ELEMENTARY ABELIAN 2-GROUP ACTIONS ON FLAG MANIFOLDS AND APPLICATIONS

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Abstract. Let $N^*$ denote the unoriented cobordism ring. Let $G = (\mathbb{Z}/2)^n$ and let $Z_*(G)$ denote the equivariant cobordism ring of smooth manifolds with smooth $G$-actions having finite stationary points. In this paper we show that the unoriented cobordism class of the (real) flag manifold $M = O(m)/(O(m_1) \times \cdots \times O(m_s))$ is in the subalgebra generated by $\bigoplus_{i<2^n} N_i$, where $m = \sum m_i$, and $2^{n-1} < m \leq 2^n$. We obtain sufficient conditions for indecomposability of an element in $Z_*(G)$. We also obtain a sufficient condition for algebraic independence of any set of elements in $Z_*(G)$. Using our criteria, we construct many indecomposable elements in the kernel of the forgetful map $Z_d(G) \rightarrow N_d$ in dimensions $2 \leq d < n$, for $n > 2$, and show that they generate a polynomial subalgebra of $Z_*(G)$.

1. Introduction

Let $G = (\mathbb{Z}/2)^n$, $n \geq 2$. Denote by $Z_*(G)$ the equivariant cobordism ring of (smooth) closed manifolds with smooth $G$-actions having finite stationary point sets $[2], [1]$. The cobordism class of a manifold $M$, along with an action $\phi$ of $G$ having finite stationary point set, will be denoted by $[M, \phi]$. Let $R_q(G)$ denote the vector space over $\mathbb{Z}_2$, with basis the set of isomorphism classes of $RG$-modules of dimension $q$. If $R_*(G) = \sum_{q \geq 0} R_q(G)$, then $R_*(G)$ is a graded commutative $\mathbb{Z}_2$-algebra with unit. The multiplication in $R_*(G)$ is given by $[V] \cdot [W] = [V \oplus W]$. One can identify $R_*(G)$ with the graded polynomial algebra over $\mathbb{Z}_2$ generated by $\hat{G} = \text{Hom}_{\mathbb{Z}_2}(G, \mathbb{Z}_2)$ (cf. [1]).

One has an algebra homomorphism $\eta_* : Z_*(G) \rightarrow R_*(G)$ where $\eta_*([M, \phi]) = \sum [T_\phi M]$, the sum being taken over the (finite) set of stationary points of $M$. By a theorem of Stong [11], we know that $\eta_*$ is a monomorphism. One has the ‘forgetful’ homomorphism $\varepsilon_* : Z_*(G) \rightarrow N_*$, the unoriented cobordism ring, $[M, \phi] \mapsto [M]$. Let $T^n$ denote the subalgebra of $N_*$ generated by $\bigoplus_{1<2^n} N_i$. Then tom Dieck [3] has shown that $\text{Im} \varepsilon_* = T^n$ (cf. Kosniowski and Stong, section 4 of [5]). In this paper we prove

**Theorem 1.1.** Let $G(m_1, \ldots, m_k)$ denote the flag manifold

$$O(m)/(O(m_1) \times \cdots \times O(m_k)),$$

$m = \sum m_i$. Let $2^{n-1} < m \leq 2^n$. Then $[G(m_1, \ldots, m_k)] \in T^n$.

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The problem of determining which flag manifolds are unoriented boundaries was addressed in [10]. The case of Grassmann manifolds was completely settled in [9], which also considers the case of complex and quaternionic Grassmannians. See also Stong [12]. The above theorem gives perhaps the best known result in general. It is easy to see that if two of the numbers \( m_1, \ldots, m_k \) are equal, then \( G(m_1, \ldots, m_k) \) admits a fixed point free involution and hence bounds. If \( m \) is even, then the above theorem does not yield the best possible result (cf. Theorem 2.1 in [10]). But by remark 2.3(iii) in [10], the general problem of determining which flag manifolds bound has been reduced to the consideration of only the case \( m = \sum m_i \) is odd, \( m_1, \ldots, m_k \) being distinct.

Theorem 1.1 is proved by exhibiting certain \( (\mathbb{Z}_2)^n \)-actions on \( G(m_1, \ldots, m_k) \) with finitely many stationary points.

It is well-known that when \( n = 2 \), \( \mathbb{Z}_2(G) \) is isomorphic to the polynomial algebra with one generator \( [\mathbb{P}^2, \phi] \), where the action is given as follows: \( t_1([x, y, z]) = [-x, y, z], \ t_2([x, y, z]) = [x, -y, z] \). Here \( t_1, t_2 \) denotes a set of generators of \( G \). Hence \( \varepsilon_* : \mathbb{Z}_2(G) \to T^*_2 \) is an isomorphism (cf. [1]). However, for \( n \geq 3 \), \( \varepsilon_* \) is not a monomorphism [8] and the structure of \( \mathbb{Z}_2(G) \) is not known.

A \( G \)-manifold \( (M, \phi) \) with finite stationary point set is equivariantly indecomposable if \( [M, \phi] \) is an indecomposable element in \( \mathbb{Z}_2(G) \). Clearly, if \( [M, \phi] \in \mathbb{Z}_2(G) \) and \( [M] \in \mathcal{N}_* \) is indecomposable, then \( [M, \phi] \in \mathbb{Z}_2(G) \) is indecomposable. An important step towards understanding the structure of \( \mathbb{Z}_2(G) \) is to know the indecomposable elements, as they generate \( \mathbb{Z}_2(G) \) as a \( \mathbb{Z}_2 \)-algebra. In section 3 we obtain a sufficient criterion for an element in \( \mathbb{Z}_2(G) \) to be indecomposable. Since \( T^*_n \) is a polynomial algebra, and since by tom Dieck’s theorem [3] \( \text{Im} \varepsilon_* = T^*_n \), it follows that the exact sequence \( 0 \to \ker \varepsilon_* \to \mathbb{Z}_2(G) \to T^*_n \to 0 \) splits. Therefore one can clearly ‘lift’ indecomposable elements from \( T^*_n \) to obtain indecomposable elements in \( \mathbb{Z}_2(G) \). However, we apply our indecomposability criterion to construct indecomposable elements in \( \mathbb{Z}_2(G) \) which belong to \( \mathcal{K}_* = \ker \varepsilon_* \) in each dimension \( 2 \leq m \leq n \) except possibly in dimension \( n \) when \( n \) is even. For the precise statement see Theorem 3.6. This is in striking contrast to the situation in the unoriented cobordism ring \( \mathcal{N}_* \), where there is no generator in dimensions \( 2^l - 1 \). Note that indecomposable elements in \( \mathcal{K}_* \) cannot arise by ‘lifting’ indecomposable elements from \( \mathcal{N}_* \). We also prove a sufficient criterion for a set of elements in \( \mathbb{Z}_2(G) \) to be algebraically independent and use it to show that suitable indecomposable elements in \( \mathcal{K}_* \) generate a polynomial subalgebra of \( \mathbb{Z}_2(G) \). In the last section we generalize a result of Conner and Floyd regarding the number of stationary points \( x \) such that an irreducible representation occurs at the tangential representation \( T_x M \) with a given multiplicity.

2. Action of \( (\mathbb{Z}_2)^n \) on flag manifolds

Let \( 2 \leq m \leq 2^n \), where \( n \geq 2 \). In this section we shall exhibit certain \( G = (\mathbb{Z}_2)^n \)-actions on the flag manifold \( G(m_1, \ldots, m_k) \cong O(m)/(O(m_1) \times \cdots \times O(m_k)) \), \( \sum m_i = m \). Indeed this is a routine generalization of the actions of \( G \) on projective spaces considered by tom Dieck in [3]. We recall the action of \( G \) on the Milnor manifold also considered by tom Dieck [3] as it will be needed in section 3.

Let \( n = \{1, 2, \ldots, n\} \). We regard \( \{e_{\alpha} \mid \alpha \in n\} \) as the ‘standard basis’ of \( \mathbb{R}^2 \), with its usual inner product. For \( 1 \leq i \leq n \), let \( t_i : \mathbb{R}^{2^n} \to \mathbb{R}^{2^n} \) be the \( \mathbb{R} \)-linear map
defined by
\[ t_i(e_\alpha) = \begin{cases} -e_\alpha & \text{if } i \in \alpha, \\ e_\alpha & \text{if } i \notin \alpha. \end{cases} \]

Then it is readily checked that \( t_i^2 = \text{Id} \), and \( t_i t_j = t_j t_i \) for \( 1 \leq i, j \leq n \). Therefore we obtain a linear action of \( G \) on \( \mathbb{R}^n \).

**Lemma 2.1.** (i) The decomposition \( \mathbb{R}^{2^n} = \sum_{\alpha \subseteq \mathbb{N}} \mathbb{R}e_\alpha \) expresses \( \mathbb{R}^{2^n} \) as a sum of mutually non-isomorphic irreducible \( G \)-submodules of \( \mathbb{R}^{2^n} \).

(ii) If \( V \) is any \( G \)-submodule of \( \mathbb{R}^{2^n} \), then \( V = \sum_{\alpha \subseteq S} \mathbb{R}e_\alpha \) for some \( S \subseteq \mathcal{P}(\mathbb{N}) \) with \( \dim V = \# S \).

**Proof of (i).** It is clear that each \( \mathbb{R}e_\alpha \) is a \( G \)-submodule. We only have to prove that if \( \alpha \neq \beta \), then \( \mathbb{R}e_\alpha \nsubseteq \mathbb{R}e_\beta \). Let \( \alpha \neq \beta \), and let \( i \in \alpha \Delta \beta \). Say, \( i \in \alpha \) and \( i \notin \beta \). Then \( t_\alpha e_\alpha = -e_\alpha \), and \( t_\beta e_\beta = e_\beta \). This shows that \( \mathbb{R}e_\alpha \) cannot be isomorphic to \( \mathbb{R}e_\beta \) as \( G \)-modules. Part (ii) is an immediate consequence of part (i).

Let \( S \subseteq \mathcal{P}(\mathbb{N}) \), \( \# S = m \). Let \( V = \sum_{\alpha \subseteq S} \mathbb{R}e_\alpha \subseteq \mathbb{R}^{2^n} \). Since the \( G \)-action on \( \mathbb{R}^{2^n} \) preserves innerproduct and since \( V \) is a \( G \)-submodule, we obtain a \( G \)-action on any flag manifold \( G(m_1, \ldots, m_k) \), \( \sum m_j = m \), modelled on \( V \). Explicitly, if \( (A_1, \ldots, A_k) \in G(m_1, \ldots, m_k) \), and \( t \in G \), then \( t(A_1, \ldots, A_k) = (tA_1, \ldots, tA_k) \). Clearly \( (A_1, \ldots, A_k) \) is a stationary point if and only if each \( A_j, 1 \leq j \leq k \), is a \( G \)-submodule of \( V \). We conclude from Lemma 2.1 that there are only finitely many stationary points. We shall denote this action on \( G(m_1, \ldots, m_k) \) by \( \phi_S \) or simply by \( \phi \) when there is no risk of confusion. Thus \( [G(m_1, \ldots, m_k) \times \mathbb{N}] \subseteq Z_*(G) \) for every \( S \subseteq \mathcal{P}(\mathbb{N}) \), \( \# S = \sum m_j \). In the case of a Grassmann manifold \( G_{m,k} \) with \( G \)-action \( \phi_S \), the stationary points are \( E_\alpha = \langle e_{\alpha_1}, \ldots, e_{\alpha_k} \rangle \), the span of \( e_{\alpha_1}, \ldots, e_{\alpha_k} \) where \( \alpha = \{ \alpha_1, \ldots, \alpha_k \} \) is any \( k \)-element subset of \( S \).

Next, we recall the action of \( G \) on Milnor manifolds considered by tom Dieck [3]. Let \( S \subseteq \mathcal{P}(\mathbb{N}) \), \( \# S = k + 1 \), and let \( T \subseteq S \), \( \# T = l + 1 \). Then we can form the product \( (P^l \times P^k, \phi_T \times \phi_S) \). Let \( H_{l,k} \) be the Milnor manifold

\[
\left\{ \left( \sum_{\alpha \subseteq T} x_\alpha e_\alpha, \sum_{\beta \subseteq S} y_\beta e_\beta \right) \mid \sum_{\alpha \subseteq T} x_\alpha y_\alpha = 0 \right\}.
\]

Then \( H_{l,k} \) is a \( G \)-stable submanifold of \( (P^l \times P^k, \phi_T \times \phi_S) \). It is obvious that there are only finitely many stationary points for the \( G \)-action on \( H_{l,k} \). We denote this action on \( H_{l,k} \) by \( \phi_{T,S} \) or simply by \( \phi \).

**Proof of Theorem 1.1.** Since \( M = G(m_1, \ldots, m_k) \), \( m = \sum m_i \leq 2^n \), was shown to admit a \( G = (\mathbb{Z}_2)^n \) action with finite stationary point set, it follows from the result of tom Dieck [3] that \( [M] \subseteq T_*^{m} \). This completes the proof.

**Example 2.2.** Take \( M = G(40, 42, 45) \). Then the dimension of \( M \) is 5370. The above theorem says that \( [M] \in T_*^{5370} = \mathbb{Z}_2[x_2, x_4, \ldots, x_{126}] \).

Let \( 1 \leq k < m \leq 2^n \). For any \( S \subseteq \mathcal{P}(\mathbb{N}) \), \( \# S = m \), we obtain an element \([G_{m,k} \times \mathbb{N}] \subseteq Z_*(G) \), \( G = (\mathbb{Z}_2)^n \). In general distinct choices of \( S \) do not necessarily lead to distinct elements \([G_{m,k} \times \mathbb{N}] \subseteq Z_*(G) \). In fact we have the following rather amusing example.
Example 2.3. Let \( m = 2^n - 2 \), \( 1 \leq k < m \), \( k \) odd. Then for any \( S \subset P(n) \), \( \#S = m \), one has \([G, m, k, \phi S] = 0 \) in \( Z_n(G) \).

Proof. First note that \( P(n) \) has the structure of a Boolean algebra where addition is given by symmetric difference and multiplication, by intersection. For this reason, we write \( \alpha + \mu \) to mean \( \lambda \Delta \mu \) in the proof. Let \( S \subset P(n) \), \( \#S = 2^n - 2 = m \). Let \( \alpha, \beta \) be elements of \( P(n) \) not in \( S \). Let \( \gamma = \alpha + \beta \). Then \( \gamma \neq \varnothing \). It is easy to see that \( f : \delta \mapsto \delta + \gamma \) defines a bijection of \( S \) onto itself such that \( f^2 \) is the identity, and \( f \) is fixed point free. It follows that \( f \) induces a bijection from the set \((\mathbb{Z}/2)\) of \( k \)-element subsets of \( S \) to itself, denoted by \( F \), where \( F(\{\alpha_1, \ldots, \alpha_k\}) = \{\alpha_1 + \gamma, \ldots, \alpha_k + \gamma\} \). Since \( k \) is odd, and \( f : S \to S \) is fixed point free, it follows that \( f \) is fixed point free.

Recall that the stationary points for the \( G \)-action \( \phi \) on \( G_{m,k} \) are

\[
E_\alpha = \langle e_{\alpha_1}, \ldots, e_{\alpha_k} \rangle,
\]

where \( \alpha = \{\alpha_1, \ldots, \alpha_k\} \subset S \), \( \#\alpha = k \). From Lam’s [6] description of the tangent bundle of Grassmannians, it is easy to see that the tangential representation \( T_\alpha G_{m,k} \) of \( G \) at \( E_\alpha \) is isomorphic to

\[
\bigoplus_{1 \leq i \leq k} \mathbb{R} e_{\alpha_i} \otimes \mathbb{R} e_j \cong \bigoplus_{1 \leq i \leq k} \mathbb{R} e_{\alpha_i + \beta_j},
\]

where \( S \setminus \alpha = \{\beta_1, \ldots, \beta_{m-k}\} \). Since \( \alpha_i + \beta_j = \alpha_i + \gamma + \beta_j + \gamma = f(\alpha_i) + f(\beta_j) \) for any \( \alpha_i, \beta_j \in S \), it is seen that \( T_\alpha G_{m,k} \cong T_{F(\alpha)} G_{m,k} \) as \( G \)-modules. Since \( F \) has no fixed point, \( \eta_\ast (G_{m,k}, \phi S) = 0 \). By Stong’s theorem [11], it follows that \([G, m, k, \phi S] = 0 \).

3. Indecomposability

In this section we obtain a sufficient condition for indecomposability of an element in \( Z_n(G) \), \( G = (\mathbb{Z}_2)^n \). We apply our criterion to show the existence of indecomposable elements in the kernel \( K_n \) of the forgetful homomorphism \( \varepsilon_\ast : Z_n(G) \to N_n \). We make use of the equivariant characteristic numbers of a \( G \)-manifold constructed by tom Dieck [4].

Let \( B = \mathbb{Z}_2[y_1, \ldots, y_n, \ldots] \) be the graded \( \mathbb{Z}_2 \)-algebra with deg \( b_m = m \), for \( m \geq 1 \). Let \( L_\ast \) denote the \( B \)-algebra \( B[[y_1, \ldots, y_n]] \) of formal power series in \( y_1, \ldots, y_n \) with \( \deg y_i = 1, 1 \leq i \leq n \). Let \( K_\ast \) denote the fraction field of \( L_\ast \). Recall that \( R_\ast (G) \) is the polynomial algebra over \( \widehat{G} = \text{Hom}_{\mathbb{Z}_2}(G, \mathbb{Z}_2) \).

As usual we denote the generators of \( G \) by \( t_1, \ldots, t_n \). For \( A \subset \mathbb{N} \), denote by \( Y_A \) the irreducible representation of \( G \) given by the character \( \chi_A \in \widehat{G} \). Here

\[
\chi_A(t_i) = \begin{cases} 1 & \text{if } i \in A, \\ 0 & \text{if } i \notin A. \end{cases}
\]

Thus \( R_\ast (G) \cong \mathbb{Z}_2[R_A \mid A \subset \mathbb{N}] \). One has the subalgebra \( \tilde{R}_\ast (G) \) of \( R_\ast (G) \) generated by \( \{Y_A \mid A \subset \mathbb{N}, A \neq \varnothing \} \). Note that the algebra map \( \eta_\ast : Z_n(G) \to \tilde{R}_\ast (G) \) actually has image in \( R_\ast (G) \), and that \( \eta_\ast \) is well known to be a monomorphism [11].

Let \( \gamma : R_\ast (G) \to K_n \) denote the \( \mathbb{Z}_2 \)-algebra homomorphism defined by \( \gamma(Y_A) = \prod_{i \in A} b_i y_i A \in K_n \), where \( b_0 = 1, y_i A = \sum_{i \in A} y_i \). Then tom Dieck [4] shows that \( \gamma \circ \eta \) is a monomorphism and that \( \text{Im}(\gamma \circ \eta_\ast) \) is contained in \( L_\ast \).
Let \( \lambda = \lambda_1, \ldots, \lambda_r, r \geq 0 \), be a non-increasing sequence of natural numbers. We denote by \( b_\lambda \) the product \( b_\lambda = b_{\lambda_1}b_{\lambda_2} \cdots b_{\lambda_r} \in B \) \((b_\lambda = 1 \text{ if } \lambda = () \), the empty sequence). For any class \([M, \phi] \in \mathbb{Z}_d(G)\), one can express \( \gamma \eta([M, \phi]) \) as \( \sum_{|\lambda| \geq 2} \psi_\lambda([M, \phi]) \cdot b_\lambda \), with \( \psi_\lambda([M, \phi]) \in \mathbb{Z}_2[y_1, \ldots, y_n] \) being a homogeneous element of degree \(|\lambda| - d\), where \(|\lambda| = \sum_j \lambda_j \). The ‘integrality theorem’ of tom Dieck [4] implies that \( \psi_\lambda([M, \phi]) = 0 \) if \(|\lambda| < d\), and that \( \psi_\lambda([M, \phi]) \in \mathbb{Z}_2[y_1, \ldots, y_n] \) if \(|\lambda| \geq d\). One can regard \( \psi_\lambda \) as defining a \( \mathbb{Z}_2 \)-linear map \( \psi_\lambda : \mathbb{Z}_d(G) \to \mathbb{Z}_2[y_1, \ldots, y_n] \). When \(|\lambda| = d\), \( \psi_\lambda([M, \phi]) \) coincides with the Stiefel-Whitney s-numbers \( s_\lambda[M] \in \mathbb{Z}_2 \) (cf. Kosniewski-Stong [5]). Just as the nonvanishing of \( s_\lambda(M) \) implies the indecomposability of \([M]\) in \( \mathcal{N}_s \), one can expect that \( \psi_k([M, \phi]) \), \( k \) any integer greater than \( d \), may detect the indecomposability of \([M, \phi] \in \mathbb{Z}_s(G)\). In fact we have the following

**Proposition 3.1.** Let \([M, \phi] \in \mathbb{Z}_d(G)\). Suppose that for some \( k > d \), either \( \psi_k([M, \phi]) \neq 0 \) or \( \psi_{k-1}([M, \phi]) \neq 0 \); then \([M, \phi] \in \mathbb{Z}_s(G)\) is indecomposable.

We need the following lemma to prove the above proposition.

**Lemma 3.2.** Let \([M, \phi] = [M_1, \phi_1] \cdot [M_2, \phi_2] \in \mathbb{Z}_d(G)\), and let \( \lambda = \lambda_1, \ldots, \lambda_r \). Then

\[
\psi_\lambda([M, \phi]) = \sum_{\lambda = \mu \cdot \nu} \psi_\mu([M_1, \phi_1]) \cdot \psi_\nu([M_2, \phi_2])
\]

where \( \mu \cdot \nu \) denotes juxtaposition of \( \mu \) and \( \nu \) arranged in the non-increasing order.

**Proof of Lemma 3.2.** This is a straightforward consequence of the fact that \( \gamma \) is an algebra homomorphism. Indeed we have

\[
\sum_\lambda \psi_\lambda([M, \phi]) b_\lambda = \gamma([M, \phi]) = \gamma([M_1, \phi_1]) \cdot \gamma([M_2, \phi_2]) = \left( \sum_\mu \psi_\mu([M_1, \phi_1]) b_\mu \right) \left( \sum_\nu \psi_\nu([M_2, \phi_2]) b_\nu \right) = \sum_\mu \sum_\nu \psi_\mu([M_1, \phi_1]) \cdot \psi_\nu([M_2, \phi_2]) b_\mu b_\nu = \sum_\lambda \left\{ \sum_{\mu \cdot \nu = \lambda} \psi_\mu([M_1, \phi_1]) \cdot \psi_\nu([M_2, \phi_2]) \right\} b_\lambda.
\]

Hence, comparing the coefficients of \( b_\lambda \), we obtain

\[
\psi_\lambda([M, \phi]) = \sum \psi_\mu([M_1, \phi_1]) \cdot \psi_\nu([M_2, \phi_2]) \in \mathbb{Z}_2[y_1, \ldots, y_n].
\]

**Proof of Proposition 3.1.** Let \( \lambda = k - 1, 1 \). If \( \lambda = \mu \cdot \nu \), then either \( \mu = 1 \) of \( \nu = 1 \). Note that \( Z_1(G) = 0 \), so that for any \([N, \theta] \in \mathbb{Z}_s(G)\), \( \psi_1([N, \theta]) = 0 \). By the above lemma it follows that \( \psi_{k-1,1} \) vanishes on any decomposable element of \( \mathbb{Z}_s(G) \). Therefore \([M, \phi] \) is indecomposable if \( \psi_{k-1,1}([M, \phi]) \neq 0 \). The proof for the case \( \lambda = k \) is similar.

The following three examples will be used in the proof of Theorem 3.6.

**Example 3.3.** Let \( 3 \leq d = 2^k - 1 \leq n \). Let \( S = \{ \emptyset, \{1\}, \{2\}, \ldots, \{d\} \} \). Then \( \sigma = [P^d, \phi_S] \) is indecomposable in \( \mathbb{Z}_s(G) \). (Note that \( [P^d] = 0 \) in \( \mathcal{N}_s \).)
We prove this by showing that $\psi_{d+2}([P^d, \phi_S]) \neq 0$. Let us write $Y_i$ for $Y_{\{i\}}$, and $Y_{ij}$ for $Y_{\{i,j\}}$, $1 \leq i, j \leq d$. A straightforward calculation shows that

$$\eta_\ast(\sigma) = \sum_{i=1}^{d} \left( \prod_{j=1, j \neq i}^{d} Y_{ij} \right) Y_i + \prod_{i=1}^{d} Y_i.$$  

Let $D = \prod_{k=1}^{d} y_k \prod_{1 \leq p < q \leq d} (y_p + y_q)$, and for $1 \leq i \leq d$, let

$$N_i = \sum_{p=1}^{d} y_p^{d+2} + \sum_{p=1, p \neq i}^{d} (y_i^{d+1}y_p + y_p^{d+1}y_i), \quad C_i = \prod_{p=1, p \neq i}^{d} y_p \prod_{1 \leq p < q \leq d, p \neq i} \prod_{q \neq i} (y_p + y_q),$$

$$N_0 = \sum_{p=1}^{d} y_p^{d+2}, \quad C_0 = \prod_{1 \leq p < q \leq d} (y_p + y_q).$$

Then $\psi_{d+2}(\sigma) = \frac{1}{\mathcal{D}}(\sum_{j=0}^{d} N_j C_j)$ and so

$$D \psi_{d+2}(\sigma) = \sum_{j=0}^{d} N_j C_j.$$  

One can show that the monomial $y_1^{d+1}y_2^{d+2}y_3^{d-2} \cdots y_d^{d-1}1 \cdots y_d$ occurs exactly once on the right hand side of (1). In fact it occurs in $N_1C_1$ exactly once and in no other $N_jC_j$ ($j \neq 1$). Hence $\psi_{d+2}(\sigma) \neq 0$. 

**Example 3.4.** Let $d \leq n$, $d$ even. As before, let $S = \{\varnothing, \{1\}, \ldots, \{d\}\}$. Let $\sigma = [P^d, \phi_S] \in Z_\ast(G)$. Note that $\sigma$ is indecomposable, since $\varepsilon_\ast(\sigma) = [P^d] \in N_\ast$ is indecomposable. We claim that $\psi_N([P^d, \phi_S]) \neq 0$ for $N = 2^m$, $m$ sufficiently large.

To see this, note that

$$\eta_\ast(\sigma) = \sum_{i=1}^{d} \left( \prod_{j=1, j \neq i}^{d} Y_{ij} \right) Y_i + \prod_{i=1}^{d} Y_i.$$  

Let $D = \prod_{p=1}^{d} y_p \prod_{1 \leq p < q \leq d} (y_p + y_q)$ and for $1 \leq i \leq d$, let

$$N_i = y_i^N + \sum_{p=1, p \neq i}^{d} (y_i + y_p)^N = \sum_{p=1, p \neq i}^{d} y_p^N,$$

as $d$ is even and $N$ is a power of 2, and

$$C_i = \prod_{p=1, p \neq i}^{d} y_p \prod_{1 \leq p < q \leq d, p \neq i} \prod_{q \neq i} (y_p + y_q).$$

Let $N_0 = \sum_{p=1}^{d} y_p^N$, and $C_0 = \prod_{1 \leq p < q \leq d} (y_p + y_q)$. Then $D \cdot \psi_N(\sigma) = \sum_{j=0}^{N} N_j C_j$. One can show that for $N$ sufficiently large—say $N > \binom{d}{2}$—the monomial $y_1^{d-1}y_2^{d-2} \cdots y_{d-1}y_d^N$ occurs in $N_0C_0$ exactly once and does not occur in any other $N_jC_j$. Hence the aforementioned monomial survives in $\psi_N(\sigma)$.  

Example 3.5. Let $S = \{\emptyset, \{1\}, \ldots, \{k\}\}$, $T = \{\emptyset, \{1\}, \ldots, \{l\}\}$, $l < k \leq n$, $k, l$ both even. Recall that one has an action $\phi_{T,S}$ of $G$ on the Milnor manifold $H_{l,k}$. Let $\sigma = [H_{l,k}, \phi_{T,S}] \in Z_1(G)$. Further assume that $(k+l) \equiv 1 \pmod{2}$, so that $[H_{l,k}] \in \mathcal{N}$ is indecomposable. We claim that for $N = 2^r$ sufficiently large, $\psi_{N+1}(\sigma) \neq 0$.

To establish our claim, first note that the normal bundle to the imbedding $H_{l,k} \hookrightarrow P^l \times P^k$ is $i^*(\xi_l \otimes \xi_k)$, where $\xi_l$ is the pull-back of the Hopf line bundle over $P^l$ via the projection $P^l \times P^k \to P^l$. Using this it is easy to see that

$$\eta_*(\sigma) = \sum_{0 \leq i \leq k \atop 0 \leq j \leq l \atop i \neq j} \left( \prod_{p=0, p \neq i} \prod_{q=0, q \neq i, j}^{k} Y_{i,p} Y_{j,q} \right),$$

with the convention that $Y_{0,p} = Y_{i,p} = Y_{0,0}$. Let $D = \prod_{p=1}^{k} y_p \prod_{1 \leq p < q \leq k} (y_p + y_q)$. For $1 \leq i \leq l$, $1 \leq j \leq k$, and $i \neq j$, write

$$N_{ij} = \sum_{p=1}^{k} y_p^{N+1} + \sum_{p=1, p \neq i}^{l} (y_i^N y_p + y_i y_p^N) + \sum_{q=1}^{k} (y_j^N y_q + y_j y_q^N),$$

$$C_{ij} = \prod_{p=1, p \neq i, j}^{k} y_p \cdot \prod_{1 \leq p < q \leq k \atop p \neq i, j}^{l} (y_p + y_q) \cdot \prod_{p=l+1}^{k} (y_i + y_p).$$

For $1 \leq j \leq k$, let

$$N_{0,j} = \sum_{p=l+1}^{k} y_p^{N+1} + \sum_{p=1, p \neq j}^{l} (y_j^N y_p + y_j y_p^N),$$

$$C_{0,j} = \prod_{1 \leq p < q \leq k}^{l} (y_p + y_q) \cdot \prod_{p=l+1}^{k} y_p.$$

For $1 \leq i \leq l$, let

$$N_{i,0} = \sum_{p=l+1}^{k} y_p^{N+1} + \sum_{p=1, p \neq i}^{l} (y_i^N y_p + y_i y_p^N),$$

$$C_{i,0} = \prod_{1 \leq p < q \leq k}^{l} (y_p + y_q) \cdot \prod_{q=l+1}^{k} (y_i + y_q).$$

Then a routine verification shows that

$$(2) \quad D\psi_{N+1}(\sigma) = \sum_{0 \leq i \leq l \atop 0 \leq j \leq k \atop i \neq j} N_{ij} C_{ij}.$$

Consider the monomial $y_1 y_2^{k-2} y_3^{k-3} \cdots y_{l-1}^{k-l+1} y_l^{k-l} y_{l+1}^{k-1} \cdots y_k^{k-2} y_{k-1} y_k^N$. We claim that this monomial survives in the right hand side of (2) for sufficiently
large $N$—say $N > \binom{k}{
}$, $N = 2^n$. First note that since $N$ is large, the only monomials which involve $y_k^N$ must come from the terms $N_k C_{1k}$, $1 \leq i \leq l$. In $N_k$ the only terms which contribute to the above monomial are $y_k^N (\sum_{p=1}^{k-1} y_p)$. It is not difficult to show that the above monomial occurs $(l-1)$-times in $y_k^N (\sum_{p=2}^{k-1} y_p) C_{1k}$, and that it does not occur in $y_k^N (\sum_{p=1}^{k-1} y_p) C_{1k}$ for $i > 1$. Since $l$ is even, it follows that the above monomial survives in $D\psi_{N+1}(\sigma)$. Hence $\psi_{N+1}(\sigma) \neq 0$.

Let $D$ denote the ideal of decomposable elements in $Z_s(G)$. Let $K_m$ denote the $\mathbb{Z}/2$-vector space $K_m/(K_m \cap D)$. The dimension of $K_m$ is the number of $\mathbb{Z}/2$-linearly independent indecomposable elements in $K_m$.

**Theorem 3.6.** Let $2^{s-1} < n < 2^s$, $n \geq 3$. Let $d = \dim_{\mathbb{Z}/2} K_m$. Then

(i) $d \geq n - 2^r + 2$, when $m = 2^r - 1$, $m \leq n$,

(ii) $d \geq n - 2^r$, when $2^r < m + 1 < 2^{r+1} \leq 2^s$, and $m$ is odd,

(iii) $d \geq n - m$, when $m$ is even, $2 \leq m < n$.

**Proof.** We shall only prove (ii), proofs for other parts being exactly analogous, and make use of Examples 3.3 and 3.4. Let $m$ be odd and $2^r < m + 1 < 2^{r+1}$, $r \leq s - 1$. Then $[H_{l,k}] \in N_\ast$ is indecomposable where $k = 2^r$, $m = l - k$. Note that $l \geq 2$, and that $k < n$. Now let $\sigma = [H_{l,k}, \phi_{S,T}]$ be as in Example 3.5. By Example 3.5, $\psi_{N+1}(\sigma) \in \mathbb{Z}[y_1, \ldots, y_k]$ is not zero. Suppose that $\psi_{N+1}(\sigma) = \sum_{t \geq 0} P_{r} y_{r}^t$, with $1 \leq t \leq k$, $P_{r} \in \mathbb{Z}[y_1, \ldots, y_k]$, is the expression for $\psi_{N+1}(\sigma)$ as a polynomial in $y_i$ with positive degree.

Now, for $1 \leq j \leq n - 2^r$, let $A_j = \{ \emptyset, \{ 1 \}, \ldots, \{ t - 1 \}, \{ t + 1 \}, \ldots, \{ k \}, \{ k + j \} \}$, and let

$$B_j = \begin{cases} T & \text{if } l < t, \\ T \cup \{k + j\} \setminus \{t\} & \text{if } t \leq l. \end{cases}$$

Then, writing $\sigma_j = [H_{l,k}, \phi_{A_j,B_j}]$, we see that $\psi_{N+1}(\sigma_j) = \sum_{r \geq 0} P_{r} y_{r}^j$. Therefore, $\psi_{N+1}(\sigma + \sigma_j) = \sum_{r \geq 0} P_{r} (y_{r}^j + y_{r}^j) \neq 0$. Hence by Proposition 3.1, it follows that $\sigma + \sigma_j$ is indecomposable. Clearly $\varepsilon_\ast(\sigma + \sigma_j) = 2[H_{l,k}] = 0$. Therefore $\sigma + \sigma_j \in K_m$. Writing $u_j = \sigma + \sigma_j$, for $1 \leq j \leq n - 2^r$, we see that for any sequence of $1 \leq j_1 < \cdots < j_p \leq n - 2^r$, one has $\psi_{N+1}(u_{j_1} + \cdots + u_{j_p}) = \sum_{1 \leq q \leq p} \psi_{N+1}(u_{j_q}) = \sum_{r \geq 0} P_{r} (y_{r}^{j_q} + \sum_{1 \leq q \leq p} y_{r}^{j_q}) \neq 0$. This proves that $u_1, \ldots, u_{n-2^r}$ are linearly independent in $K_m$, completing the proof.

**Remark 3.7.** Note that in the course of the above proof one could as well work with the smallest integer $M$ such that $\psi_{M}(\sigma) \neq 0$ in the place of the integer $N + 1$.

The above theorem suggests the following conjecture. Note that the group $\text{Aut}(\mathbb{Z}_2^n) \cong SL_n(\mathbb{Z}_2)$ acts on $Z_s(G)$ as $\mathbb{Z}_2$-algebra automorphisms. Indeed if $w \in SL_n(\mathbb{Z}_2)$, then $w([M, \phi]) = [M, \phi^w]$, where $\phi^w(t, x) = \phi(w(t), x)$ for all $x \in M$. In particular, let $\sigma \in Z_d(G)$ be indecomposable. Then so is $w(\sigma)$, and $\sigma + w(\sigma) \in K_d$.

**Conjecture.** If $w(\sigma) \neq \sigma$, then $\sigma + w(\sigma)$ is an indecomposable element in $K_d$.

Let $u_1, u_2, \ldots, u_k$ be homogeneous elements in $Z_s(G)$. Assume that there exists an integer $N_i$ such that $f_i := \psi_{N_i}(u_i) \neq 0$. We may suppose that $N_i$ is the smallest such integer. By relabelling the $u_i$’s if necessary, we assume that $N_1 \leq N_2 \leq \cdots \leq$
Proposition 3.8. With the above notation, assume that \( f_1, \ldots, f_k \in \mathbb{Z}_2[y_1, \ldots, y_n] \) are algebraically independent. Then \( u_1, \ldots, u_k \) are algebraically independent.

Proof. Suppose that \( P(u_1, \ldots, u_k) = 0 \) where \( P \) is a homogeneous polynomial of degree \( d \). Then we can write \( P \) as \( P = \sum \varepsilon_i u_1^{r_1} \cdots u_k^{r_k} \) where the sum is over all sequences \( r = (r_1, \ldots, r_k) \in \mathbb{N}^k \) with \( \sum_{i=1}^k r_i |u_i| = d \), \( \varepsilon_i \in \{0, 1\} \), \( \mathbb{N} \) being the set of non-negative integers. Let \( |r| = r_1 + \cdots + r_k. \) Let \( S = \{ r \mid \varepsilon_r = 1 \} \), and let \( s = \min \{|r| \mid r \in S \} \). Note that \( P = \sum_{r \in S} u_1^{r_1} \cdots u_k^{r_k} \). For \( r \in S \), let \( r^j = r_{i_{j-1}+1} + \cdots + r_{i_j}, \) for \( 1 \leq j \leq p \). Let \( S_p = \{ r \in S \mid |r| = s \} \), and let \( s_p = \min \{|r^p| \mid r \in S_p \} \). Similarly, let \( S_{p-1} = \{ r \in S_p \mid r^p = s_p \} \), and set \( s_{p-1} = \min \{|r^{p-1}| \mid r \in S_{p-1} \} \). Having defined \( S_p, S_{p-1}, \ldots, S_{p-j+1} \) and \( s_p, s_{p-1}, \ldots, s_{p-j+1} \), we define \( s_{p-j} \) as \( S_{p-j} = \{ r \in S_{p-j+1} \mid r^{p-j+1} = s_{p-j+1} \} \) and \( s_{p-j} = \min \{|r^{p-j}| \mid r \in S_{p-j} \} \). Note that \( S_1 \) consists precisely of those \( r \in S_p \) such that the sequence \( (r^p, \ldots, r^1) = (s_p, \ldots, s_1) \) is the smallest in the lexicographic ordering of \( \mathbb{N}^p \), as \( r \) varies in \( S_p \).

Now let

\[ \lambda = N_i^{s_1} \cdots N_i^{s_p} = \underbrace{N_i, \ldots, N_i}_{s_1} \cdots \underbrace{N_i, \ldots, N_i}_{s_p}. \]

Claim. For \( r \in S \),

\[ \psi_\lambda(u_1^{r_1} \cdots u_k^{r_k}) = \begin{cases} f_1^{r_1} \cdots f_k^{r_k} & \text{if } r \in S_1, \\ 0 & \text{otherwise.} \end{cases} \]

It is easily seen that \( \psi_\lambda(u_1^{r_1} \cdots u_k^{r_k}) = 0 \) if \( |r| > s = |\lambda| \). Also it is trivial to see that \( \psi_{N_i^{s_j}}(u_1^{r_{i_{j-1}+1}} \cdots u_k^{r_{i_j}}) = f_{i_{j-1}+1}^{r_{i_{j-1}+1}} \cdots f_{i_j}^{r_{i_j}}, 1 \leq j \leq p \). Furthermore \( \psi_{M_1 \cdots M_q}(u_1^{r_1} \cdots u_k^{r_k}) = 0, q = i_j - i_{j-1} - 1 \), if some \( M_k < N_{i_j} \) by minimality of \( N_j \).

By Lemma 3.2 the claim follows. To complete the proof of the lemma, we apply \( \psi_\lambda \) to \( P(u_1, \ldots, u_k) \) = 0. By the claim above

\[ 0 = \psi_\lambda(P(u_1, \ldots, u_k)) = \sum_{r \in S_1} \psi_\lambda(u_1^{r_1} \cdots u_k^{r_k}) = \sum_{r \in S_1} f_1^{r_1} \cdots f_k^{r_k} \]

which contradicts the hypothesis that the \( f_i \)'s are algebraically independent. \( \square \)

Let \( u_1, \ldots, u_k \) be homogeneous elements in \( K \) constructed as in the proof of Theorem 3.6. By Remark 3.7 there exist \( M_i \) smallest such that \( \psi_{M_i}(u_i) \neq 0 \). Write \( \psi_{M_i}(u_i) = P_i(y_1, \ldots, y_k) \) where \( P_i \) involves \( y_k \).

Theorem 3.9. With the above notation, suppose \( u_1, \ldots, u_r \in K \) are such that \( 1 \leq k_1, \ldots, k_r \leq n \) are all distinct. Then \( u_1, \ldots, u_r \) are algebraically independent.

Proof. The theorem follows immediately from the above proposition as \( \psi_{M_i}(u_i) = P_i(y_1, \ldots, y_k), i = 1, \ldots, r \), are clearly algebraically independent. \( \square \)

In particular, it follows from Theorem 3.6 that \( K_m \) has at least \( r \) elements which generate a polynomial algebra on \( r \) variables where

(i) \( r = n - 2^q + 2 \), when \( m = 2^q - 1, m \leq n \),

(ii) \( r = n - 2^q \), when \( 2^q < m + 1 < 2^{q+1} \leq 2^n \), and \( m \) is odd,

(iii) \( r = n - m \), when \( m \) is even, \( 2 \leq m < n \).
We conclude this section with the following questions. Let as usual $G = (\mathbb{Z}_2)^n$.

**Question 1.** Is $Z_\ast(G)$ finitely generated as a $\mathbb{Z}_2$-algebra?

**Question 2.** Are there indecomposable elements in $Z_\ast(G)$ beyond dimension $2^n - 2$?

4. **Tangential representations**

An important step in determining the structure of $Z_\ast(\mathbb{Z}_2^n)$ is the following observation of Conner and Floyd (Lemma 32.3, [1]). Let $(M, \phi)$ be a smooth closed $\mathbb{Z}_2 \times \mathbb{Z}_2$-manifold, and let $\alpha \in \{1, 2\}$. If $Y_\alpha$ occurs in the representation $T_x M$ with multiplicity $p$, at some stationary point $x$. Then it occurs with the same multiplicity at an even number of stationary points. We shall show in this section that this phenomenon happens for any $(\mathbb{Z}_2)^n$-manifold with finite stationary point set.

**Theorem 4.1.** Let $n \geq 2$ and $(M, \phi)$ be a closed $(\mathbb{Z}_2)^n$-manifold with finite stationary point set. Suppose that for some stationary point $x$, and $\alpha \subset \mathbb{Z}_2$, $Y_\alpha$ occurs with multiplicity $p$ in $T_x M$. Then it occurs with the same multiplicity at an even number of stationary points.

**Proof.** Let $d = \text{dim} M$. We use the notation of section 3. Let $V = Y^{p_1}_{\alpha_1} \cdots Y^{p_k}_{\alpha_k}$, $\sum p_i = d$, $\alpha_1, \ldots, \alpha_k$ distinct. Then

$$\gamma(V) = \frac{1}{y_{\alpha_1}^{p_1} \cdots y_{\alpha_k}^{p_k}} (1 + b_1 y_{\alpha_1} + b_2 y_{\alpha_1}^2 + \cdots b_{p_1} + \cdots (1 + b_1 y_{\alpha_k} + b_2 y_{\alpha_k}^2 + \cdots b_{p_k})$$

Let $r < d$. The coefficient $C_{d-r}(V)$ of $b_{d-r}^1$ in the expression for $\gamma(V)$ can be calculated as follows: If $p_1 = p \geq r$, then

$$C_{d-r}(V) = \frac{(p)}{y_{\alpha_1}^r} + P_{\alpha_1}(V)$$

where $P_{\alpha_1}(V)$ is a polynomial in $(1/y_{\alpha_1})$ of degree less than $r$ with coefficients in $\mathbb{Z}_2[1/y_{\alpha_1}]$. If $p < r$, then $C_{d-r}(V) = Q_{\alpha_1}(V)$, a polynomial in $(1/y_{\alpha_1})$ of degree less than $r$ with coefficients in $\mathbb{Z}_2[1/y_{\alpha_1}]$. Let $S_{\alpha,r} = \{x \in S \mid Y_\alpha \text{ occurs with multiplicity } r\}$. Fix $\alpha = \alpha_1$. We need to show that $|S_{\alpha,r}|$ is even for each $r, 1 \leq r \leq d$. Now suppose that $r = s$ is the largest integer such that $Y_\alpha$ occurs with multiplicity $s$, among $T_x M$, as $x$ varies over the (finite) set $S$ of stationary points. One knows that $s < d$. Then

$$0 = \psi_{1, \ldots, 1}(M, \phi) = \sum_{x \in S} C_{d-s}(T_x M).$$

Multiplying both sides of (3) by $y_\alpha^s$ we obtain

$$0 = \sum_{x \in S} (1 + y_\alpha^s P_\alpha(T_x M)) + \sum_{x \in S \setminus S_{\alpha,r}} y_\alpha^s Q_\alpha(T_x M).$$

Note that $y_\alpha^s P_\alpha(T_x M)$, $y_\alpha^s Q_\alpha(T_x M)$ are polynomials in $y_\alpha$ without constant terms having coefficients in $\mathbb{Z}_2[1/y_\beta \mid \beta \neq \alpha]$. Clearing the denominators in (4), we obtain
Again note that \( y \) terms and having coefficients in \( \mathbb{Z} \) variables, we see that
\[ y \alpha = 0 \text{ in the above expression. Then we obtain } 0 = |S_{\alpha,s}| \cdot \bigl[ y_{\beta_1} \cdots y_{\beta_t} \bigr] \text{ in } \mathbb{Z}_2[y_1, \ldots, y_n]/\langle \gamma_0 \rangle. \] As \( \mathbb{Z}_2[y_1, \ldots, y_n]/\langle \gamma_0 \rangle \) is again a polynomial algebra in \((n-1)\)-variables, we see that \( |S_{\alpha,s}| \) must be even.

Inductively assume that \( Y^r_\alpha \) occurs in \( T_x M \), if at all, at an even number of stationary points for any \( d > p > r \). We must show that \( Y^r_\alpha \) occurs in \( T_x M \), if at all, for an even number of stationary points. Proceeding exactly as before we see that
\[ 0 = \sum_{p > r} \left( \sum_{x \in S_{\alpha,p}} \left( \begin{array}{c} p \\ r \end{array} \right) + y_{\alpha}^r P_T(x M) \right) + \sum_{x \in S \setminus \{x \in S, p \geq r \}} y_{\alpha}^r Q_T(x M). \]

Again note that \( y_{\alpha}^r P_T(x M) \) and \( y_{\alpha}^r Q_T(x M) \) are polynomials in \( y_\alpha \) without constant terms and having coefficients in \( \mathbb{Z}_2[1/y_\beta \mid \beta \neq \alpha] \). As before we see that \( \sum_{p > r} |S_{\alpha,p}| \left( \begin{array}{c} p \\ r \end{array} \right) = 0 \text{ (mod 2)} \). Hence \( |S_{\alpha,r}| \equiv \sum_{p > r} |S_{\alpha,p}| \left( \begin{array}{c} p \\ r \end{array} \right) \equiv 0 \text{ (mod 2)} \), since by the induction hypothesis each \( |S_{\alpha,p}| \), \( p > r \), is even.

Remark 4.2. It may be noted that our method can be used to interpret the vanishing of \( \psi(M, \phi) \) for \( |\lambda| < d = \dim M \) in terms of the irreducible representations \( Y^r_\alpha \), \( \alpha \in \mathbb{Z}_2 \), which occur in \( T_x M \) as \( x \) varies in the (finite) set of stationary points of \( (M, \phi) \).

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