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A NOTE ON THE COHOMOLOGY OF FINITARY MODULES

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Abstract. Let G be a group, D a division ring and V a DG-module. V is called finitary provided that $V/C_V(g)$ is finite dimensional for all $g \in G$. We investigate the first and second degree cohomology of finitary modules in terms of a local system for G.

In this note we prove the following three theorems on the cohomology of finitary modules in terms of the cohomology of a local system of subgroups:

Theorem 1. *Let G be a group, K a field, V a finitary KG-module and $\mathcal{L}$ a local system of subgroups of G. Suppose that, for all $H \in \mathcal{L}$, V is completely reducible as a KH-module. Then $[V, G]$ is completely reducible as a KG-module.*

Theorem 2. *Let G be a group, D a division ring, V a finitary DG-module, $\mathcal{L}$ a local system of subgroups of G and H an extension of V by G (i.e. $H/V \cong G$). Suppose that the following holds for all L in $\mathcal{L}$:*

(i) *The extension of V by L in H splits.*
(ii) *$V/C_V(L)$ is finite dimensional.*
(iii) *$H^1(L, V)$ is finite dimensional.*

Then H splits over V.

Theorem 3. *Let G be a group, D a division ring, $\mathcal{L}$ a local system of subgroups of G, W a DG-module and V a DG-submodule of W such that $W = V + C_W(H)$ for all $H \in \mathcal{L}$. Then there exists a canonical DG-monomorphism from $W/C_W(G)$ to $[V^*, G]^*$, where Y^* denotes the dual of a module Y.*

We remark that conditions (ii) and (iii) in Theorem 2 are automatically fulfilled if all members of $\mathcal{L}$ are finite groups generated by elements whose order is coprime to the characteristic of D.

Proof of Theorem 1. Let $H \in \mathcal{L}$. Then $[V, H] = [V, H, H]$ and so $[V, G] = [V, G, G]$. Hence we may assume that $V = [V, G]$. Let W be the sum of all the irreducible KG-submodules in V, where $W = 0$ if G has no irreducible submodules in V. We need to show that $W = V$.

So suppose that $V \neq W$. Then $[V, G] \not\leq W$ and we may assume that $[V, H] \not\leq W$ for all $H \in \mathcal{L}$. Let $H \in \mathcal{L}$ and let I_H be the set of irreducible KH-submodules I in $[V, H]$ with $I \not\leq W$. For $I \in I_H$ let $m(I)$ be the supremum of all positive integers t such that I^t is isomorphic to a KH-submodule of V. Pick $h \in H$ with

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$[I,h] \neq 0$. Then $m(I) \cdot \deg_I(h) \leq \deg_V(h)$. In particular, $m(I)$ is finite. Note that there exists a unique KH-submodule $\hat{I}$ in V isomorphic to $I^{m(I)}$, namely $\hat{I}$ is the submodule generated by all the H-submodules in V isomorphic to I. Let $K(I) = \mathrm{Hom}_{KH}(I,I)$ and $d(I) = \dim_K K(I)$. Since $\dim_K [I,h] = \dim_{K(I)}[I,h] \cdot \dim_K K(I)$, $d(I) \leq \deg_V(h)$ and so $d(I)$ is finite. Let m be the minimum of all $m(I)$, $I \in I_H$, $H \in \mathcal{L}$, and d the minimum of all $d(I)$, $I \in I_H$, $H \in \mathcal{L}$, $m(I) = m$.

Pick $H \in \mathcal{L}$ and $I \in I_H$ with $m(I) = m$ and $d(I) = d$. Without loss $H \leq F$ for all $F \in \mathcal{L}$. Let $F \in \mathcal{L}$. Since V is completely reducible as a KF-module, there exists $J \in I_F$ such that I is isomorphic to a KH-submodule of J. Let a be a positive integer such that I^a is isomorphic to a KH-submodule of J. Then $I^{a \cdot m(J)}$ is isomorphic to a KH-submodule of V and so $a \cdot m(J) \leq m$. By minimal choice of $m, m \leq m(J)$. Thus $a = 1$ and $m(J) = m$. In particular, $\hat{I} \leq \hat{J}$ and there exists a unique KH-submodule U in J isomorphic to I. Hence $K(J)$ acts on U and the restriction $K(J)|_U$ of $K(J)$ to U is contained in $K(U)$. Since $\dim_K K(U) = \dim_K K(I) = d \leq \dim_K K(J)$, we conclude that $K(J)|_U = K(U)$. It is now easy to see that every irreducible KH-submodule of $\hat{I}$ lies in an irreducible KF-submodule of $\hat{J}$. Hence $\langle I^F \rangle$ is an irreducible KF-module for all $F \in \mathcal{L}$ and $\langle I^G \rangle$ is an irreducible KG-submodule in V not contained in W. This contradiction completes the proof of Theorem 1. □

The following definition and lemma are used in the proof of Theorem 2.

Definition 4. (a) Let R be a ring, A a set, M an R-module and for $a \in A$ let $\rho_a \colon A \to M$ be a bijection. Then A is called an affine R-module provided that for all a, b, c in A, $\rho_a(b) + \rho_b(c) = \rho_a(c)$.

(b) Let R be a ring, A and B affine R-modules and $\pi \colon A \to B$. Then π is called an affine R-homomorphism if for some a in A and b in B, $\rho_b \pi \rho_a^{-1}$ is an R-homomorphism of modules.

(c) Let R be a ring and A an affine R-module. A subset B of A is called an affine R-submodule if $\rho_a(B)$ is an R-submodule of M for some a in A.

Remark. Let M be an R-module and define $\rho_x \colon M \to M$, $y \to y - x$. Then M is an affine R-module. Moreover, if A is any affine R-module with M as underlying module, then for all a in A, ρ_a is an isomorphism of affine R-modules. Finally if a, b are in A and C is a subset of A, then $\rho_a(C) = \rho_b(C) + \rho_a(b)$ and so C is an affine submodule if and only if $\rho_a(C)$ is the coset of an R-submodule in M.

Lemma 5. *Let G be a group, R a ring and V an RG-module. Let A_G be the set of complements to V in $V \rtimes G$. Then*

(a) *A_G is an affine R-module.*
(b) *Let $H \leq G$. Then the canonical map from A_G to A_H is affine.*
(c) *Let $I_G = \{G^v | v \in V\}$. Then I_G is an affine RG submodule of A_G, $I_G \cong V/C_V(G)$ and $A_G/I_G \cong H^1(G,V)$.*

Proof of the lemma. Identify V and G with their images in the semidirect product $V \rtimes G$. So $V \rtimes G = VG$.

(a) Let M_G be the set of functions $f \colon VG/V \to V$ with $f(ab) = f(a)^{b^{-1}} + f(b)$ for all a, b in VG/V, i.e. M_G is the set of derivations for G on V. Note that M_G is an R-module via $(r \cdot f)(a) = r \cdot f(a)$. For K, L in A_G define $\rho_K(L) \in M_G$ by $\rho_K(L)(Va) = v$, whenever $a \in K$ and $v \in V$ with $va \in L$. Then ρ_K is a bijection from A_G onto M_G (see for example [As, 17.1]).

Let K, L, N be in A_G and a in K. Put $b = \rho_K(L)(Va)a$ and $c = \rho_L(N)(Vb)b$. Then $Va = Vb = Vc$, $b \in L$, $c \in N$ and $c = \rho_L(N)(Va)\rho_K(L)(Va)a$. Thus $\rho_K(L)+\rho_L(N) = \rho_K(N)$. (Here we write the binary operation on V multiplicatively whenever V is regarded as a subgroup of $V \rtimes G$.)

(b) For L in A_G let $\pi(L) = L \cap VH$. Then it is easy to check that $\rho_H \pi \rho_G^{-1}$ is just the restriction map $M_G \to M_H$, $\phi \to \phi_{VH/V}$. Thus π is affine.

(c) Define $\alpha\colon V \to M$ by $\alpha(v)(a) = v^a - v$. Then $\ker \alpha = C_V(G)$ and $\alpha(V) = \rho_G(I_G)$ is the set of inner derivations. In particular $H^1(G, V) = M/\alpha(V) \cong A_G/I_G$ and (c) holds. □

Proof of Theorem 2. Let $L \in \mathcal{L}$. By (i) we may view $V \rtimes L$ as a subgroup of H and by part (a) of the Lemma, A_L is an affine D-module and by (ii), (iii) and part (c) of the Lemma, A_L is finite dimensional. For L and K in $\mathcal{L}$ with $L \leq K$ let $\pi_{K,L}$ be the affine map defined in part (b) of the Lemma. We claim that the inverse limit of $(\pi_{K,L})_{L \leq K}$ is not empty. Note that finite dimensional affine D-modules fulfill the descending chain condition on affine subspaces and so a set of affine subspaces whose intersection is empty has a finite subset whose intersection is empty. Moreover, images and inverse images of affine subspaces under affine maps are affine. Now the proof in [KW, 1K1] that inverse limits of non-empty finite sets are not empty carries over word for word, except that "subset" has to be replaced by "affine subspace". Let $(C_L)_{L \in \Lambda}$ be an element in the inverse limit. Then $\bigcup\{C_L | L \in \mathcal{L}\}$ is a complement to V in H and Theorem 2 is proved. □

Proof of Theorem 3. For $X \leq V^*$ let $X^\perp = \{v \in V | x(v) = 0 \text{ for all } x \in X\}$. We will first prove that:

$$(*) \qquad \text{For all } K \leq G, [V^*, K]^\perp = C_V(K).$$

Indeed, let $x \in V^*$, $k \in K$ and $v \in V$. Then

$$[x, k](v) = (x^k - x)(v) = x^k(v) - x(v) = x(v^{k^{-1}}) - x(v) = x([v, k^{-1}]).$$

It follows that $v \in [V^*, K]^\perp$ if and only if $[v, K] \leq V^{*\perp} = 0$ and so if and only if $v \in C_V(K)$.

Let $H \in \mathcal{L}$. Define a map $a_H\colon W \to [V^*, H]^*$ by $a_H(w)(x) = x(u)$ where $x \in [V^*, H]$, $w \in W$ and $u \in V$ with $w \in u + C_W(H)$. Note that by $(*)$ this definition does not depend on the choice of u. If $K \leq H$ with $K \in \mathcal{L}$, then $C_W(H) \leq C_W(K)$ and so $w \in u + C_W(K)$ and $a_H(w)(x) = a_K(w)(x)$ for all $x \in [V^*, K]$. Define $a\colon W \to [V^*, G]^*$ by $a(w)(x) = a_H(w)(x)$ whenever $w \in W$, $x \in [V^*, G]$ and $H \in \mathcal{L}$ with $x \in [V^*, H]$. By the preceding observation and since $\mathcal{L}$ is a local system this definition does not depend on the choice of H. Let $w \in W$ with $a(w) = 0$. Then $a_H(w) = 0$ for all $H \in \mathcal{L}$ and so $u \in [V^*, H]^\perp$, where u is as above. By $(*)$, $u \in C_V(H)$ and so $w \in C_W(H)$ for all $H \in \mathcal{L}$. Thus $\ker a = C_W(G)$. It remains to show that a is a DG-homomorphism. Clearly a is D-linear. Let w, x, u and H be as above and $g \in G$. We may assume without loss that $g \in H$. Then $w^g \in u^g + C_W(H)$ and so

$$\begin{aligned} a(w^g)(x) &= a_H(w^g)(x) = x(u^g) = x^{g^{-1}}(u) \\ &= a_H(w)(x^{g^{-1}}) = a(w)(x^{g^{-1}}) = a(w)^g(x). \end{aligned}$$

Thus $a(w^g) = a(w)^g$ and a is a DG-homomorphism, completing the proof of Theorem 3. □

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