A NOTE ON THE COHOMOLOGY OF FINITARY MODULES

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Abstract. Let $G$ be a group, $D$ a division ring and $V$ a $DG$-module. $V$ is called finitary provided that $V/C_V(g)$ is finite dimensional for all $g \in G$. We investigate the first and second degree cohomology of finitary modules in terms of a local system for $G$.

In this note we prove the following three theorems on the cohomology of finitary modules in terms of the cohomology of a local system of subgroups:

**Theorem 1.** Let $G$ be a group, $K$ a field, $V$ a finitary $KG$-module and $\mathcal{L}$ a local system of subgroups of $G$. Suppose that, for all $H \in \mathcal{L}$, $V$ is completely reducible as a $KH$-module. Then $[V,G]$ is completely reducible as a $KG$-module.

**Theorem 2.** Let $G$ be a group, $D$ a division ring, $V$ a finitary $DG$-module, $\mathcal{L}$ a local system of subgroups of $G$ and $H$ an extension of $V$ by $G$ (i.e. $H/V \cong G$). Suppose that the following holds for all $L \in \mathcal{L}$:

(i) The extension of $V$ by $L$ in $H$ splits.
(ii) $V/C_V(L)$ is finite dimensional.
(iii) $H^1(L,V)$ is finite dimensional.

Then $H$ splits over $V$.

**Theorem 3.** Let $G$ be a group, $D$ a division ring, $\mathcal{L}$ a local system of subgroups of $G$, $W$ a $DG$-module and $V$ a $DG$-submodule of $W$ such that $W = V + C_W(H)$ for all $H \in \mathcal{L}$. Then there exists a canonical $DG$-monomorphism from $W/C_W(H)$ to $[V^*,G]^*$, where $Y^*$ denotes the dual of a module $Y$.

We remark that conditions (ii) and (iii) in Theorem 2 are automatically fulfilled if all members of $\mathcal{L}$ are finite groups generated by elements whose order is coprime to the characteristic of $D$.

**Proof of Theorem 1.** Let $H \in \mathcal{L}$. Then $[V,H] = [V,H,H]$ and so $[V,G] = [V,G,G]$. Hence we may assume that $V = [V,G]$. Let $W$ be the sum of all the irreducible $KG$-submodules in $V$, where $W = 0$ if $G$ has no irreducible submodules in $V$. We need to show that $W = V$.

So suppose that $V \neq W$. Then $[V,G] \not\subseteq W$ and we may assume that $[V,H] \not\subseteq W$ for all $H \in \mathcal{L}$. Let $H \in \mathcal{L}$ and let $I_H$ be the set of irreducible $KH$-submodules $I$ in $[V,H]$ with $I \not\subseteq W$. For $I \in I_H$ let $m(I)$ be the supremum of all positive integers $t$ such that $I^t$ is isomorphic to a $KH$-submodule of $V$. Pick $h \in H$ with $m(I) = m(I^h)$.

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Then $m(I) \cdot \deg_I(h) \leq \deg_V(h)$. In particular, $m(I)$ is finite. Note that there exists a unique $KH$-submodule $I$ in $V$ isomorphic to $I^m(I)$, namely $I$ is the submodule generated by all the $H$-submodules in $V$ isomorphic to $I$. Let $K(I) = \text{Hom}_K(H(I), I)$ and $d(I) = \dim_K K(I)$. Since $\dim_K[I, h] = \dim_K[I, h] \cdot \dim_K K(I)$, $d(I) \leq \deg_V(h)$ and so $d(I)$ is finite. Let $m$ be the minimum of all $m(I)$, $I \in I_H$, $H \in \mathcal{L}$, and $d$ the minimum of all $d(I)$, $I \in I_H$, $H \in \mathcal{L}$, $m(I) = m$.

Pick $H \in \mathcal{L}$ and $I \in I_H$ with $m(I) = m$ and $d(I) = d$. Without loss $H \leq F$ for all $F \in \mathcal{L}$. Let $F \in \mathcal{L}$. Since $V$ is completely reducible as a $KF$-module, there exists $J \in I_F$ such that $I$ is isomorphic to a $KH$-submodule of $J$. Let $a$ be a positive integer such that $I^a$ is isomorphic to a $KH$-submodule of $J$. Then $I^{a \cdot m(J)}$ is isomorphic to a $KH$-submodule of $V$ and so $a \cdot m(J) \leq m$. By minimal choice of $m, m \leq m(J)$. Thus $a = 1$ and $m(J) = m$. In particular, $I \leq J$ and there exists a unique $KH$-submodule $U$ in $J$ isomorphic to $I$. Hence $K(J)$ acts on $U$ and the restriction $K(J)|_U$ of $K(J)$ to $U$ is contained in $K(U)$. Since $\dim_K K(U) = \dim_K K(I) = d \leq \dim_K K(J)$, we conclude that $K(J)|_U = K(U)$. It is now easy to see that every irreducible $KH$-submodule of $I$ lies in an irreducible $KF$-submodule of $J$. Hence $(I^F)$ is an irreducible $KF$-module for all $F \in \mathcal{L}$ and $\langle I^G \rangle$ is an irreducible $KG$-submodule in $V$ not contained in $W$. This contradiction completes the proof of Theorem 1.

The following definition and lemma are used in the proof of Theorem 2.

**Definition 4.** (a) Let $R$ be a ring, $A$ a set, $M$ an $R$-module and for $a \in A$ let $\rho_a : A \rightarrow M$ be a bijection. Then $A$ is called an affine $R$-module provided that for all $a, b, c$ in $A$, $\rho_a(b) + \rho_b(c) = \rho_a(c)$.

(b) Let $R$ be a ring, $A$ and $B$ affine $R$-modules and $\pi : A \rightarrow B$. Then $\pi$ is called an affine $R$-homomorphism if for some $a$ in $A$ and $b$ in $B$, $\rho_b \pi \rho_a^{-1}$ is an $R$-homomorphism of modules.

(c) Let $R$ be a ring and $A$ an affine $R$-module. A subset $B$ of $A$ is called an affine $R$-submodule if $\rho_a(B)$ is an $R$-submodule of $M$ for some $a$ in $A$.

**Remark.** Let $M$ be an $R$-module and define $\rho_z : M \rightarrow M, y \mapsto y - x$. Then $M$ is an affine $R$-module. Moreover, if $A$ is any affine $R$-module with $M$ as underlying module, then for all $a$ in $A$, $\rho_a$ is an isomorphism of affine $R$-modules. Finally if $a, b$ are in $A$ and $C$ is a subset of $A$, then $\rho_a(C) = \rho_b(C) + \rho_a(b)$ and so $C$ is an affine submodule if and only if $\rho_a(C)$ is the coset of an $R$-submodule in $M$.

**Lemma 5.** Let $G$ be a group, $R$ a ring and $V$ an $RG$-module. Let $A_G$ be the set of complements to $V$ in $V \times G$. Then

(a) $A_G$ is an affine $R$-module.

(b) Let $H \leq G$. Then the canonical map from $A_G$ to $A_H$ is affine.

(c) Let $I_G = \{G^v | v \in V\}$. Then $I_G$ is an affine $RG$-submodule of $A_G, I_G \cong V/C_V(G)$ and $A_G/I_G \cong H^1(G, V)$.

**Proof of the lemma.** Identify $V$ and $G$ with their images in the semidirect product $V \times G$. So $V \times G = V G$.

(a) Let $M_G$ be the set of functions $f : VG/V \rightarrow V$ with $f(ab) = f(a)^{b^{-1}} + f(b)$ for all $a, b$ in $VG/V$, i.e. $M_G$ is the set of derivations for $G$ on $V$. Note that $M_G$ is an $R$-module via $(r \cdot f)(a) = r \cdot f(a)$. For $K, L$ in $A_G$ define $\rho_K(L) \in M_G$ by $\rho_K(L)(va) = v$, whenever $a \in K$ and $v \in V$ with $va \in L$. Then $\rho_K$ is a bijection from $A_G$ onto $M_G$ (see for example [As, 17.1]).
Let $K, L, N$ be in $A_G$ and $a$ in $K$. Put $b = \rho_K(L)(V)a$ and $c = \rho_L(N)(V)b$. Then $Va = Vb = Vc$, $b \in L$, $c \in N$ and $c = \rho_L(N)(V)a\rho_K(L)(V)a$. Thus $\rho_K(L)+\rho_L(N) = \rho_K(N)$. (Here we write the binary operation on $V$ multiplicatively whenever $V$ is regarded as a subgroup of $V \rtimes G$.)

(b) For $L$ in $A_G$ let $\pi(L) = L \cap VH$. Then it is easy to check that $\rho_H \pi \rho_G^{-1}$ is just the restriction map $M_G \to M_H$, $\phi \to \phi_{VH/V}$. Thus $\pi$ is affine.

(c) Define $\alpha : V \to M$ by $\alpha(v)(a) = v^a - v$. Then $\ker \alpha = C_V(G)$ and $\alpha(V) = \rho_G(I_G)$ is the set of inner derivations. In particular $H^1(G, V) = M/\alpha(V) \cong A_G/I_G$ and (c) holds. 

\[ \square \]

**Proof of Theorem 2.** Let $L \in \mathcal{L}$. By (i) we may view $V \rtimes L$ as a subgroup of $H$ and by (a) of the Lemma, $A_L$ is an affine $D$-module and by (ii), (iii) and part (c) of the Lemma, $A_L$ is finite dimensional. For $L$ and $K$ in $\mathcal{L}$ with $L \leq K$ let $\pi_{K, L}$ be the affine map defined in part (b) of the Lemma. We claim that the inverse limit of $(\pi_{K, L})_{L \leq K}$ is not empty. Note that finite dimensional affine $D$-modules fulfill the descending chain condition on affine subspaces and so a set of affine subspaces whose intersection is empty has a finite subset whose intersection is empty. Moreover, images and inverse images of affine subspaces under affine maps are affine. Now the proof in [KW, 1K1] that inverse limits of non-empty finite sets are not empty carries over word for word, except that “subset” has to be replaced by “affine subspace”. Let $(C_L)_{L \in \mathcal{L}}$ be an element in the inverse limit. Then $\bigcup\{C_L|L \in \mathcal{L}\}$ is a complement to $V$ in $H$ and Theorem 2 is proved. 

\[ \square \]

**Proof of Theorem 3.** For $X \leq V^\perp$ let $X^\perp = \{v \in V| x(v) = 0 \text{ for all } x \in X\}$. We will first prove that:

\[ (*) \quad \text{For all } K \leq G, [V^*, K]^\perp = C_V(K). \]

Indeed, let $x \in V^*$, $k \in K$ and $v \in V$. Then

\[ [x, k](v) = (x^k - x)(v) = x^k(v) - x(v) = x(v - x(v)) = x([v, k]). \]

It follows that $v \in [V^*, K]^\perp$ if and only if $[v, K] \leq V^\perp = 0$ and so if and only if $v \in C_V(K)$.

Let $H \in \mathcal{L}$. Define a map $a_H : W \to [V^*, H]^*$ by $a_H(w)(x) = x(u)$ where $x \in [V^*, H]$, $w \in W$ and $u \in V$ with $w \in u + C_W(H)$. Note that by $(*)$ this definition does not depend on the choice of $u$. If $K \leq H$ with $K \in \mathcal{L}$, then $C_W(H) \leq C_W(K)$ and so $w \in u + C_W(K)$ and $a_H(w)(x) = a_K(w)(x)$ for all $x \in [V^*, K]$. Define $a : W \to [V^*, G]^*$ by $a(w)(x) = a_H(w)(x)$ whenever $w \in W$, $x \in [V^*, G]$ and $H \in \mathcal{L}$ with $x \in [V^*, H]$. By the preceding observation and since $\mathcal{L}$ is a local system this definition does not depend on the choice of $H$. Let $w \in W$ with $a(w) = 0$. Then $a_H(w) = 0$ for all $H \in \mathcal{L}$ and so $u \in [V^*, H]^\perp$, where $u$ is as above. By $(*)$, $u \in C_V(H)$ and so $w \in C_W(H)$ for all $H \in \mathcal{L}$. Thus $\ker a = C_W(G)$.

It remains to show that $a$ is a $DG$-homomorphism. Clearly $a$ is $D$-linear. Let $w$, $x$, $u$ and $H$ be as above and $g \in G$. We may assume without loss that $g \in H$. Then $w^g \in w^g + C_W(H)$ and so

\[ a(w^g)(x) = a_H(w^g)(x) = x(w^g) = x^g^{-1}(u) \]

\[ = a_H(w)(x^g^{-1}) = a(w)(x^g^{-1}) = a(w^g)(x). \]

Thus $a(w^g) = a(w^g)$ and $a$ is a $DG$-homomorphism, completing the proof of Theorem 3. 

\[ \square \]
REFERENCES


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