A BOUND FOR THE NILPOTENCY OF A GROUP OF SELF HOMOTOPY EQUIVALENCES

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Abstract. Let \( E_{\Omega}(X) \) be the group of homotopy classes of self-homotopy equivalences of \( X \) such that \( \Omega f \simeq 1_{\Omega X} \). We prove that \( E_{\Omega}(X) \) is a nilpotent group and that \( \text{nil} E_{\Omega}(X) \leq \text{cat}(X) - 1 \).

Given a pointed space \( X \) of the homotopy type of a CW-complex, let \( \mathcal{E}(X) \) denote the group of based homotopy classes of self homotopy equivalences of \( X \) ([1] is an excellent survey on this object). A considerable amount of work has been dedicated to obtaining finiteness properties, not only of \( \mathcal{E}(X) \), but also of certain interesting subgroups which preserve additional geometrical structure (see for example [2],[5],[6],[8]). This note goes in this direction: Let \( E_{\Omega}(X) \) be the kernel of the obvious map \( \mathcal{E}(X) \to \mathcal{E}(\Omega X) \) (i.e. homotopy classes of equivalences \( f: X \to X \) such that \( \Omega f \simeq 1_{\Omega X} \)) and, as usual, denote by \( \text{cat}(X) \) the Lusternik-Schnirelmann category of \( X \). Then we prove:

Theorem. If \( \text{cat} X \) is finite then \( E_{\Omega}(X) \) is a nilpotent group and \( \text{nil} E_{\Omega}(X) \leq \text{cat}(X) - 1 \).

Remarks. (a) Observe that \( E_{\Omega}(X) \) is a subgroup of the group \( \mathcal{E}_{\#}(X) \) consisting of homotopy classes of equivalences inducing the identity on the homotopy groups of \( X \). Therefore it is known to be nilpotent for finite complexes in view of [4, Thm. B]. Note also that, in general, this inclusion is proper as is shown in the following example comunicated to us by F. Cohen: It is known [4, Cor. 1.3] that, given a prime \( p \geq 3 \) and \( n \geq 1 \), \( p^n \) is an exponent for \( S^{2n+1} \) at \( p \). Therefore, if we consider \( \rho \) the \( p^n \)-th power map on the space \( X = (\Omega^{2n-3}S^{2n+1}(2n+1))_\rho(p) \) and call \( \sigma = 1 + \rho \), it follows that \( \pi_*(\sigma) = 1_{\pi_*(X)} \). On the other hand \( \Omega(\rho) \) is essential [9, Thm. 1] and thus \( \Omega(\sigma) \) cannot be homotopic to the identity.

(b) However, for rational spaces it is well known that \( E_{\Omega}(X) = \mathcal{E}_{\#}(X) \) since in this case \( \Omega X \) has the homotopy type of a product of Eilenberg-Mac Lane spaces of type \((n_i, \mathbb{Q})\) in which the integers \( \{n_i\} \) describe the degrees of a basis of \( \pi_*(X) \). Hence, the theorem above could be seen as a generalization of [6, Thm. 1].

The rest of the paper is devoted to the proof of the theorem above. To simplify the notation we shall not distinguish between a homotopy class and a map which represents it. Also, equality of homotopy classes (or maps) will often mean homotopy between its representatives.

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To start, let us recall the characterization of the LS category of a space $X$ given in [7]. The $n$-th Ganea fibration of $X$, $F_n(X) \rightarrow E_n(X) \xrightarrow{p_n} X$, is defined by an inductive procedure in the following way: $p_0$ is just the path fibration $\Omega X \rightarrow PX \xrightarrow{p_0} X$. Next consider $C$ the homotopy cofibre of the inclusion $F_{n-1}(X) \rightarrow E_{n-1}(X)$ and extend $p_{n-1}$ to a map $C \rightarrow X$. The associated fibration to this map $F_n(X) \rightarrow E_n(X) \xrightarrow{p_n} X$ is by definition the $n$-th Ganea fibration of $X$. $E_n(X)$ is called the $n$-th Ganea space for $X$. As a general picture we have:

$$
\begin{array}{c}
\Omega X \\
\downarrow \\
PX \xrightarrow{i_1} E_1(X) \\
\downarrow \\
p_0 \\
X
\end{array}
\quad
\begin{array}{c}
F_1(X) \\
\downarrow \\
\cdots
\end{array}
\quad
\begin{array}{c}
F_{n-1}(X) \\
\downarrow \\
p_{n-1} \\
E_n(X)
\end{array}
\quad
\begin{array}{c}
F_n(X) \\
\downarrow \\
p_n \\
X
\end{array}

Then, we shall make use of the following facts:

1. $	ext{cat } X \leq n$ if and only if $p_n$ admits a homotopy section.
2. $F_n(X)$ has the homotopy type of the join of $n + 1$ copies of $\Omega X$.
3. For each space $X$ and each integer $n$, the fibration $E_n(X) \xrightarrow{p_n} X$ defines an augmented functor, that is to say, given $f: X \rightarrow Y$, there exists a (functorial) map $E_n(f): E_n(X) \rightarrow E_n(Y)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
E_n(X) & \xrightarrow{E_n(f)} & E_n(Y) \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
$$

Next, given a map $f: X \rightarrow X$ representing an element on $E_{\Omega}(X)$ define the length of $f$, $l(f)$, as the biggest integer $n$ for which $fp_n = p_n$. Since $E_1(X)$ has the homotopy type of $\Sigma \Omega X$ and (up to homotopy) $p_1: \Omega X \rightarrow X$ is the adjoint to the identity, $l(f)$ is at least 1. Also, observe that if $l(f) = n$, then $fp_m = f$ for any $m \leq n$. Next, define $G_n$ as the subgroup of $E_\Omega(X)$ consisting of equivalences of length at least $n$. Clearly $G_1 = E_\Omega(X)$ and $G_{n+1} \subset G_n$.

**Lemma.** $[G_1, G_n] \subset G_{n+1}$.

**Proof.** First, recall [10] that given a cofibration sequence $Y \rightarrow Z \rightarrow C$, the coaction $\nu: C \rightarrow \Sigma Y \vee C$ induces a natural action of the group $[\Sigma Y, X]$ on $[C, X]$. Explicitly, given $\beta \in [\Sigma Y, X]$ and $\alpha \in [C, X]$, define $\alpha^\beta = (\beta, \alpha) \circ \nu$. The orbits of this action are precisely $i_*^{-1}(h)$, $h \in [Z, X]$, with $i_*: [C, X] \rightarrow [Z, X]$ induced by $i$. That is to say, given maps $\alpha_1, \alpha_2: C \rightarrow X$, $\alpha_1i \sim \alpha_2i$ if and only if there exists $\beta: \Sigma Y \rightarrow X$ such that $\alpha_1^\beta = \alpha_2^\beta$.

Note also that given $\gamma \in [X, W]$ and $\varphi \in [C, C]$, $\gamma^\alpha = (\gamma \alpha)^\gamma$ and $\alpha^\beta \varphi = (\alpha \varphi)^{\beta_\varphi}$, with $\Sigma \varphi \in [\Sigma Y, \Sigma Y]$ induced by $\varphi$ by collapsing $Z$. We return to the proof of the lemma. Let $f, g: X \rightarrow X$ be maps satisfying $fp_1 = p_1$ and $gp_n = p_n$. We will prove that $fgp_{n+1} = gf p_{n+1}$. For that we shall apply the considerations above to the cofibration sequence $F_n(X) \rightarrow E_n(X) \xrightarrow{i_n} E_{n+1}(X)$. Since $gp_n = p_n$
and $p_n = p_{n+1}i_{n+1}$, there exists $h: \Sigma F_n(X) \to X$ such that $gp_{n+1} = p_n^h$. Observe that:

(i) Since $h$ factors as the composite $\Sigma F_n(X) \xrightarrow{k} \Sigma \Omega X \xrightarrow{p_1} X$, we have $fh = fp_1k = p_1k = h$.

(ii) On the other hand, since $\Omega f = 1$, via (2), it follows that $F_n(f) = *^{n+1}\Omega f = 1$.

Finally we can write:

$$fgp_{n+1} = fp_{n+1}h = (fp_{n+1})h = (fp_{n+1})^{h\Sigma F_n(f)} = (p_{n+1}E_{n+1}(f))^{h\Sigma F_n(f)} = p_{n+1}^{hE_{n+1}(f)} = gp_{n+1}E_{n+1}(f) = gfp_{n+1}. \quad \square$$

Proof of the theorem. Observe that if $\text{cat} \; X = m$ then $G_m = \{1\}$. Indeed, given $f \in G_n$ and in view of (1), $f = fp_m\sigma = p_m\sigma = 1$ with $\sigma$ section of $p_n$. Hence, by lemma above we have a finite decreasing sequence of normal subgroups

$$E_{\Omega}(X) = G_1 \supset G_2 \supset \ldots \supset G_m = \{1\}$$

in which $[G_1, G_n] \subset G_{n+1}$ and thus the theorem follows. \hfill \square

References


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