PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 2, February 1998, Pages 523–530
S 0002-9939(98)04164-1

SEMICLASSICAL LIMIT OF THE NONLINEAR SCHRÖDINGER EQUATION IN SMALL TIME

E. GRENIER

(Communicated by Jeffrey B. Rauch)

ABSTRACT. We study the semi-classical limit of the nonlinear Schrödinger equation for initial data with Sobolev regularity, before shocks appear in the limit system, and in particular the validity of the WKB method.

1. INTRODUCTION

We study the behaviour as h goes to zero of solutions ψ_h of the nonlinear Schrödinger equation (NLS)

$$-ih\partial_t\psi_h - \frac{h^2}{2}\Delta_x\psi_h + f(|\psi_h|^2)\psi_h = 0 \tag{1.1}$$

where $x \in \mathbb{R}^d$ and $t \in \mathbb{R}^+$, with initial data

$$\psi_h(0,x) = a^0(x,h)e^{iS^0(x)/h}, \tag{1.2}$$

where $f \in C^\infty(\mathbb{R}^+,\mathbb{R})$, S^0 is a function of $H^s(\mathbb{R}^d)$ (Sobolev spaces) for s large enough, and a^0 is a function, polynomial in h, with coefficients of Sobolev regularity in x.

The usual WKB method leads us to look for solutions of the form

$$\psi_h(t,x) = a(t,x,h)e^{iS(t,x)/h} \tag{1.3}$$

where

$$a(t,x,h) = \sum_{j=0}^{+\infty} a_j(t,x)h^j. \tag{1.4}$$

The functions a_j satisfy a hierarchy of equations that can be solved in small time (see [1] for more details).

On the other hand [1], with the change of variables

$$v = \nabla_x S + h(\bar{a}\nabla_x a - a\nabla_x\bar{a})/2i\rho \qquad \text{and} \qquad \rho = |a|^2,$$

the NLS equation (1.1) is transformed to

$$\partial_t\rho + \nabla_x.(\rho v) = 0,$$

Received by the editors August 13, 1996.
1991 *Mathematics Subject Classification.* Primary 35Q55, 35C20.
Key words and phrases. Nonlinear Schrödinger equations, semiclassical limit.

$$\partial_t v + \nabla_x\Big(\frac{|v|^2}{2} + f(\rho)\Big) = \frac{h^2}{2}\nabla_x\Big(\frac{1}{\sqrt{\rho}}\Delta_x\sqrt{\rho}\Big),$$

which is a perturbation of the Euler equations of compressible isentropic fluid mechanics

$$\partial_t\rho + \nabla_x.(\rho v) = 0, \tag{1.5}$$

$$\partial_t v + \nabla_x\Big(\frac{|v|^2}{2} + f(\rho)\Big) = 0. \tag{1.6}$$

If $f' > 0$, this system has smooth solutions on a time interval $[0, T^\star[$, for initial data with Sobolev regularity [5].

A natural question is then to show the existence of smooth solutions ψ_h of (1.1) on a time interval $[0, T]$ independent on h, for initial data a^0 and S^0 with Sobolev regularity, and to justify the WKB expansion on the same interval of time, T being linked to the existence time of a smooth solution to (1.5,1.6). The case of initial data with analytic regularity has been treated by P. Gérard in [1], and the case $f(\rho) = \rho$ has been investigated in one space dimension for all time by Jin, Levermore and Mc Laughlin in [4], using the integrability of the cubic nonlinear Schrödinger equation.

Notice that with particular assumptions on d and f, equation (1.1) has global smooth solutions [2]. This kind of result can however not be used, since the main problem is to obtain bounds *uniform* in h on the solution ψ_h, in order to be able to pass to the limit and to justify the WKB hierarchy.

Let us now state the three main theorems.

Theorem 1.1. *Let $f \in C^\infty(\mathbb{R}^+, \mathbb{R})$, with $f' > 0$. Let $s > d/2 + 2$, let $S^0(x) \in H^s(\mathbb{R}^d)$ and $a^0(x, h)$ be a sequence of functions uniformly bounded in $H^s(\mathbb{R}^d)$. Then there exist $T > 0$ and solutions $\psi_h(t, x) = a_h(t, x)\exp(iS_h(t, x)/h)$ to equation (1.1), with initial data (1.2). Moreover, a_h and S_h are bounded in $L^\infty([0, T], H^s)$ uniformly in h.*

We can then link T to the existence time of a smooth solution to (1.5,1.6):

Theorem 1.2. *Under the assumptions of Theorem 1.1, if moreover $a^0(x, h)$ converges to a^0 in $H^s(\mathbb{R}^d)$ as h goes to 0, and if system (1.5,1.6) with initial data $\rho(0, x) = |a^0(x)|^2$ and $v(0, x) = \nabla_x S^0(x)$ has a solution in $L^\infty([0, T], H^{s+2}(\mathbb{R}^d))$, then, for h small enough, there exist solutions to equation (1.1) of the form $\psi_h(t, x) = a_h(t, x)\exp(iS_h(t, x)/h)$ on $[0, T]$, where a_h and S_h are bounded in $L^\infty([0, T], H^s)$ uniformly in h.*

We then justify the WKB expansions :

Theorem 1.3. *Under the assumptions of Theorem 1.2, if moreover $a^0(x, h)$ has an expansion of the form*

$$a^0(x, h) = \sum_{j=0}^{N} a^0_j(x)h^j + h^N r_N(x, h) \tag{1.7}$$

where

$$\lim_{h\to 0}\|r_N\|_{H^s(\mathbb{R}^d)} = 0$$

for $N \in \mathbb{N}$ and $s - 2N - 2 - d/2 > 0$, then, on the time interval $[0, T]$ given by Theorem 1.2,

$$a_h(t,x)\exp(iS_h(t,x)/h) = \sum_{j=0}^{N} a_j(t,x)h^j \exp(iS(t,x)/h) + h^N r_N(t,x,h) \tag{1.8}$$

as h goes to zero, where S and a_j are given by the WKB method, and where

$$\lim_{h\to 0} \|r_N\|_{L^\infty([0,T],H^{s-2N-2-d/2}(\mathbb{R}^d))} = 0.$$

The results of this paper have been announced in [3]. The proofs rely on the following simple idea : instead of looking as usual at solutions ψ_h of the form

$$\psi_h(t,x) = a_h(t,x)e^{iS(t,x)/h}$$

where S is independent of h, we allow S to depend on h, in order to get better equations for a_h and S_h. Namely we will look for solutions ψ_h of the form

$$\psi_h(t,x) = a_h(t,x)e^{iS_h(t,x)/h}. \tag{1.9}$$

2. Proof of Theorem 1.1

Putting (1.9) in (1.1), we get

$$-ih\partial_t a_h + \partial_t S_h a_h - \frac{h^2}{2}\Delta_x a_h - ih\nabla_x S_h.\nabla_x a_h - \frac{ih}{2}a_h\Delta_x S_h + \frac{1}{2}a_h|\nabla_x S|^2 + a_h f(|a_h|^2) = 0,$$

that we split into

$$\partial_t S_h + \frac{|\nabla_x S_h|^2}{2} + f(|a_h|^2) = 0,$$

and

$$\partial_t a_h - \frac{ih}{2}\Delta_x a_h + \nabla_x S_h.\nabla_x a_h + \frac{1}{2}a_h\Delta_x S_h = 0.$$

The change of variable $w_h = \nabla_x S_h$ then leads to

$$\partial_t w_h + (w_h.\nabla_x)w_h + f'(|a_h|^2)\nabla_x|a_h|^2 = 0,$$

$$\partial_t a_h + (w_h.\nabla_x)a_h + \frac{1}{2}a_h\nabla_x.w_h = \frac{ih}{2}\Delta_x a_h.$$

Let a_h^1 be the real part of a_h and a_h^2 be its imaginary part. We have

$$\partial_t a_h^1 + \sum_{j=1}^{d} w_h^j\partial_j a_h^1 + \frac{1}{2}a_h^1\sum_{j=1}^{d}\partial_j w_h^j = -\frac{h}{2}\Delta_x a_h^2, \tag{2.1}$$

$$\partial_t a_h^2 + \sum_{j=1}^{d} w_h^j\partial_j a_h^2 + \frac{1}{2}a_h^2\sum_{j=1}^{d}\partial_j w_h^j = \frac{h}{2}\Delta_x a_h^1, \tag{2.2}$$

$$\partial_t w_h^i + f'(|a_h^1|^2 + |a_h^2|^2)(2a_h^1\partial_i a_h^1 + 2a_h^2\partial_i a_h^2) + \sum_{i=1}^{d} w_h^j\partial_j w_h^i = 0, \tag{2.3}$$

where w_h^i is the i^{th} component of w_h. This system can be written in the form

$$\partial_t u_h + \sum_{i=1}^{d} A^i(u_h)\partial_i u_h = hL(u_h), \tag{2.4}$$

$$u_h = \begin{pmatrix} a_h^1 \\ a_h^2 \\ w_h^1 \\ \cdots \\ w_h^d \end{pmatrix}, \qquad L(u_h) = \begin{pmatrix} -\frac{1}{2}\Delta_x a_h^2 \\ \frac{1}{2}\Delta_x a_h^1 \\ 0 \\ \cdots \\ 0 \end{pmatrix}$$

and

$$A(u_h,\xi) = \sum_{j=1}^{d} \xi_j A^j(u_h)$$

$$= \begin{pmatrix} \sum_{i=1}^d \xi_i w_h^i & 0 & \xi_1 a_h^1/2 & \xi_2 a_h^1/2 & \cdots \\ 0 & \sum_{i=1}^d \xi_i w_h^i & \xi_1 a_h^2/2 & \xi_2 a_h^2/2 & \cdots \\ 2\xi_1 a_h^1 f' & 2\xi_1 a_h^2 f' & \sum_{i=1}^d \xi_i w_h^i & 0 & \cdots \\ 2\xi_2 a_h^1 f' & 2\xi_2 a_h^2 f' & 0 & \sum_{i=1}^d \xi_i w_h^i & \cdots \\ \cdots & \cdots & \cdots & \cdots & \ddots \end{pmatrix}.$$

The matrix $A(u_h,\xi)$ can be symmetrized by

$$S = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1/4f' & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1/4f' \end{pmatrix}$$

which is symmetric and positive since $f' > 0$.

Let $u_h = (a_h^1, a_h^2, w_h)$ be a solution of system (2.4). We will make classical energy estimates (see for instance [5]), and keep track of the h dependence.

We want to bound $(S\partial_x^\alpha u_h, \partial_x^\alpha u_h)$ where α is a multi index of length $\le s$, and $(.,.)$ is the usual L^2 scalar product. We have

$$\partial_t(S\partial_x^\alpha u_h, \partial_x^\alpha u_h) = (\partial_t S\partial_x^\alpha u_h, \partial_x^\alpha u_h) + 2(S\partial_t\partial_x^\alpha u_h, \partial_x^\alpha u_h)$$

since S is symmetric.

The first term can be bounded by

$$(\partial_t S\partial_x^\alpha u_h, \partial_x^\alpha u_h) \le |\partial_t S|_{L^\infty} ||\partial_x^\alpha u_h||_{L^2}^2,$$

but

$$|\partial_t S|_{L^\infty} \le C(|u_h|_{L^\infty})|\partial_t u_h|_{L^\infty}$$

and by Sobolev injections and equation (2.4),

$$|\partial_t u_h|_{L^\infty} \le C(||u_h||_s)||u_h||_s$$

where $s > d/2 + 2$ and where

$$||u_h||_s^2 = \sum_{|\alpha|\le s} ||\partial_x^\alpha u_h||_{L^2}^2.$$

For the second term we use

$$(S\partial_t\partial_x^\alpha u_h, \partial_x^\alpha u_h) = h(SL(\partial_x^\alpha u_h), \partial_x^\alpha u_h) - (S\partial_x^\alpha(\sum_{i=1}^d A^i(u_h)\partial_i u_h), \partial_x^\alpha u_h). \tag{2.5}$$

But

$$(SL(\partial_x^\alpha u_h), \partial_x^\alpha u_h) = -\frac{h}{2}\int \partial_x^\alpha a_h^1 \Delta_x \partial_x^\alpha a_h^2 - \partial_x^\alpha a_h^2 \Delta_x \partial_x^\alpha a_h^1 = 0$$

by integration by parts.

The second term of (2.5) can be rewritten under the form

$$(S\partial_x^\alpha(\sum_{i=1}^d A^i(u_h)\partial_i u_h), \partial_x^\alpha u_h) = (S\sum_{i=1}^d A^i(u_h)\partial_i\partial_x^\alpha u_h, \partial_x^\alpha u_h)$$

$$+(S\Big(\partial_x^\alpha(\sum_{i=1}^d A^i(u_h)\partial_i u_h) - \sum_{i=1}^d A^i(u_h)\partial_i\partial_x^\alpha u_h\Big), \partial_x^\alpha u_h).$$

But

$$(S\sum_{i=1}^d A^i(u_h)\partial_i\partial_x^\alpha u_h, \partial_x^\alpha u_h) = -\sum_{i=1}^d (\partial_i(SA^i(u_h))\partial_x^\alpha u_h, \partial_x^\alpha u_h)$$

$$-\sum_{i=1}^d (SA^i(u_h)\partial_i\partial_x^\alpha u_h, \partial_x^\alpha u_h)$$

since $SA^i(u_h)$ is a symmetric matrix. Therefore

$$|(S\sum_{i=1}^d A^i(u_h)\partial_i\partial_x^\alpha u_h, \partial_x^\alpha u_h)| \le C(|u_h|_{L^\infty})||\partial_x^\alpha u_h||_{L^2}^2 |\nabla_x u_h|_{L^\infty}$$

$$\le C(|u_h|_{L^\infty})||\partial_x^\alpha u_h||_{L^2}^2 ||u_h||_s.$$

The usual estimates on commutators lead to

$$|(S\Big(\partial_x^\alpha(\sum_{i=1}^d A^i(u_h)\partial_i u_h) - \sum_{i=1}^d A^i(u_h)\partial_i\partial_x^\alpha u_h\Big), \partial_x^\alpha u_h)| \le C(||u_h||_s)||u_h||_s^2.$$

Thus

$$\partial_t \sum_{|\alpha|\le s} (S\partial_x^\alpha u_h, \partial_x^\alpha u_h) \le C(||u_h||_s)||u_h||_s^2$$

for $s > 2 + d/2$.

This energy estimate is independent on h, and the end of the proof of Theorem 1.1 is straightforward with the help of Gronwall's Lemma.

3. Proof of Theorem 1.2

Assume now that there exists a solution (ρ, v) in $L^\infty([0,T], H^{s+2}(\mathbb{R}^d))$ to equations (1.5,1.6), on a time interval $[0,T]$, with $s > d/2 + 2$, for the initial data

$$\rho = |\lim_{h\to 0} a_h^0|^2 \qquad \text{and} \qquad v = \nabla_x S^0. \tag{3.1}$$

We want to show that there exists a solution to equation (2.4) on the same time interval, for h small enough, a solution which is uniformly bounded in $L^\infty([0,T], H^s)$. The limit system of (2.4) is

$$\partial_t u + \sum_{i=1}^d A^i(u)\partial_i u = 0 \tag{3.2}$$

where $u = (a_1, a_2, w)$, and admits a solution on a maximal time interval $[0, T'[$. Let us prove that $T' > T$. Assume $T' \leq T$ and let $\rho = |a_1|^2 + |a_2|^2$ and $v = w$. As (ρ, v) satisfy (1.5,1.6) with initial data (3.1), we know that ρ and v are in $L^\infty([0,T'], H^s(\mathbb{R}^d))$; therefore $w \in L^\infty([0,T'], H^s(\mathbb{R}^d))$. Using (2.1,2.2), we get that a_1 and a_2 are in $L^\infty([0,T'], H^{s-1}(\mathbb{R}^d))$, which is impossible since T' is assumed to be the maximal time of existence. Therefore $T' > T$, and system (3.2) has a smooth solution on the time interval $[0,T]$.

Setting $v_h = u_h - u$, we get

$$\partial_t v_h + \sum_{i=1}^d A^i(u+v_h)\partial_i v_h + \sum_{i=1}^d (A^i(u+v_h) - A^i(u))\partial_i u = hL(v_h) + hL(u).$$

The matrix $\sum_{i=1}^d A^i(u+v_h)\xi_i$ is symmetrisable and we can make again the energy estimates of the previous section. The term

$$\sum_{i=1}^d A^i(u+v_h)\partial_i v_h$$

can be treated as above;

$$(S\partial_x^\alpha \Big(\sum_{i=1}^d (A^i(u+v_h) - A^i(u))\partial_i u\Big), \partial_x^\alpha v_h)$$

is bounded by

$$C(||u||_s, ||v_h||_s)||v_h||_s^2.$$

Moreover

$$(hSL(\partial_x^\alpha v_h) + hSL(\partial_x^\alpha u), \partial_x^\alpha v_h) = (hL(\partial_x^\alpha u), \partial_x^\alpha v_h)$$

is bounded by

$$h||u||_{\alpha+2}||v_h||_\alpha.$$

Thus

$$\partial_t \sum_{|\alpha|\leq s} (S\partial_x^\alpha v_h, \partial_x^\alpha v_h) \leq C(||v_h||_s, ||u||_{s+2})^2||v_h||_s^2 + h||u||_{s+2}||v_h||_s,$$

for $s > d/2+2$. Moreover $||v_h(t=0)||_s$ goes to zero as h goes to 0; therefore Gronwall's Lemma shows that for h small enough, there exists a constant $C(h)$ such that

$$||v_h||_s \leq C(h) \quad \text{on} \quad [0,T],$$

with $C(h) \to 0$ as $h \to 0$, which ends the proof of Theorem 1.2.

4. Proof of Theorem 1.3

4.1. **Zeroth order.** From Theorem 1.2, we know that a_h and w_h are bounded in $L^\infty([0,T],H^s(\mathbb{R}^d))$; thus $\partial_t a_h$ and $\partial_t w^h$ are bounded in $L^\infty([0,T],H^{s-2}(\mathbb{R}^d))$. Therefore, for a subsequence, a_h and w_h converge uniformly in $L^\infty([0,T],H^{s-2}(\mathbb{R}^d))$ to a'_0 and w'_0, solutions of

$$\partial_t w'_0 + (w'_0.\nabla_x)w'_0 + f'(|a'_0|^2)\nabla_x|a'_0|^2 = 0,$$

$$\partial_t a'_0 + w'_0.\nabla_x a'_0 + \frac{1}{2}a'_0\nabla_x.w'_0 = 0,$$

with initial data $a'_0 = \lim_{h\to 0} a^0(h)$ and $w'_0 = \nabla S^0$.

This system admits an unique solution; therefore in fact all the sequence (a_h, w_h) converges.

4.2. **First order.** As in the previous section, we set $v_h = u_h - u$, and we show by the same energy estimate that

$$||v_h(t)||_{H^{s-2}} \leq hC(||u||_{L^\infty([0,T],H^s)})$$

for all $t \leq T$. Let then $\tilde{v}_h = v_h/h$. We see that $\tilde{v}_h$ is bounded in $L^\infty([0,T],H^{s-2})$, and that $\partial_t\tilde{v}_h$ is bounded in $L^\infty([0,T],H^{s-4})$. Thus, for a subsequence, $\tilde{v}_h$ converges strongly in $L^\infty([0,T],H^{s-4})$ to a function u'_1. Taking the limit of the equations for $\tilde{v}_h$, we obtain that u'_1 solves the linearized problem

$$\partial_t u'_1 + \sum_{i=1}^d A^i(u'_0)\partial_i u'_1 + \sum_{i=1}^d (\nabla A^i(u'_0)u'_1)\partial_i u'_0 = L(u'_0),$$

with initial data

$$u'_1 = \lim_{h\to 0} \frac{u_h(0) - u'_0}{h}.$$

As the solution of this problem is unique, we in fact get the convergence of the whole sequence $\tilde{v}_h$ to u'_1.

4.3. **Higher order terms.** Assume that we already obtained an asymptotic expansion to the order N

$$u_h = \sum_{j=0}^N u'_j h^j + o(h^N),$$

the functions u'_j being in $L^\infty([0,T],H^{s-2j}(\mathbb{R}^d))$. Let

$$\tilde{u}_h = \sum_{j=0}^N u'_j h^j$$

and

$$v_h = u_h - \sum_{j=0}^{N} u'_j h^j.$$

We write the equation for u_h, making a Taylor expansion on A_i to get

$$\partial_t v_h + \sum_{i=1}^{d} A^i(\tilde{u}_h + v_h)\partial_i v_h - \sum_{i=1}^{d} (A^i(\tilde{u}_h) - A^i(\tilde{u}_h + v_h))\partial_i \tilde{u}_h$$
$$= hL(v_h) + h^{N+1}B_h^N,$$

where B_h^N is a function which depends on $\tilde{u}_h$ and is bounded in $L^\infty([0,T], H^\sigma)$, with $\sigma = s - 2N - 2$, uniformly in h (see below). Moreover, by the assumption on the initial data, $h^{-N-1}v_h(0)$ is bounded in H^s. Making again the energy estimates of the previous section, we obtain that $h^{-N-1}v_h$ is bounded in $L^\infty([0,T], H^{s-2N-2})$.

We then set $\tilde{v}_h = v_h/h^{N+1}$, and as above, we get that $\tilde{v}_h$ converges to some function u'_{N+1} as h goes to 0. To find the equation for u'_{N+1}, write the term of order h^{N+1} in

$$\partial_t(\tilde{u}_h + h^{N+1}w_h) + \sum_{i=1}^{d} A^i(\tilde{u}_h + h^{N+1}w_h)\partial_i(\tilde{u}_h + h^{N+1}w_h) - hL(\tilde{u}_h)$$

is 0 (which enables us to verify that B_h^{N+1} is bounded).

4.4. **WKB expansion.** We have obtained the formal expansion of a_h and of S_h to an arbitrarily high order. To get back the usual WKB expansion we have only to write the identity of the two following formal series:

$$\sum_{j=0}^{+\infty} a_j(t,x)h^j e^{iS(t,x)/h} = (\sum_{k=0}^{+\infty} a'_k h^k)e^{i\sum_{k=0}^{+\infty} S'_k h^k}.$$

For instance, $S = S'_0$, $a_0 = a'_0 e^{iS'_1}$, and $a_1 = e^{iS'_1}(a'_1 + iS'_2 a'_0)$.

References

1. P. Gérard, *Remarques sur l'analyse semi-classique de l'équation de Schrödinger non linéaire*, Séminaire EDP de l'École Polytechnique, Palaiseau, France (1992-93), lecture n^o XIII. MR **94i:**35157
2. J. Ginibre, G. Velo, *On the global Cauchy problem for some nonlinear Schrödinger equation*, Ann. Inst. H. Poincaré, Anal. non linéaire **1** (1984), 309-323. MR **87a:**35164
3. E. Grenier, *Limite semiclassique de l'équation de Schrödinger non linéaire en temps petit*, C.R. Acad. Sci. Paris, Série I **320** (1995), 691-694. MR **96a:**35191
4. S. Jin, C. D. Levermore, D.W. Mc Laughlin, *The behaviour of solutions of the NLS equation in the semiclassical limit*, Singular limits of dispersive waves, NATO ASI, Series B : Physics, vol. 320, 1994, pp. 235-256. MR **95k:**35191
5. A. Majda, *Compressible fluid flow and systems of conservation laws in several space variables*, Appl. Math. Sci 53, Springer, 1984. MR **85e:**35077

Laboratoire d'Analyse Numérique, CNRS - URA 189, Université Paris 6, 4 place Jussieu, 75252 Paris Cedex 05, France

E-mail address: grenier@ann.jussieu.fr