in \mathcal{G}$.*

We say that $N \in \boldsymbol{L}^{\#}(\mathcal{D})$ is *formally normal* if $N^{\#}N = NN^{\#}$ or equivalently if $\|Nf\| = \|N^{\#}f\|$ for each $f \in \mathcal{D}$. If $N \in \boldsymbol{L}^{\#}(\mathcal{D})$ is formally normal, then

$$\|N^n f\|^2 = \langle N^{n-1}f, N^{\#}N^n f\rangle \leq \|N^{n-1}f\|\,\|N^{n+1}f\|, \qquad f \in \mathcal{D},\ n \geq 1,$$

which means that the sequence $\{\|N^n f\|\}_{n=0}^{\infty}$ is logarithmically convex for every $f \in \mathcal{D}$. Consequently, for every $f \in \mathcal{D}$, the limit $\mu_N(f) := \lim_{n\to\infty} \|N^n f\|^{1/n}$ exists at least in the closed interval $[0, \infty]$. Below we collect some properties of the function $\mu_N : \mathcal{D} \longrightarrow [0, \infty]$ (cf. [13], [4], [20], [17] for related results).

Lemma 1.4. *If $N \in \boldsymbol{L}^{\#}(\mathcal{D})$ is formally normal, then*

(i) $\|Nf\| \leq \mu_N(f)\,\|f\|$ *for $f \in \mathcal{D}$,*
(ii) $\mu_N(f + g) \leq \max\{\mu_N(f), \mu_N(g)\}$ *for $f, g \in \mathcal{D}$,*
(iii) $\mu_N(\lambda f) = \mu_N(f)$ *for $\lambda \in \mathbb{C} \setminus \{0\}$ and $f \in \mathcal{D}$,*
(iv) $\mu_N(Nf) = \mu_N(N^{\#}f) = \mu_N(f)$ *for $f \in \mathcal{D}$,*
(v) *for any $a \geq 0$, the set $\{f \in \mathcal{D} : \mu_N(f) \leq a\}$ is a linear subspace of $\mathcal{D}$ which is invariant for N and $N^{\#}$.*

Proof. (i). By the formal normality of N, we have

$$\|Nf\| = \langle N^{\#}Nf, f\rangle^{1/2} \leq \|f\|^{1/2}\,\|N^{\#}Nf\|^{1/2} = \|f\|^{1/2}\,\|N^2 f\|^{1/2}, \quad f \in \mathcal{D}.$$

Applying the above inequality to the formally normal operator N^2, we get

$$\|Nf\| \leq \|f\|^{1/2}\,\|N^2 f\|^{1/2} \leq \|f\|^{1/2}\,\|f\|^{1/2^2}\,\|N^{2^2}f\|^{1/2^2}, \quad f \in \mathcal{D}.$$

Now an induction procedure leads to

$$\|Nf\| \leq \|f\|^{(1/2)+\cdots+(1/2^n)}\|N^{2^n}f\|^{1/2^n}, \quad f \in \mathcal{D}.$$

Passing with n to ∞ we get (i).

(ii). If $t > \max\{\mu_N(f), \mu_N(g)\}$, then there is $s > 0$ such that $\|N^n f\| \leq st^n$ and $\|N^n g\| \leq st^n$ for $n \geq 1$. Consequently $\|N^n(f + g)\| \leq 2st^n$ for $n \geq 1$. This in turn implies that $\mu_N(f + g) \leq t$. Passing with t to $\max\{\mu_N(f), \mu_N(g)\}$ we get (ii).

(iii) is obvious. Since N is formally normal, we have

$$\|N^n(Nf)\| = \|N^n(N^{\#}f)\| = \|N^{n+1}f\|, \quad f \in \mathcal{D},\ n \geq 0,$$

which in turn implies (iv). (v) follows from (ii), (iii) and (iv). □

Given $A \in \boldsymbol{L}(\mathcal{D})$ we denote by $\mathcal{Q}(A)$ the set of all *quasianalytic vectors* of A (i.e. $f \in \mathcal{Q}(A)$ if and only if $\sum_{n=1}^{\infty} \|A^n f\|^{-1/n} = \infty$).

Lemma 1.5. *If $N \in \boldsymbol{L}^{\#}(\mathcal{D})$ is formally normal and $\mathcal{D}$ is the linear span of $\mathcal{Q}(N)$, then $\overline{N}$ and $\overline{N^{\#}}$ are normal operators and $\overline{N} = (N^{\#})^*$.*

Proof. Since N is formally normal, it must be $\mathcal{Q}(N) = \mathcal{Q}(N^{\#})$. Hence, by [16, Theorem 1], both the operators $\overline{N}$ and $\overline{N^{\#}}$ are normal. Consequently N^* is normal. However $\overline{N^{\#}} \subseteq N^*$, so it must be $\overline{N^{\#}} = N^*$ or equivalently $(N^{\#})^* = \overline{N}$. □

2. H–positive definiteness

In this section we characterize pairs of operators, which are H–positive definite. Given a pair $\boldsymbol{A} = (A_1, A_2) \in \boldsymbol{L}(\mathcal{D})^2$, we denote by $\digamma(\boldsymbol{A})$ the set of all pairs (N_1, N_2) of formally normal operators $N_j \in \boldsymbol{L}^{\#}(\mathcal{D}_j)$, $j = 1, 2$, which satisfy the following conditions:

(F1) $\mathcal{D} \subseteq \mathcal{D}_1 \subseteq \mathcal{D}_2$,
(F2) $A_j \subseteq N_j$ for $j = 1, 2$,
(F3) $N_2(\mathcal{D}_1) \subseteq \mathcal{D}_1$,
(F4) $N_2 N_1^{\#} f = N_1^{\#} N_2 f$ for $f \in \mathcal{D}_1$,
(F5) $\mathcal{D}_1$ is the linear span of $\{N_1^{\#k} f : f \in \mathcal{D},\ k \geq 0\}$,
(F6) $\mathcal{D}_2$ is the linear span of $\{N_2^{\#k} f : f \in \mathcal{D}_1,\ k \geq 0\}$.

H–positive definiteness can be described in terms of the set $\digamma(\boldsymbol{A})$ as follows.

Theorem 2.1. *The pair $\boldsymbol{A} = (A_1, A_2) \in \boldsymbol{L}(\mathcal{D})^2$ is H–positive definite if and only if $\digamma(\boldsymbol{A}) \neq \varnothing$.*

Proof. If $(N_1, N_2) \in \digamma(\boldsymbol{A})$, then applying (F1)÷(F4) we get

$$\begin{aligned} \langle \boldsymbol{A}^\alpha f, \boldsymbol{A}^\beta g\rangle &= \langle N_1^{\alpha_1} N_2^{\alpha_2} f, N_1^{\beta_1} N_2^{\beta_2} g\rangle \\ &= \langle N_1^{\#\beta_1} N_2^{\alpha_2} f, N_1^{\#\alpha_1} N_2^{\beta_2} g\rangle \\ &= \langle N_2^{\alpha_2} N_1^{\#\beta_1} f, N_2^{\beta_2} N_1^{\#\alpha_1} g\rangle \\ &= \langle N_2^{\#\beta_2} N_1^{\#\beta_1} f, N_2^{\#\alpha_2} N_1^{\#\alpha_1} g\rangle, \quad f, g \in \mathcal{D},\ \alpha, \beta \in \mathbb{Z}_+^2, \end{aligned} \tag{2.1}$$

so

$$\sum_{\alpha,\beta\in\mathbb{Z}_+^2} \langle \boldsymbol{A}^\alpha f(\beta), \boldsymbol{A}^\beta f(\alpha)\rangle = \Big\| \sum_{\alpha\in\mathbb{Z}_+^2} N_2^{\#\alpha_2} N_1^{\#\alpha_1} f(\alpha)\Big\|^2 \geq 0, \quad f \in \mathcal{F}(\mathbb{Z}_+^2, \mathcal{D}).$$

Suppose now that $\boldsymbol{A}$ is H–positive definite. Then the kernel $\varPhi_{\boldsymbol{A}} : \mathbb{Z}_+^2 \times \mathbb{Z}_+^2 \longrightarrow \boldsymbol{S}(\mathcal{D})$ defined by $\varPhi_{\boldsymbol{A}}(\alpha, \beta)(f, g) = \langle \boldsymbol{A}^\alpha f, \boldsymbol{A}^\beta g\rangle$ $(f, g \in \mathcal{D},\ \alpha, \beta \in \mathbb{Z}_+^2)$ is positive definite. By Theorem 1.2 there exist an inner product space $\mathcal{D}_2$ and a function $\varPsi : \mathbb{Z}_+^2 \longrightarrow \boldsymbol{L}(\mathcal{D}, \mathcal{D}_2)$ such that

$$\langle \boldsymbol{A}^\alpha f, \boldsymbol{A}^\beta g\rangle = \langle \varPsi(\beta) f, \varPsi(\alpha) g\rangle, \quad f, g \in \mathcal{D},\ \alpha, \beta \in \mathbb{Z}_+^2, \tag{2.2}$$

$$\mathcal{D}_2 \text{ is the linear span of } \{\varPsi(\alpha) f : f \in \mathcal{D},\ \alpha \in \mathbb{Z}_+^2\}. \tag{2.3}$$

Inserting $\alpha = \beta = \mathbf{0} := (0, 0)$ into (2.2) we conclude that the linear operator $\varPsi(\mathbf{0})$ is an isometry. Therefore, replacing $\mathcal{D}$ by $\varPsi(\mathbf{0})\mathcal{D}$, A_j by $\varPsi(\mathbf{0}) A_j \varPsi(\mathbf{0})^{-1}$ and $\varPsi(\alpha)$ by $\varPsi(\alpha)\varPsi(\mathbf{0})^{-1}$, we can assume without loss of generality that $\mathcal{D} \subseteq \mathcal{D}_2$ and

$$\varPsi(\mathbf{0}) \subseteq I_{\mathcal{D}_2}. \tag{2.4}$$

Set $\varOmega_1 = \mathbb{Z}_+ \times \{0\}$, $\varOmega_2 = \mathbb{Z}_+^2$, $e_1 = (1, 0) \in \varOmega_1$ and $e_2 = (0, 1) \in \varOmega_2$. Define $\mathcal{D}_1$ as the linear span of $\{\varPsi(\alpha) f : \alpha \in \varOmega_1,\ f \in \mathcal{D}\}$. According to (2.4), the condition

(F1) holds. It follows from (2.2) that

$$\begin{aligned}\langle \varPsi(\alpha+e_j)f, \varPsi(\beta)g\rangle &= \langle \boldsymbol{A}^{\beta} f, \boldsymbol{A}^{\alpha} A_j g\rangle \\ &= \langle \varPsi(\alpha)f, \varPsi(\beta)A_j g\rangle, \quad f,g\in\mathcal{D},\ \alpha,\beta\in\varOmega_j,\ j=1,2.\end{aligned} \tag{2.5}$$

One can deduce from (2.3) and (2.5) that for every $j\in\{1,2\}$ there are operators $S_j, N_j \in \boldsymbol{L}(\mathcal{D}_j)$ such that

$$N_j\varPsi(\beta)g = \varPsi(\beta)A_j g, \quad g\in\mathcal{D},\ \beta\in\varOmega_j,\ j=1,2, \tag{2.6}$$

$$S_j\varPsi(\alpha)f = \varPsi(\alpha+e_j)f, \quad f\in\mathcal{D},\ \alpha\in\varOmega_j,\ j=1,2, \tag{2.7}$$

$$\langle S_j f, g\rangle = \langle f, N_j g\rangle, \quad f,g\in\mathcal{D}_j,\ j=1,2. \tag{2.8}$$

The equality (2.8) implies that $S_j, N_j \in \boldsymbol{L}^{\#}(\mathcal{D}_j)$ and $S_j = N_j^{\#}$. According to (2.6) and (2.7), we have $S_jN_j = N_jS_j$. Hence the operator N_j is formally normal. Moreover, by (2.7), the operator $N_j^{\#}$ acts as follows:

$$N_j^{\#}\varPsi(\alpha)f = \varPsi(\alpha+e_j)f, \quad f\in\mathcal{D},\ \alpha\in\varOmega_j,\ j=1,2. \tag{2.9}$$

It is a matter of direct verification that (2.4), (2.6) and the definition of $\mathcal{D}_1$ yield (F2) and (F3). The condition (F4) can easily be deduced from (2.6) and (2.9). Applying (2.4) and (2.9), we get

$$\varPsi(\alpha)f = N_2^{\#\alpha_2}N_1^{\#\alpha_1}f, \quad f\in\mathcal{D},\ \alpha\in\mathbb{Z}_+^2,$$

which together with (2.3) implies (F5) and (F6). Thus $(N_1,N_2)\in\mathcal{F}(\boldsymbol{A})$. □

It turns out that, up to unitary equivalence, the set $\mathcal{F}(\boldsymbol{A})$ contains at most one element.

Proposition 2.2. *If $(N_1,N_2),(N_1',N_2')\in\mathcal{F}(\boldsymbol{A})$, then there is a unitary operator $U:\mathcal{D}_2\longrightarrow\mathcal{D}_2'$ such that*
(F7) $U|_{\mathcal{D}} = I_{\mathcal{D}}$,
(F8) $U(\mathcal{D}_1)=\mathcal{D}_1'$,
(F9) $U|_{\mathcal{D}_1}N_1 = N_1'\,U|_{\mathcal{D}_1}$,
(F10) $UN_2 = N_2'\,U$.

Proof. Since $(N_1,N_2),(N_1',N_2')\in\mathcal{F}(\boldsymbol{A})$, we conclude from (2.1) that

$$\begin{aligned}\langle N_2^{\#\beta_2}N_1^{\#\beta_1}f, N_2^{\#\alpha_2}N_1^{\#\alpha_1}g\rangle = \langle (N_2')^{\#\beta_2}(N_1')^{\#\beta_1}f, (N_2')^{\#\alpha_2}(N_1')^{\#\alpha_1}g\rangle,\\ f,g\in\mathcal{D},\ \alpha,\beta\in\mathbb{Z}_+^2.\end{aligned} \tag{2.10}$$

By (F5), (F6) and (2.10), there is a unique unitary operator $U:\mathcal{D}_2\longrightarrow\mathcal{D}_2'$ such that

$$U(N_2^{\#k}N_1^{\#l}f) = (N_2')^{\#k}(N_1')^{\#l}f, \quad f\in\mathcal{D},\ k,l\geq 0. \tag{2.11}$$

Now the conditions (F7)÷(F10) can be inferred from (2.11) via (F1)÷(F6). □

We conclude this section with the following observation whose proof is left to the reader.

Remark. Let $(N_1,N_2)\in\mathcal{F}(\boldsymbol{A})$. If (N_1',N_2') is a pair of operators $N_j'\in\boldsymbol{L}(\mathcal{D}_j')$, $j=1,2$, and $U:\mathcal{D}_2\longrightarrow\mathcal{D}_2'$ is a unitary operator such that (F7)÷(F10) hold, then $(N_1',N_2')\in\mathcal{F}(\boldsymbol{A})$.

3. Bounded components

In this section we prove that every n–tuple of bounded operators, which is H–positive definite, is commutative. The method of proof works also for some pairs of operators whose first component is unbounded and the other one bounded.

To begin with we characterize those pairs $(N_1, N_2) \in \mathcal{F}(\boldsymbol{A})$, $\boldsymbol{A} \in \boldsymbol{L}(\mathcal{D})^2$, whose components are bounded.

Lemma 3.1. *If $\boldsymbol{A} = (A_1, A_2) \in \boldsymbol{L}(\mathcal{D})^2$ and $(N_1, N_2) \in \mathcal{F}(\boldsymbol{A})$, then*

(i) *N_1 is bounded if and only if so is A_1; moreover $\|N_1\| = \|A_1\|$,*
(ii) *N_2 is bounded if and only if $a := \sup_{f\in\mathcal{D}} \sup_{k\geq 0} \lim_{n\to\infty} \|A_1^k A_2^n f\|^{1/n} < \infty$; moreover $\|N_2\| = a$.*

Proof. (i). If A_1 is bounded, then by (F2) we have $\mu_{N_1}(f) \leq \|A_1\|$ for $f \in \mathcal{D}$. Applying (F5) and Lemma 1.4 (v), we get $\mu_{N_1}(f) \leq \|A_1\|$ for $f \in \mathcal{D}_1$. Hence, by Lemma 1.4 (i), the operator N_1 is bounded and $\|N_1\| \leq \|A_1\|$. The converse implication and inequality follow from (F2).

(ii). Assume that a is finite. By the formal normality of N_1 and (F1)÷(F4), we have

$$\|N_2^n(N_1^{\#k} f)\| = \|N_1^{\#k} N_2^n f\| = \|N_1^k N_2^n f\| = \|A_1^k A_2^n f\|, \quad f \in \mathcal{D},\ k, n \geq 0, \tag{3.1}$$

so $\mu_{N_2}(N_1^{\#k} f) \leq a$ for $f \in \mathcal{D}$ and $k \geq 0$. By (F5) and Lemma 1.4 (v), this is equivalent to $\mu_{N_2}(f) \leq a$ for $f \in \mathcal{D}_1$. Applying (F6) and once more Lemma 1.4 (v), we get $\mu_{N_2}(f) \leq a$ for $f \in \mathcal{D}_2$; so, by Lemma 1.4 (i), N_2 is bounded and $\|N_2\| \leq a$. The converse implication and inequality can easily be deduced from (3.1). □

We are now in a position to prove that Theorem 1.1 remains true without assuming commutativity of the operators involved.

Theorem 3.2. *If $\boldsymbol{A} = (A_1, \ldots, A_n) \in \boldsymbol{B}(\mathcal{D})^n$ $(n \geq 2)$ is H–positive definite, then $\boldsymbol{A}$ is subnormal.*

Proof. According to Theorem 1.1, it suffices to show that the operators $A_1, \ldots, A_n$ commute. Since obviously each pair (A_i, A_j), $1 \leq i < j \leq n$, is H–positive definite, we can restrict ourselves to the case $n = 2$.

It follows from Theorem 2.1 that there is $(N_1, N_2) \in \mathcal{F}(\boldsymbol{A})$ with $N_j \in \boldsymbol{L}^{\#}(\mathcal{D}_j)$, $j = 1, 2$. Since $\lim_{n\to\infty} \|A_1^k A_2^n f\|^{1/n} \leq \|A_2\| \lim_{n\to\infty} (\|A_1^k\| \|f\|)^{1/n} \leq \|A_2\|$ for $f \in \mathcal{D}$ and $k \geq 0$, we conclude from Lemma 3.1 that the operators N_1 and $M_2 := N_2|_{\mathcal{D}_1}$ are bounded. Notice that $M_2 \in \boldsymbol{B}(\mathcal{D}_1)$ due to (F3). By (F4), we have $M_2 N_1^{\#} = N_1^{\#} M_2$, so $\overline{M}_2 N_1^* = N_1^* \overline{M}_2$. Since $\overline{N}_1$ is normal, the Fuglede theorem (cf. [5, Theorem I]) yields $\overline{M}_2 \overline{N}_1 = \overline{N}_1 \overline{M}_2$. Hence $A_2 A_1 = A_1 A_2$ due to (F2). □

It follows from the proof of Theorem 3.2 that if $\boldsymbol{A} = (A_1, A_2) \in \boldsymbol{L}(\mathcal{D})^2$ and $(N_1, N_2) \in \mathcal{F}(\boldsymbol{A})$, then the operators A_1 and A_2 are simultaneously bounded if and only if so are N_1 and N_2. Notice also that analysis similar to that in the proof of Theorem 3.2 enables us to prove in an elementary way Theorem 1.1 without using the Heinz inequality (as well as without using the Fuglede theorem). Indeed if $n = 2$, then the operator N_1 can be defined on the whole space $\mathcal{D}_2$ by the same formula as in (2.6), i.e.

$$N_1 \Psi(\beta) g = \Psi(\beta) A_1 g, \quad g \in \mathcal{D}, \beta \in \mathbb{Z}_+^2,$$

and the pair $(\overline{N}_1, \overline{N}_2)$ turns out to be a commutative bounded normal extension of $\boldsymbol{A}$ (the same reasoning applies to the case $n > 2$). This idea of proving Theorem 1.1 is similar to that from [18].

Theorem 3.2 can be generalized to the case of pairs of operators whose first component is unbounded as follows.

Theorem 3.3. *If $\boldsymbol{A} = (A_1, A_2) \in \boldsymbol{L}(\mathcal{D})^2$ is H–positive definite, $\mathcal{D}$ is the linear span of $\mathcal{Q}(A_1)$ and $\sup_{f\in\mathcal{D}} \sup_{k\geq 0} \lim_{n\to\infty} \|A_1^k A_2^n f\|^{1/n} < \infty$, then $\boldsymbol{A}$ is subnormal.*

Proof. It follows from Theorem 2.1 and Lemma 3.1(ii) that there is $(N_1, N_2) \in \digamma(\boldsymbol{A})$ with $N_j \in \boldsymbol{L}^\#(\mathcal{D}_j)$, $j = 1, 2$, such that $N_2 \in \boldsymbol{B}(\mathcal{D}_2)$. Since evidently $\mathcal{Q}(A_1) \subseteq \mathcal{Q}(N_1)$ and by virtue of [16, Proposition 2] $N_1^\#(\mathcal{Q}(N_1)) \subseteq \mathcal{Q}(N_1)$, we conclude from (F5) that $\mathcal{D}_1$ is the linear span of $\mathcal{Q}(N_1)$. According to Lemma 1.5, $\overline{N_1^\#}$ is normal and $\overline{N}_1 = (N_1^\#)^*$. By (F4), we have $M_2 N_1^\# = N_1^\# M_2$, where $M_2 := N_2|_{\mathcal{D}_1}$. Since $M_2 \in \boldsymbol{B}(\mathcal{D}_1)$, we get $\overline{M}_2\, \overline{N_1^\#} \subseteq \overline{N_1^\#}\, \overline{M}_2$. The Fuglede theorem yields $\overline{M}_2 (N_1^\#)^* \subseteq (N_1^\#)^* \overline{M}_2$ or equivalently $\overline{M}_2 \overline{N}_1 \subseteq \overline{N}_1 \overline{M}_2$. Hence $A_2 A_1 = A_1 A_2$. Since $A_2 \in \boldsymbol{B}(\mathcal{D})$ and, consequently, $\mathcal{Q}(A_2) = \mathcal{D}$, the subnormality of $\boldsymbol{A}$ follows from [16, Theorem 10]. □

Notice that the situation described in Theorem 3.3 can happen if and only if $\boldsymbol{A}$ is subnormal, $\mathcal{D}$ is the linear span of $\mathcal{Q}(A_1)$ and A_2 is bounded (because if A_1 and A_2 commute and A_2 is bounded, then $\sup_{f\in\mathcal{D}} \sup_{k\geq 0} \lim_{n\to\infty} \|A_1^k A_2^n f\|^{1/n} < \infty$).

4. Symmetric and unitary components

In this section we concentrate on pairs of operators whose first component is either symmetric or unitary while the other one is arbitrary. Recall that $A \in \boldsymbol{L}(\mathcal{D})$ is symmetric (resp. unitary) if and only if $A \in \boldsymbol{L}^\#(\mathcal{D})$ and $A = A^\#$ (resp. $A^{-1} = A^\#$).

Proposition 4.1. *If $\boldsymbol{A} = (A_1, A_2) \in \boldsymbol{L}(\mathcal{D})^2$ is H–positive definite and A_1 is either symmetric or unitary, then $\boldsymbol{A}$ is commutative.*

Proof. According to Theorem 2.1, there is $(N_1, N_2) \in \digamma(\boldsymbol{A})$ with $N_j \in \boldsymbol{L}^\#(\mathcal{D}_j)$, $j = 1, 2$.

Assume first that A_1 is symmetric. Then, by (2.1), we have

$$\tag{4.1} \begin{aligned} \langle N_2^{\#\beta_2} N_1^{\#(\beta_1+1)} f, N_2^{\#\alpha_2} N_1^{\#\alpha_1} g\rangle &= \langle \boldsymbol{A}^\alpha f, \boldsymbol{A}^{\beta+e_1} g\rangle \\ &= \langle \boldsymbol{A}^{\alpha+e_1} f, \boldsymbol{A}^\beta g\rangle \\ &= \langle N_2^{\#\beta_2} N_1^{\#\beta_1} f, N_2^{\#\alpha_2} N_1^{\#(\alpha_1+1)} g\rangle, \\ & \qquad f, g \in \mathcal{D},\ \alpha, \beta \in \mathbb{Z}_+^2. \end{aligned}$$

One can deduce from (F5), (F6) and (4.1) that there is a symmetric operator $M_1 \in \boldsymbol{L}^\#(\mathcal{D}_2)$ such that

$$\tag{4.2} M_1(N_2^{\#\alpha_2} N_1^{\#\alpha_1} g) = N_2^{\#\alpha_2} N_1^{\#(\alpha_1+1)} g, \quad g \in \mathcal{D},\ \alpha \in \mathbb{Z}_+^2.$$

It follows from (F5), (F6) and (4.2) that $N_1^\# \subseteq M_1$ and M_1 commutes with $N_2^\#$. Since M_1 is symmetric, we conclude that N_1 is symmetric and M_1 commutes with N_2. Hence N_1 and $N_2|_{\mathcal{D}_1}$ commute. By (F2), $\boldsymbol{A}$ is commutative.

Assume now that A_1 is unitary. Then, by the formal normality of N_1 and (F2), we have

$$\begin{aligned}\langle N_1 N_1^{\#}(N_1^{\#k} f), N_1^{\#l} g\rangle &= \langle A_1^{l+1} f, A_1^{k+1} g\rangle \\ &= \langle A_1^{l} f, A_1^{k} g\rangle \qquad\qquad f, g \in \mathcal{D},\ k, l \geq 0. \\ &= \langle N_1^{\#k} f, N_1^{\#l} g\rangle\end{aligned}$$

This and (F5) imply $N_1 N_1^{\#} = N_1^{\#} N_1 = I_{\mathcal{D}_1}$, so $N_1^{-1} = N_1^{\#}$. According to (F4), we have $M_2 N_1^{\#} = N_1^{\#} M_2$ with $M_2 := N_2|_{\mathcal{D}_1}$. Thus $N_1 M_2 = M_2 N_1$, which by virtue of (F2) implies $A_1 A_2 = A_2 A_1$. ☐

The following result concerning subnormality can easily be inferred from Proposition 4.1 and [16, Theorem 10′].

Corollary 4.2. *Suppose that $\boldsymbol{A} = (A_1, A_2) \in \boldsymbol{L}(\mathcal{D})^2$ is H–positive definite and $\mathcal{D}$ is the linear span of $\mathcal{Q}(A_2)$. If one of the following two conditions holds,*

(i) *A_1 is symmetric and $\mathcal{D}$ is the linear span of $\mathcal{Q}(A_1)$,*

(ii) *A_1 is unitary,*

then $\boldsymbol{A}$ is subnormal.

Combining Theorems 3.2 and 3.3 with Proposition 4.1, we get new sufficient conditions for an n–tuple of operators to be commutative or subnormal (e.g. an H–positive definite 4–tuple $\boldsymbol{A} = (A_1, A_2, A_3, A_4)$ with symmetric A_1 and A_2 and bounded A_3 and A_4 is commutative).

5. S–positive definiteness

In this section we show that an n–tuple $\boldsymbol{A} \in \boldsymbol{L}(\mathcal{D})^n$ is positive definite in the sense of Sz.–Nagy if and only if $\boldsymbol{A}$ is commutative and H–positive definite. In other words, positive definiteness in the sense of Sz.–Nagy seems to be more appropriate for subnormality of arbitrary a priori noncommutative n–tuples.

Denote by $\mathcal{N}_n$ the $*$–semigroup $\mathbb{Z}_+^n \times \mathbb{Z}_+^n$ with the coordinatewise addition as the semigroup operation and the involution defined by $(\alpha, \beta)^* = (\beta, \alpha)$ for $\alpha, \beta \in \mathbb{Z}_+^n$. We say that an n–tuple $\boldsymbol{A} \in \boldsymbol{L}(\mathcal{D})^n$ is *S–positive definite* if the function $\Phi_{\boldsymbol{A}} : \mathcal{N}_n \longrightarrow \boldsymbol{S}(\mathcal{D})$ defined by $\Phi_{\boldsymbol{A}}(\alpha, \beta)(f, g) = \langle \boldsymbol{A}^\alpha f, \boldsymbol{A}^\beta g\rangle$ ($f, g \in \mathcal{D}$, $\alpha, \beta \in \mathbb{Z}_+^n$) is $*$–positive definite.

Proposition 5.1. *An n–tuple $\boldsymbol{A} = (A_1, \ldots, A_n) \in \boldsymbol{L}(\mathcal{D})^n$ ($n \geq 1$) is S–positive definite if and only if it is commutative and H–positive definite.*

Proof. To prove the “if” part of the conclusion, take $f \in \mathcal{F}(\mathcal{N}_n, \mathcal{D})$ and set $g(\beta) = \sum_{\alpha \in \mathbb{Z}_+^n} \boldsymbol{A}^\alpha f(\alpha, \beta)$ for $\beta \in \mathbb{Z}_+^n$. Then evidently $g \in \mathcal{F}(\mathbb{Z}_+^n, \mathcal{D})$ and, by the commutativity of $\boldsymbol{A}$, we have

$$\begin{aligned}&\sum_{\alpha', \beta' \in \mathbb{Z}_+^n} \sum_{\alpha, \beta \in \mathbb{Z}_+^n} \Phi_{\boldsymbol{A}}((\alpha', \beta')^* + (\alpha, \beta))(f(\alpha, \beta), f(\alpha', \beta')) \\ &\qquad = \sum_{\alpha', \beta' \in \mathbb{Z}_+^n} \sum_{\alpha, \beta \in \mathbb{Z}_+^n} \langle \boldsymbol{A}^{\beta' + \alpha} f(\alpha, \beta), \boldsymbol{A}^{\beta + \alpha'} f(\alpha', \beta')\rangle \\ &\qquad\qquad = \sum_{\beta, \beta' \in \mathbb{Z}_+^n} \langle \boldsymbol{A}^{\beta'} g(\beta), \boldsymbol{A}^{\beta} g(\beta')\rangle \geq 0.\end{aligned}$$

Assume now that $\boldsymbol{A}$ is S–positive definite. It follows from Theorem 1.3 that there is an inner product space $\mathcal{E} \supseteq \mathcal{D}$ and an involution preserving semigroup homomorphism $\Pi : \mathcal{N}_n \longrightarrow \boldsymbol{L}^{\#}(\mathcal{E})$ such that

$$\langle \boldsymbol{A}^\alpha f, \boldsymbol{A}^\beta g\rangle = \langle \Pi(\alpha,\beta) f, g\rangle, \quad f,g \in \mathcal{D},\ \alpha,\beta \in \mathbb{Z}_+^n. \tag{5.1}$$

Set $e_j = (\delta_{1,j}, \dots, \delta_{2n,j})$ and $N_j = \Pi(e_j)$ for $j = 1,\dots,n$ ($\delta_{k,l}$ is the Kronecker symbol). Then, by the commutativity of $\mathcal{N}_n$, we see that

$$\text{the operators } N_1,\dots,N_n, N_1^{\#},\dots,N_n^{\#} \text{ commute.} \tag{5.2}$$

Since $e_j^* = e_{j+n}$ for $j = 1,\dots,n$, we obtain $\Pi(\alpha,\beta) = \boldsymbol{N}^{\#\beta}\boldsymbol{N}^\alpha$ for $\alpha,\beta \in \mathbb{Z}_+^n$, where $\boldsymbol{N} = (N_1,\dots,N_n)$ and $\boldsymbol{N}^{\#} = (N_1^{\#},\dots,N_n^{\#})$. Hence, by (5.1) and (5.2), we have

$$\langle \boldsymbol{A}^\alpha f, \boldsymbol{A}^\beta g\rangle = \langle \boldsymbol{N}^\alpha f, \boldsymbol{N}^\beta g\rangle, \quad f,g \in \mathcal{D},\ \alpha,\beta \in \mathbb{Z}_+^n. \tag{5.3}$$

One can conclude from (5.3) that there is an isometry $V : \mathcal{D} \longrightarrow \mathcal{E}$ such that

$$V\boldsymbol{A}^\alpha f = \boldsymbol{N}^\alpha f, \quad f \in \mathcal{D},\ \alpha \in \mathbb{Z}_+^n. \tag{5.4}$$

Inserting $\alpha = \boldsymbol{0}$ into (5.4), we get $Vf = f$ for $f \in \mathcal{D}$. This and (5.4) lead to $A_j \subseteq N_j$ for $j = 1,\dots,n$. Since $N_1,\dots,N_n$ commute, so do $A_1,\dots,A_n$.

Finally, (5.2) and (5.3) give us

$$\sum_{\alpha,\beta\in\mathbb{Z}_+^n} \langle \boldsymbol{A}^\alpha f(\beta), \boldsymbol{A}^\beta f(\alpha)\rangle = \Big\| \sum_{\alpha\in\mathbb{Z}_+^n} \boldsymbol{N}^{\#\alpha} f(\alpha)\Big\|^2 \geq 0, \quad f \in \mathcal{F}(\mathbb{Z}_+^n, \mathcal{D}),$$

which completes the proof. □

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