UNIQUENESS OF POSITIVE SOLUTIONS FOR STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS

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ABSTRACT. Sufficient conditions for the uniqueness of positive solutions of singular Sturm-Liouville boundary value problems

\[(E) \left( |u'|^{m-2}u'' \right)' + f(t, u, u') = 0, \quad \theta_1 < t < \theta_2, \quad m \geq 2, \]

\{(BC) \begin{align*}
\alpha_1 u(\theta_1) - \beta_1 u'(\theta_1) &= 0, \\
\alpha_2 u(\theta_2) + \beta_2 u'(\theta_2) &= 0,
\end{align*}\]

where \(\alpha_i, \beta_i \geq 0\) and \(\alpha_i^2 + \beta_i^2 \neq 0\) (\(i = 1, 2\)), are established.

1. Introduction

In this paper, we are concerned with the uniqueness of positive solutions of boundary value problems for the nonlinear differential equation

\[(E) \left( |u'|^{m-2}u'' \right)' + f(t, u, u') = 0, \quad \theta_1 < t < \theta_2, \quad m \geq 2, \]

subject to one of the following sets of boundary conditions:

\{(BC.1) \quad u(\theta_1) = \xi_1 \geq 0, \quad u'(\theta_2) = \xi_2 \geq 0, \}

\{(BC.2) \quad u'(\theta_1) = \xi_1 \leq 0, \quad u(\theta_2) = \xi_2 \geq 0, \}

\{(BC.3) \quad u(\theta_1) = \xi_1 \geq 0, \quad u(\theta_2) = \xi_2 \geq 0, \}

where \(m \geq 2, (\theta_1, \theta_2) \subseteq (-\infty, \infty)\) and \(f: (\theta_1, \theta_2) \times (0, \infty) \times (-\infty, \infty) \to (0, \infty)\) satisfies

- \(f(t, x, y)\) is locally Lipschitz continuous for \((x, y)\) in \((0, \infty) \times \{(\infty, 0) \cup (0, \infty)\}\); \(f(t, x, y)/x^{m-1}\)

- is strictly decreasing with respect to \(x \in (0, \infty)\) for each fixed \((t, y) \in (\theta_1, \theta_2) \times (-\infty, \infty)\); and

- \(\text{sgn}(y)f(t, x, y)\) is decreasing with respect to \(y \in (-\infty, \infty)\) for each fixed \((t, x) \in (\theta_1, \theta_2) \times (0, \infty)\).
Furthermore, we use the uniqueness theorems of (E) with respect to the boundary conditions (BC.1)–(BC.3) to show that

\[
\begin{aligned}
\text{(BVP)}
\begin{cases}
\text{(E)} \quad (|u'|^{m-2}u')' + f(t, u, u') = 0, & \text{in } (\theta_1, \theta_2), \ m \geq 2, \\
\text{(BC)} \quad \alpha_1 u(\theta_1) - \beta_1 u'(\theta_1) = 0, \\
\quad \alpha_2 u(\theta_2) + \beta_2 u'(\theta_2) = 0,
\end{cases}
\end{aligned}
\]

where \(\alpha_i, \beta_i \geq 0\) and \(\alpha_i^2 + \beta_i^2 \neq 0\) \((i = 1, 2)\) has at most one positive solution in \(C^1([\theta_1, \theta_2])\).

Equations of the type (E) arise in studies of radially symmetric solutions (i.e., solutions \(u\) that depend only on the variable \(r = |x|\)) of the \(m\)-Laplace equation,

\[
(E_1) \quad \nabla \cdot (|\nabla u|^{m-2}\nabla u) + g(|x|, u, \nabla u) = 0, \quad R_0 < |x| < R_1, \ x \in \mathbb{R}^N, \ N \geq 2.
\]

A radially symmetric solution of (E_1) satisfies the ordinary differential equation

\[
(E_2) \quad (|u'|^{m-2}u')' + \frac{N-1}{r} |u'|^{m-2}u' + g(r, u, u') = 0, \quad R_0 < r < R_1.
\]

With the change of variables \(t = r \frac{m-N}{m} \) (for \(m \neq N\)) or \(t = \log r \) (for \(m = N\)), equation (E_2) can be reduced to an equation of the type (E) or

\[
(E^*) \quad (m-1)|u'|^{m-2}u'' + f(t, u, u') = 0, \quad \theta_1 < t < \theta_2, \ m \geq 2.
\]

Conditions for the existence of solutions of equation (E) with respect to (BC.1)–(BC.3) were studied by many authors; see for instance, De Figueiredo, Lions and Nussbaum [6], Granas, Guenther and Lee [9], Kaper, Knaap and Kwong [12], Lions [16], del Pino, Elgueta and Manasevich [19], Rabinowitz [21], Wong [24], and the references therein. The uniqueness problem concerning (E), for the case \(m = 2\), has been studied by many authors. For example, Gatica, Oliker and Waltman [7], Kwong [14], Dalmasse [4, 5], Brezis and Oswald [3], Krasnoselkii [13], and the excellent book by Agarwal and Lakshmikantham [1]. However, it seems that very little is known for the case \(m \neq 2\). Recently, Naito [17] considered the case \(f(t, u, u') = p(t)f(u)\) and established some excellent conditions for uniqueness by using the generalized Prüfer transformation and comparison theorems. In this article, the author attempts to afford a concise approach to study the uniqueness of positive solutions of (E) with boundary conditions (BC.1)–(BC.3) and (BC).

For other related results, we refer the reader to Bobisud [2], Dalmasse [4, 5], Guedda and Veron [10], Naito [17], del Pino and Manasevich [20], O’Regan [22], and Wong and Yu [23].

2. MAIN RESULT

Let \(u\) and \(v\) be two distinct positive solutions of (E). We define

\[
w(t) := \{u(t)\}^{m-1}(|v'(t)|^{m-2}v'(t)) - (|u'(t)|^{m-2}u'(t))\{v(t)\}^{m-1}
\]

for \(t \in [a, b] \subseteq [\theta_1, \theta_2]\).
It is clear that \( w(t) \) satisfies
\[
w'(t) = u^{m-1}|u'|^{m-2}u' - (|u'|^{m-2}u')'v^{m-1} \\
+ (m-1)u'v|u'|^{m-2}u - |u'|^{m-2}v^{m-2} \\
= u^{m-1}\{-f(t,v,v')\} - \{-f(t,u,u')\}v^{m-1} \\
+ (m-1)u'v|u'|^{m-2}u - |u'|^{m-2}v^{m-2} \\
(2) \\
= (uv)^{m-1}\left\{ \frac{f(t,u,u')}{v^{m-1}} - \frac{f(t,v,v')}{v^{m-1}} \right\} \\
+ (m-1)u'v|u'|^{m-2}u - |u'|^{m-2}v^{m-2} \\
= (uv)^{m-1}\left\{ \frac{f(t,u,u')}{v^{m-1}} - \frac{f(t,v,v')}{v^{m-1}} + f(t,v,u') - f(t,v,v') \right\} \\
+ (m-1)u'v|u'|^{m-2}u - |u'|^{m-2}v^{m-2} \]
for \( t \in (a,b) \subseteq (\theta_1, \theta_2) \).

In order to treat our main results, we need the following:

**Lemma 2.1.** Let \( u \) and \( v \) be distinct positive solutions of \((E)\)–(BC) in \( C^1([\theta_1, \theta_2]) \) and \( u > v \) on \((\theta_1, \theta_2)\) for \( i = 1, 2, 3 \). Then \( w'(t) < 0 \) in \((\theta_1, \theta_2)\), that is, \( w(t) \) is strictly decreasing in \([\theta_1, \theta_2]\).

**Proof.** We separate the proof into the following cases:

**Case (1).** Suppose that \( u \) and \( v \) are two distinct positive solutions of \((E)\)–(BC.1).

First, we claim that \( u'(t)v(t) \geq v'(t)u(t) \geq 0 \) in \([\theta_1, \theta_2]\).

**(1°)** Assume that there exists \( t_1 \in (\theta_1, \theta_2) \) such that
\[ u'(t_1)v(t_1) > v'(t_1)u(t_1) \geq 0 \] and \( u'(t_1)v(t_1) = v'(t_1)u(t_1) \geq 0 \),
which imply \( u'(t) \geq v'(t) \geq 0 \) on \([\theta_1, t_1]\). It follows from (2), \( f(t,x,y)/x^{m-1} \) is strictly decreasing with respect to \( x \in (0, \infty) \) and \( f(\cdot, \cdot, y) \) is decreasing in \((0, \infty)\) that \( w'(t) < 0 \) on \((\theta_1, t_1)\). Thus \( w(t) \) is a strictly decreasing function on \([\theta_1, t_1]\).

Therefore
\[ 0 = (v'(t_1)u(t_1))^{m-1} - (u'(t_1)v(t_1))^{m-1} = w(t_1) \]
which gives a contradiction.

**(2°)** Assume that there exists a strictly decreasing sequence \( \{t_n\}_{n=1}^\infty \) satisfying
\[ \lim_{n \rightarrow \infty} t_n = \theta_1, \quad (u'v - v'u)(t_n) = 0, \quad (u'v - v'u)(t_2n) = (u''v - v''u)(t_2n) \leq 0 \]
and \( (u'v - v'u)'(t_{2n-1}) = (u''v - v''u)(t_{2n-1}) \geq 0 \) for all \( n \in \mathbb{N} \). It follows from \((E^*)\), \((H)\) and \( u'(t_{2n}) \geq v'(t_{2n}) \geq 0 \) that
\[ 0 \geq (m-1)|u'|^{m-2}v^{m-2}(u''v - v''u)(t_{2n}) \\
= (uv)^{m-1}(t_{2n}) \left\{ \frac{f(t_{2n}, u(t_{2n}), u'(t_{2n}))}{u^{m-1}(t_{2n})} - \frac{f(t_{2n}, v(t_{2n}), v'(t_{2n}))}{v^{m-1}(t_{2n})} \right\} \\
= (uv)^{m-1}(t_{2n}) \left\{ \frac{f(t_{2n}, v(t_{2n}), v'(t_{2n}))}{v^{m-1}(t_{2n})} - \frac{f(t_{2n}, u(t_{2n}), u'(t_{2n}))}{u^{m-1}(t_{2n})} \right\} \\
+ (uv)^{m-1}(t_{2n}) \left\{ \frac{f(t_{2n}, u(t_{2n}), u'(t_{2n}))}{u^{m-1}(t_{2n})} - \frac{f(t_{2n}, u(t_{2n}), u'(t_{2n}))}{u^{m-1}(t_{2n})} \right\} > 0. \]
This gives a contradiction. Hence, we have \( u'(t)v(t) \geq v'(t)u(t) \geq 0 \) in \([\theta_1, \theta_2]\), which implies \( w'(t) < 0 \) in \([\theta_1, \theta_2]\).
Case (2). Suppose that \( u \) and \( v \) are two distinct positive solutions of (E)–(BC.2). The proof is quite similar to Case (1), thus we omit the details.

Case (3). Suppose that \( u \) and \( v \) are two distinct positive solutions of (E)–(BC.3). By virtue of Case (1) and Case (2), we need only consider the case \( u'(\theta_1) \neq v'(\theta_1) \) and \( u'(\theta_2) \neq v'(\theta_2) \). Without loss of generality, we may assume that \( \xi_1 \leq \xi_2 \) (resp. \( \xi_2 \leq \xi_1 \)), which implies \( u'(\theta_1) > v'(\theta_1) \geq 0 \) (resp. \( u'(\theta_2) < v'(\theta_2) \leq 0 \)).

\((3')\) Assume that there exist \( t_1, t_2 \in (\theta_1, \theta_2) \) such that \( u'(t_1) = v'(t_2) = 0 \). Since \( |u'|^{m-2}u' \) and \( |v'|^{m-2}v' \) are strictly decreasing in \((\theta_1, \theta_2)\), \( t_1 \) and \( t_2 \) are determined uniquely. If \( t_1 < t_2 \), it follows from \( u'(t_1) > v'(t_1) \geq 0 \) and \( u'(t_1) = 0 < v'(t_1) \) that there exists \( t_3 \in (\theta_1, t_1) \) satisfying
\[ u'(t) > v'(t) > 0 \text{ on } [\theta_1, t_3] \quad \text{and} \quad u'(t) = v'(t) > 0. \]

It follows from \( u(\theta_1) = v(\theta_1) = \xi_1 \geq 0 \), \( u'(t_3) = v'(t_3) > 0 \) and Case (1) that \( w'(t) < 0 \) on \((\theta_1, t_3)\). Thus \( w(t) \) is a strictly decreasing function on \([\theta_1, t_3]\). Therefore
\[
0 < \{u'(t_3)\}^{m-1}[\{u(t_3)\}^{m-1} - \{v(t_3)\}^{m-1}] = w(t_3)
\]
\[
< w(\theta_1) = \{u(\theta_1)\}^{m-1}[|v'(\theta_1)|^{m-2}v'(\theta_1) - |u'(\theta_1)|^{m-2}u'(\theta_1)] \leq 0,
\]
which gives a contradiction. If \( t_1 > t_2 \), it follows from \( u'(t_2) < v'(t_2) \leq 0 \) and \( v'(t_1) < 0 = u'(t_1) \) that there exists \( t_4 \in (t_1, \theta_2) \) such that
\[
 u'(t_4) = v'(t_4) < 0 \quad \text{and} \quad u'(t) < v'(t) \leq 0 \text{ on } (t_4, \theta_2].
\]

It follows from \( u(\theta_2) = v(\theta_2) = \xi_2 \geq 0 \), \( u'(t_4) = v'(t_4) < 0 \) and Case (2) that \( w'(t) < 0 \) on \((t_4, \theta_2)\). Thus \( w(t) \) is a strictly decreasing function on \([t_4, \theta_2]\). Therefore
\[
0 \leq \{u(t_4)\}^{m-1}[|v'(t_2)|^{m-2}v'(t_2) - |u'(t_2)|^{m-2}u'(t_2)] = w(t_2)
\]
\[
< w(t_4) = |u'(t_4)|^{m-2}u'(t_4)[\{u(t_4)\}^{m-1} - \{v(t_4)\}^{m-1}] \leq 0,
\]
which gives a contradiction, too. Thus, \( t_1 = t_2 \). By Cases (1)–(2), we see that \( w'(t) < 0 \) in \((\theta_1, \theta_2)\).

\((4')\) Assume that there exists \( t_1 \in (\theta_1, \theta_2) \) such that \( u'(t_1) = 0 \) and \( v'(t) \neq 0 \) in \((\theta_1, \theta_2)\). It follows from \( u'(t_1) > v'(t_1) \geq 0 \) and \( u'(t_1) = 0 < v'(t_1) \) that there exists \( t_5 \in (\theta_1, t_1) \) satisfying
\[
 u'(t) > v'(t) > 0 \text{ on } [\theta_1, t_5] \quad \text{and} \quad u'(t) = v'(t) > 0.
\]

Just as in the proof in \((3')\), we get a contradiction.

\((5')\) Assume that there exists \( t_2 \in (\theta_1, \theta_2) \) such that \( v'(t_2) = 0 \) and \( u'(t) \neq 0 \) in \((\theta_1, \theta_2)\). Therefore, we obtain \( 0 < u'(t_2) < v'(t_2) \leq 0 \), which gives a contradiction.

\((6')\) Assume that \( u'(t) \neq 0 \) and \( v'(t) \neq 0 \) in \((\theta_1, \theta_2)\). It follows from \( u'(t_1) > v'(t_1) \geq 0 \) and \( u'(t_2) < v'(t_2) \geq 0 \) that there exists \( t_6 \in (\theta_1, \theta_2) \) satisfying
\[
 u'(t) > v'(t) > 0 \text{ on } [\theta_1, t_6] \quad \text{and} \quad u'(t) = v'(t) > 0.
\]

Just as in the proof in \((3')\), we get a contradiction.

**Theorem 2.2.** The boundary value problem (E)–(BC.1) has at most one positive solution in \( C^1([\theta_1, \theta_2]) \).

**Proof.** Assume to the contrary that \( u \) and \( v \) are two distinct positive solutions of (E)–(BC.1). We claim that \( u \) and \( v \) intersect in \((\theta_1, \theta_2)\). Suppose, on the contrary,
that \( u(t) > v(t) \) in \((\theta_1, \theta_2)\). It follows from Lemma 2.1 and \( u'(\theta_1) \geq v'(\theta_1) \geq 0 \) that
\[
0 \leq \{ u(\theta_2) \}^{m-1} - \{ v(\theta_2) \}^{m-1} \mid v'(\theta_2) \mid^m - \mid u'(\theta_2) \mid^m = w(\theta_2)
\]
\[
< w(\theta_1) = \{ u(\theta_1) \}^{m-1} \mid v'(\theta_1) \mid^m - \mid u'(\theta_1) \mid^m \leq 0,
\]
which gives a contradiction. Hence, there exists \( t_1 \in (\theta_1, \theta_2) \) such that \( u(t_1) = v(t_1) > 0 \). Following from \( u(t_1) = v(t_1) > 0 \), \( u'(\theta_2) = v'(\theta_2) = \xi_2 \geq 0 \), and repeating the same process as above, we obtain a \( t_2 \in (t_1, \theta_2) \) such that \( u(t_2) = v(t_2) > 0 \).

Now, we claim that \( u \) and \( v \) intersect in \((t_1, t_2)\). Assume, on the contrary, that \( u(t) > v(t) \) in \((t_1, t_2)\); then \( u'(t_1) \geq v'(t_1) \geq 0 \) and \( 0 \leq u'(t_2) \leq v'(t_2) \). From Lemma 2.1 we see that
\[
0 \leq \{ u(t_2) \}^{m-1} \mid v'(t_2) \mid^m - \mid u'(t_2) \mid^m = w(t_2)
\]
\[
< w(\theta_1) = \{ u(\theta_1) \}^{m-1} \mid v'(\theta_1) \mid^m - \mid u'(\theta_1) \mid^m \leq 0,
\]
which gives a contradiction, too. Hence, there exists \( t_3 \in (t_1, t_2) \) such that \( u(t_3) = v(t_3) > 0 \). Repeating the same argument, we obtain a strictly decreasing sequence \( \{ t_n \}_{n=3}^\infty \subset (t_1, t_2) \subset (\theta_1, \theta_2) \) such that \( t_n \in (t_1, t_{n-1}) \) and \( u(t_n) = v(t_n) \) for all \( n = 3, 4, \ldots \). By the Bolzano-Weierstrass theorem, we see that \( \{ t_n \}_{n=3}^\infty \) has an accumulation point, say \( \eta \), in \([t_1, t_2]\). It is clear that \( u(\eta) = v(\eta) > 0 \) and \( u'(\eta) = v'(\eta) > 0 \). Since \( f(t, x, y) \) satisfies (H), it follows from the uniqueness of the non-zero initial value problem that \( u(t) = v(t) \) in \([\theta_1, \theta_2]\) (see, for example, Hartman [11]).

**Theorem 2.3.** The boundary value problem (E)–(BC.2) has at most one positive solution in \( C^1((\theta_1, \theta_2)) \).

**Proof.** Assume to the contrary that \( u \) and \( v \) are two distinct positive solutions of (E)–(BC.2). Just as in the proof of Theorem 2.2, we claim that \( u \) and \( v \) intersect in \((\theta_1, \theta_2)\). Suppose, on the contrary, that \( u(t) > v(t) \) in \((\theta_1, \theta_2)\). It follows from Lemma 2.1 and \( u'(\theta_2) \leq v'(\theta_2) \geq 0 \) that
\[
0 \leq \{ u(\theta_2) \}^{m-1} \mid v'(\theta_2) \mid^m - \mid u'(\theta_2) \mid^m = w(\theta_2)
\]
\[
< w(\theta_1) = \{ u(\theta_1) \}^{m-1} \mid v'(\theta_1) \mid^m - \mid u'(\theta_1) \mid^m \leq 0,
\]
which gives a contradiction. Hence, there exists \( t_1 \in (\theta_1, \theta_2) \) such that \( u(t_1) = v(t_1) > 0 \). Since \( u(t_1) = v(t_1) > 0 \) and \( u'(\theta_2) = v'(\theta_2) = \xi_2 \geq 0 \), it follows from Theorem 2.2 that \( u(t) = v(t) \) on \([t_1, \theta_2] \). Therefore, \( u(t) = v(t) \) in \([\theta_1, \theta_2]\).

**Theorem 2.4.** The boundary value problem (E)–(BC.3) has at most one positive solution in \( C^1((\theta_1, \theta_2)) \).

**Proof.** Assume to the contrary that \( u \) and \( v \) are two distinct positive solutions of (E)–(BC.3). By virtue of Theorems 2.2 and 2.3, we see that \( u'(\theta_1) \neq v'(\theta_1) \) and \( u'(\theta_2) \neq v'(\theta_2) \). Without loss of generality, we may assume that \( u(t) > v(t) \) in \((\theta_1, \theta_2)\). Thus \( u'(\theta_1) > v'(\theta_1) \). Define \( t_1 \) and \( t_2 \) so that \( u'(t_1) = v'(t_2) = 0 \). Similar to the proof of Lemma 2.1, we have that \( t_1 = t_2 \). Applying Theorems 2.2 and 2.3, we obtain \( u \equiv v \) on \([\theta_1, t_1]\) and \( u \equiv v \) on \([t_1, \theta_2]\). Therefore, we obtain the desired results.

**Theorem 2.5.** The boundary value problem (BVP) has at most one positive solution in \( C^1((\theta_1, \theta_2)) \).
Proof. Assume to the contrary that $u$ and $v$ are two distinct positive solutions of (BVP). We split the proof into the following cases.

Case (1). Assume that $\alpha_1 = 0$, that is, $u'(\theta_1) = v'(\theta_1) = 0$. Since $|u'|^{m-2}u'$ and $|v'|^{m-2}v'$ are strictly decreasing in $(\theta_1, \theta_2)$, $u'(t) < 0$ and $v'(t) < 0$ in $(\theta_1, \theta_2)$. Now, we claim that $u$ and $v$ intersect in $(\theta_1, \theta_2)$. Suppose to the contrary that $u(t) > v(t) > 0$ in $(\theta_1, \theta_2)$.

(1') If $\alpha_2 = 0$, then $u'(\theta_2) = v'(\theta_2) = 0$. This contradicts the fact that $u'(t) < 0$ and $v'(t) < 0$ in $(\theta_1, \theta_2)$.

(2') If $\beta_2 = 0$, then $u(\theta_2) = v(\theta_2) = 0$. It follows from $u'(\theta_1) = v'(\theta_1) = 0$, $u(\theta_2) = v(\theta_2) = 0$ and Theorem 2.3 that $u(t) = v(t)$ in $(\theta_1, \theta_2)$, which gives a contradiction.

(3') If $\alpha_2 \beta_2 \neq 0$, then $u'(\theta_2) v(\theta_2) = v'(\theta_2) u(\theta_2)$. It is clear that $u(\theta_2) > v(\theta_2) > 0$, and thus $u'(\theta_2) < v'(\theta_2) < 0$. In fact, if $v(\theta_2) = 0$ (resp. $u(\theta_2) = 0$), it follows from (BC) and $u'(\theta_2) v(\theta_2) = v'(\theta_2) u(\theta_2)$ that $u(\theta_2) = v(\theta_2) = 0$, with Theorem 2.3 gives a contradiction.

Repeating the similar argument in Case (1) of Lemma 2.1, we can see that $w(t)$ is strictly decreasing on $(\theta_1, \theta_2)$. Therefore, we obtain

$$0 = (-1)^{m-2} [(u'(\theta_1) u(\theta_1))^{m-1} - (u'(\theta_1) v(\theta_1))^{m-1}] = w(\theta_1)$$

$$> w(\theta_2) = (-1)^{m-2} [(v'(\theta_2) u(\theta_2))^{m-1} - (u'(\theta_2) v(\theta_2))^{m-1}] = 0,$$

which gives a contradiction.

Hence, there is $t_2 \in (\theta_1, \theta_2)$ such that $u(t_2) = v(t_2) > 0$. Since $u'(\theta_1) = v'(\theta_1) = 0$ and $u(t_2) = v(t_2) > 0$, it follows from Theorem 2.3 that $u(t) = v(t)$ on $[\theta_1, t_2]$. Therefore, $u(t) = v(t)$ in $[\theta_1, t_2]$.

Case (2). Assume that $\beta_1 = 0$, that is, $u(\theta_1) = v(\theta_1) = 0$. It follows from $|u'|^{m-2}u'$ and $|v'|^{m-2}v'$ strictly decreasing in $(\theta_1, \theta_2)$ and $u(\theta_1) = v(\theta_1) = 0$ that $u'(\theta_1) > 0$ and $v'(\theta_1) > 0$. Now, we claim that $u$ and $v$ intersect in $(\theta_1, \theta_2)$. Suppose to the contrary that $u(t) > v(t) > 0$ in $(\theta_1, \theta_2)$, and this implies $u'(\theta_1) \geq v'(\theta_1) > 0$.

(4') If $\alpha_2 = 0$, then $u'(\theta_2) = v'(\theta_2) = 0$. It follows from Theorem 2.2 that $u(t) = v(t)$ in $[\theta_1, \theta_2]$, which gives a contradiction.

(5') If $\beta_2 = 0$, then $u(\theta_2) = v(\theta_2) = 0$. It follows from Theorem 2.4 that $u(t) = v(t)$ in $[\theta_2, t_2]$, which gives a contradiction.

(6') If $\alpha_2 \beta_2 \neq 0$, then $u'(\theta_2) v(\theta_2) = v'(\theta_2) u(\theta_2)$. Repeating the same argument in Case (1) of Lemma 2.1 (or cf. Case (1)–(3')), we see that $w(t)$ is strictly decreasing on $[\theta_1, \theta_2]$. Therefore, we obtain

$$0 = \{u(\theta_1)\}^{m-1} [(u'(\theta_1)|u'|^{m-2}v'(\theta_1) - |u'(\theta_1)|^{m-2}u'(\theta_1)] = w(\theta_1)$$

$$> w(\theta_2) = \{u(\theta_2)\}^{m-1} [(u'(\theta_2)|u'|^{m-2}v'(\theta_2) - |u'(\theta_2)|^{m-2}u'(\theta_2)] = 0,$$

which gives a contradiction.

Hence, there is $t_2 \in (\theta_1, \theta_2)$ such that $u(t_2) = v(t_2) > 0$. Since $u(\theta_1) = v(\theta_1) = 0$ and $u(t_2) = v(t_2) > 0$, it follows from Theorem 2.4 that $u(t) = v(t)$ on $[\theta_1, t_2]$. Therefore, $u(t) = v(t)$ in $[\theta_1, \theta_2]$.

Just as in the proof of Cases (1)–(2), we can exclude the possibility of $\alpha_2 = 0$ or $\beta_2 = 0$.

Case (3). Assume that $\alpha_1 \alpha_2 \beta_1 \beta_2 \neq 0$, that is, $u'(\theta_i) v(\theta_i) = v'(\theta_i) u(\theta_i)$ for $i = 1, 2$. The rest of the proof is quite similar to the proofs in Cases (1)–(3') and Cases (2)–(6'), so we omit the details.

By Cases (1)–(3), we obtain the desired results.
3. Remarks and Examples

Recently, Gatica, Oliker and Waltman [7], Kwong [14], Naito [17], Brezis and Oswald [3], and Dalmasso [5] showed the following important results:

**Theorem 3.A ([3, Theorem 1]).** Consider the problem
\[
\begin{aligned}
\Delta u + f(x, u) &= 0 \quad \text{in } \Omega, \\
u &\geq 0, \ u \neq 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

and make the following assumptions:
(A₁) For a.e. \( x \in \Omega \) the function \( u \to f(x, u) \) is continuous on \([0, \infty)\) and the function \( u \to f(x, u)/u \) is strictly decreasing in \((0, \infty)\).
(A₂) For each \( u \geq 0 \) the function \( u \to f(x, u) \) belongs to \(L^\infty(\Omega)\). Then, there exists at most one solution of (BVP.1) in \(H_0^1 \cap L^\infty(\Omega)\).

**Theorem 3.B ([5, Theorem 1]).** Let \( f \in C^1([0, R] \times [0, \infty)) \) satisfying
- \( u f_u(t, u) > f(t, u) \text{ for } (t, u) \in [0, R] \times [0, \infty) \), that is, \( f(t, u)/u \) is strictly increasing in \( u \in [0, \infty) \) for each fixed \( t \in [0, R] \);
- \( f_t(u, u) \leq 0 \text{ for } (t, u) \in [0, R] \times [0, \infty) \);
- there exists \( u > 0 \) such that \( f(R, u) \geq 0 \).

Then
\[
\begin{aligned}
u''(t) + f(|t|, u(t)) &= 0, \quad -R < t < R, \\
u(-R) = u(R) &= 0
\end{aligned}
\]

has at most one positive solution in \(C^2[-R, R]\).

**Theorem 3.C ([7, Theorem 4.1]).** Let \( k \in \{1, 2, 3, \ldots\} \), \( p \in (0, 1) \) and \( h \in C((0, 1); [0, \infty)) \) such that
\[
0 < \int_0^1 (1 - t)^{-p} h(t) dt < \infty.
\]

Then
\[
u''(t) + \frac{k}{t} u'(t) + h(t) u^{-p}(t) = 0 \quad \text{in } (0, 1)
\]

has at least one positive solution satisfying (BC.2) in \(C^1[0, 1] \cap C^2(0, 1)\).

**Theorem 3.D ([14, Theorem 2]).** Assume that \( q(t) > 0 \text{ in } (0, 1) \) and \( f(u)/u \) is decreasing in \((0, \infty)\) and not constant in any neighborhood of \( u = 0 \). Then
\[
u''(t) + q(t) f(u(t)) = 0 \quad \text{in } (0, 1)
\]

has a unique positive solution satisfying (BC.2) in \(C^1[0, 1] \cap C^2(0, 1)\).

**Theorem 3.E ([17, Theorems 1–3]).** Let
- \( (B_1) \ p \in C(\theta_1, \theta_2) \text{ and } p(t) > 0 \text{ in } (\theta_1, \theta_2); \)
- \( (B_2) \ f \in C(0, \infty), f(u) > 0 \text{ in } (0, \infty) \) and \( f(u)/u \) is decreasing in \( u \in (0, \infty) \);
- \( (B_3) \text{ for any } \lambda > 0 \text{ (resp. } \lambda < 0), \text{ a solution } u \text{ of }\)
\[
\begin{aligned}
|u'|^{m-2} u'' + p(t) f(u) &= 0, \quad \theta_1 < t < \theta_2, \ m \geq 2,
\end{aligned}
\]
satisfying $u(\theta_1) = 0$ and $u'(\theta_1) = \lambda$ (resp. $u(\theta_2) = 0$ and $u'(\theta_2) = \lambda$) is determined uniquely as long as $u'(t) > 0$ (resp. $u'(t) < 0$).

Then, (E5) has at most one positive solution in $C^1[\theta_1, \theta_2]$ satisfying (BC.1)–(BC.3).

Remark 3.F. Comparing our uniqueness theorems with the above-mentioned results, we have the following remarks:

(I) The assumption “$u \rightarrow f(x, u)$ is continuous on $[0, \infty)$” in Theorem 3.A implies $f(x, u) \neq u^p$ for $p < 0$, and “$f(u)/u$ is decreasing in $(0, \infty)$ and is not a constant in any neighborhood of $u = 0$” in Theorem 3.D is imposed to exclude the situation in which $f(u)$ behaves like a linear function in a neighborhood of $u = 0$, that is, $f(u)/u$ behaves like a strictly decreasing function near $u = 0$.

(II) It is clear that if $f(t, u)$ is (strictly) decreasing in $u \in (0, \infty)$ and

$$f(t, u) \equiv h(t)u^{-p}, h(t)u^q, u^\alpha + u^{-\alpha}, \sin(t)u^{-p} + \cos(t)u^q$$

for any given $p \in [0, \infty)$, $q \in [0, m - 1]$, $\alpha \in [0, m - 1]$ and $h \in C((0, 1); [0, \infty))$, then $f(t, u)$ satisfies “$f(t, u)/u^m$ is strictly decreasing in $u$”. But Theorems 3.A, 3.C, 3.D and 3.E cannot be applied to most of these functions, for example,

$$f(t, u) \equiv u^{1/2}, u^{-1} + u \text{ and } tu^{1/4} + e^tu^{1/2}.$$  

Furthermore, Theorem 3.C does not tell us “the uniqueness of positive solution of (E5) with (BC.2)".


(IV) The excellent uniqueness Theorem 3.B of Dalmasso [5] combines with the main results in this article. They can criticize almost all the uniqueness of positive solutions of (E)–(BC.i) $(i = 1, 2, 3)$ and (BVP).

Example 3.G. (I) It follows from Theorem 2.2 (resp. Theorems 2.3, 2.4, 2.5) that the boundary value problem

(BVP.3)

$$\begin{cases}
|u'|^{m-2}u'' + 2[t(1-t)]^2u^{-p} + \frac{\sin(t)|u'|^{2-2}}{1+\tau^2} = 0 \text{ in } (0, 1), \\
u(0) = u'(1) = 0 \\
(\text{resp. } u'(0) = u(1) = 0, u(0) = u(1) = 0, u(0) = u(1) + 2u'(1) = 0)
\end{cases}$$

has at most one positive solution in $C^1[0, 1]$, where $p \in (0, \infty)$.

(II) It follows from Theorem 2.3 (resp. Theorems 2.2, 2.4, 2.5) that the boundary value problem

(BVP.4)

$$\begin{cases}
|u'|^{m-2}u'' + \frac{1}{1+\tau^2}u^p = 0 \text{ in } (0, 1), \quad p \in (-\infty, m - 1), \\
u'(0) = u(1) = 0 \\
(\text{resp. } u(0) = u'(1) = 0, u(0) = u(1) = 0, u(0) = u(1) + 2u'(1) = 0)
\end{cases}$$

has at most one positive solution in $C^1[0, 1]$.  

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(III) It follows from Theorem 2.4 (resp. Theorems 2.2, 2.3, 2.5) that the boundary value problem
\[
\begin{align*}
\left\{ \begin{array}{l}
\left| u' \right|^{m-2} u' + \frac{1}{\Gamma(\alpha)} (u^\alpha + u^{-\alpha}) + \cos(t) \left| u' \right|^{-\beta} = 0 & \text{in } (0, 1), \\
u(0) = u(1) = 0
\end{array} \right.
\end{align*}
\]
(BVP.5)
has at most one positive solution in \(C^1[0,1]\), where \(\alpha \in [0, m - 1], \beta > 0\).

References


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