

## OSCILLATION OF ANALYTIC CURVES

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ABSTRACT. The number of zeroes of the restriction of a given polynomial to the trajectory of a polynomial vector field in  $(\mathbb{C}^n, 0)$ , in a neighborhood of the origin, is bounded in terms of the degrees of the polynomials involved. In fact, we bound the number of zeroes, in a neighborhood of the origin, of the restriction to the given analytic curve in  $(\mathbb{C}^n, 0)$  of an analytic function, linearly depending on parameters, through the stabilization time of the sequence of zero subspaces of Taylor coefficients of the composed series (which are linear forms in the parameters). Then a recent result of Gabrielov on multiplicities of the restrictions of polynomials to the trajectories of polynomial vector fields is used to bound the above stabilization moment.

### INTRODUCTION

Let  $\psi : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^n, 0)$  be an analytic curve in  $\mathbb{C}^n$ . We assume that an analytic mapping  $\psi = (\psi_1, \dots, \psi_n)$  is regular in a neighborhood of  $0 \in \mathbb{C}$ . It is known that for any compact analytic family of analytic functions  $Q_\lambda : \mathbb{C}^n \rightarrow \mathbb{C}$ , the number of isolated zeroes of  $Q_\lambda$  on  $\psi$  near the origin is uniformly bounded in  $\lambda$  (see [5]). In particular, this is true for  $Q_\lambda$ —all the polynomials of degree  $d$ .

However, in many situations it is important to have an effectively computable bound on the number of zeroes. Such a bound cannot be produced by the approach of [5]. Some additional approaches to this problem have been developed: Khovan-ski's theory of Pfaffian functions ([10]) allows one to effectively bound the number of real zeroes for solutions of some special type of differential equations. Recently, serious progress has been achieved in extending these bounds to non-Pfaffian situations and to complex zeroes ([6], [7]).

Another approach, based on Bernstein-type inequalities and their relation to the number of zeroes, has been proposed in [13], and developed for linear differential equations in [9]. However, even for  $\psi(t)$  the trajectory of a polynomial vector field and  $Q_\lambda$ -polynomials of degree  $d$ , a serious gap remains. The results of [6] (see also [11]) bound effectively only the multiplicities of zeroes of  $Q_\lambda/\psi$ , but not their number. On the other hand, the results of [9] cannot be extended directly from one differential equation to a system.

Let us stress that the main difficulty in the problem above concerns the situation where  $Q_\lambda$  is a small perturbation of  $Q_{\lambda_0}$ , for which  $Q_{\lambda_0}/\psi \equiv 0$ . The number of

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zeroes (with their multiplicities) which  $Q_\lambda/\psi$  can have near the origin for  $\lambda$  close to  $\lambda_0$  we call “cyclicity”, following [12].

In the present paper we propose a new approach for bounding the number of zeroes of  $Q_\lambda/\psi$ . It is based on the study of the algebraic properties of the Taylor coefficients of the function  $f_\lambda(t) = Q_\lambda(\psi(t))$  (considered as polynomials in the parameters  $\lambda$  of the problem). This approach, pioneered by Bautin in [1], and developed further in [2], [3] and others, has been recently extended to an effective bounding of complex zeroes of analytic families in [4].

The following results were obtained:

1. Gabrielov’s bound on the multiplicity of zeroes of the restriction of a polynomial to the trajectory of a polynomial vector field ([6]) is extended to the corresponding “cyclicities”.
2. The results of [9] are partially extended to solutions of linear systems of equations (with the bounds of [9] replaced by Gabrielov’s bounds).

The paper is organized as follows: In the first section we prove a very general result on a restriction of a linear family of functions to an analytic curve. The bounds are not effective, but the construction is transparent, and it isolates the possible sources of ineffectiveness. In the second section we combine the construction of the first one with the bounds of [6] on multiplicities, to obtain our main results. Finally, in section 3 we give a very general algebraic condition, under which the bound on multiplicities implies the same bound for cyclicities.

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## 1. CYCLICITY OF A LINEAR FAMILY ON AN ANALYTIC CURVE

As above, let  $\psi : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^n, 0)$  be an analytic curve, and let

$$(1) \quad \psi(t) = \sum_{k=1}^{\infty} a_k t^k, \quad a_k \in \mathbb{C}^n,$$

be the Taylor expansion of  $\psi$  at  $0 \in \mathbb{C}$ . We assume that the series (1) converges in a disk  $D_R \subseteq \mathbb{C}$ , and moreover, that  $|a_k| \leq K(1/R)^k$  for some  $K > 0$ .

For  $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{C}^m$ , let

$$Q_\lambda = \sum_{i=1}^m \lambda_i Q_i,$$

with  $Q_j : (\mathbb{C}^n, 0) \rightarrow \mathbb{C}$  analytic functions,  $j = 1, \dots, m$ . We assume that  $Q_j$  are analytic in the polydisk  $\Delta_\rho$  of a radius  $\rho > 0$  at  $0 \in \mathbb{C}^n$ , and are bounded in  $\Delta_\rho$  by  $C > 0$ .

Let

$$f_\lambda(t) = Q_\lambda(\psi(t)) = \sum_{i=1}^m \lambda_i Q_i(\psi(t)).$$

We have

$$Q_i(\psi(t)) = \sum_{k=0}^{\infty} b_k^i t^k,$$

and hence

$$(2) \quad f_\lambda(t) = \sum_{k=0}^{\infty} v_k(\lambda)t^k,$$

where

$$v_k(\lambda) = \sum_{i=1}^m \lambda_i b_k^i.$$

Thus the coefficients  $v_k(\lambda)$  of the series (2) are linear functions in  $\lambda$ . By the standard estimates, for any  $i = 1, \dots, m$  and any  $k$ ,

$$(3) \quad |b_k^i| \leq K_1 \left( \frac{1}{R_1} \right)^k,$$

where an explicit expression for  $K_1$  and  $R_1$ , through the constants  $K, R, \rho$  and  $C$ , defined above, can be easily written down. Hence, for any  $\lambda \in \mathbb{C}^m$ ,

$$|v_k(\lambda)| \leq mK_1 \left( \frac{1}{R_1} \right)^k |\lambda|,$$

where

$$(4) \quad |\lambda| = \max\{|\lambda_1|, \dots, |\lambda_m|\}.$$

In order to state the main theorem of this section, we have to define an integer  $d$  and a constant  $c > 0$ , associated to the sequence  $v_0(\lambda), \dots, v_q(\lambda), \dots$  of linear functions in  $\lambda \in \mathbb{C}^m$ . Let  $L_i \subseteq \mathbb{C}^m$  be a linear subspace of  $\mathbb{C}^m$ , defined by the equations  $v_0(\lambda) = 0, \dots, v_i(\lambda) = 0$ . We have

$$L_0 \supseteq L_1 \supseteq \dots \supseteq L_i \supseteq \dots$$

Hence on a certain step  $d$  this sequence stabilizes:  $L_d = L_{d+1} = \dots = L$ . We call this number  $d$  the Bautin index of the sequence  $v_0(\lambda), \dots, v_i(\lambda), \dots$  or of the series (2).

Notice that in general one cannot explicitly find the Bautin index of a given series, since the moments, when the dimension of  $L_i$  drops, are usually difficult to determine.

Since  $L$  is defined by  $v_0(\lambda) = \dots = v_d(0) = 0$ , any linear function  $l(\lambda)$ , which vanishes on  $L$ , can be expressed as a linear combination of  $v_0, \dots, v_d$ . Moreover, there exists a constant  $\tilde{c} > 0$ , depending on  $v_i(\lambda)$ , such that for any  $l(\lambda), l/L \equiv 0$ ,

$$l(\lambda) = \sum_{j=0}^d \mu_j v_j(\lambda), \quad \mu_j \in \mathbb{C},$$

with  $|\mu_j| \leq \tilde{c} \|l\|, j = 0, \dots, d$ . (Here for  $l(\lambda) = \sum_{i=1}^m \alpha_i \lambda_i, \|l\| = \max_i |\alpha_i|$ .)

We define  $c > 0$  as the minimum of  $\tilde{c}$  above. Also, an effective estimation of  $c$  is difficult, in general. However, if the Bautin index  $d$  is known,  $c$  can be found by a finite computation.

Now let a series (2) be given, satisfying (3), and let the Bautin index  $d$  and the constant  $c$  of this series be defined as above. Notice that by (4), for any  $\lambda \in \mathbb{C}^m$ ,

the series (2) converges in a disk  $D_{R_1}$ . Let

$$\begin{aligned} R' &= \frac{1}{4}R_1, \\ R'' &= \frac{1}{2 \max(\alpha, 2)}R_1, \\ R''' &= \left( \frac{1}{e^{10d+2} \cdot \max(\alpha, 2)} \right) R_1, \end{aligned}$$

where

$$\alpha = cK_1 \left[ \max \left( \frac{1}{R_1}, 1 \right) \right]^d \cdot (d + 1).$$

**Theorem 1.1.** *For any  $\lambda \in \mathbb{C}^m$ , if  $f_\lambda(t) \not\equiv 0$ , then this function can have at most  $d \log_{5/4}(4 + 2\alpha)$  zeroes in the disk  $D_{R'}$ , at most  $20d$  zeroes in  $D_{R''}$ , and at most  $d$  zeroes in  $D_{R'''}$  (zeroes are counted with multiplicities).*

*Proof.* We will use some notions and results from [12]. An analytic function  $f(t) = \sum_{i=0}^\infty a_i t^i$  is said to belong to the Bernstein class  $B_{N,R,\alpha}^2$ , with  $N$  an integer,  $R > 0$ ,  $\alpha > 0$ , if for any  $j > N$ ,

$$|a_j|R^j \leq \alpha \max_{i=0,\dots,N} \{ |a_i|R^i \}.$$

The following result is proved in [12]: Let  $f(t) \in B_{N,R,\alpha}^2$ , and let

$$\begin{aligned} R' &= \frac{1}{4}R, \\ R'' &= \frac{1}{2 \max(\alpha, 2)}R, \\ R''' &= \left( \frac{1}{e^{10N+2} \cdot \max(\alpha, 2)} \right) R. \end{aligned}$$

Then the number of zeroes of  $f$  in  $D_{R'}$  does not exceed  $N \log_{5/4}(4 + 2\alpha)$ , in  $D_{R''}$  does not exceed  $20N$ , and in  $D_{R'''}$  does not exceed  $N$ .

Hence to prove Theorem 1.1 it remains to show that  $f_\lambda(t) \in B_{d,R_1,\alpha}^2$ , with  $\alpha = cK_1[\max(1/R_1, 1)]^d$  for any  $\lambda \in \mathbb{C}^m$ . Since each of the coefficients  $v_j(\lambda)$ ,  $j \geq d$ , vanishes on  $L$ , we have

$$v_j(\lambda) = \sum_{i=0}^d \mu_i^j v_i(\lambda),$$

with  $|\mu_i^j| \leq c\|v_j\| \leq cK_1(1/R_1)^j$  by (3) above. Hence, for any  $\lambda \in \mathbb{C}^m$ ,

$$|v_j(\lambda)| \leq cK_1 \left( \frac{1}{R_1} \right)^j \cdot (d + 1) \cdot \max_{i=0,\dots,d} |v_i(\lambda)|,$$

or

$$|v_j(\lambda)|R_1^j \leq cK_1(d + 1) \cdot \max_{i=0,\dots,d} |v_i(\lambda)|,$$

or

$$|v_j(\lambda)|R_1^j \leq \alpha \max_{i=0,\dots,d} \{ |v_i(\lambda)|R_1^i \},$$

where  $\alpha = cK_1[\max(1/R_1, 1)]^d(d + 1)$ . This shows that  $f_\lambda(t) \in B_{d,R_1,\alpha}^2$ , and hence completes the proof of Theorem 1.1.

*Remark.* The arguments mentioned above, concerning a stabilization of the sequence  $L_i$  and a representation of  $v_j$  as a linear combination of  $v_0, \dots, v_d$ , in our specific case of linear  $v_j(\lambda)$ , replace the Hilbert finiteness theorem, Nullstellensatz and the Hironaka division algorithm, respectively, which are necessary in a general case of analytic or polynomial coefficients  $v_j(\lambda)$ . (Compare [4] and section 3 below.)

**Corollary 1.2.** *Let  $\psi(t)$  be an analytic curve in  $\mathbb{C}^n$ ,  $Q_\lambda : (\mathbb{C}^n, 0) \rightarrow \mathbb{C}$  an analytic family, linear in  $\lambda$ , and let the Bautin index  $d$  of the series  $f_\lambda(t) = Q_\lambda(\psi(t))$  be defined as above. Then the cyclicity of  $Q_\lambda$  on  $\psi(t)$ , i.e. the maximal number of zeroes that  $Q_\lambda$  can have in a sufficiently small neighborhood of the origin on the curve  $\psi$ , does not exceed  $d$ .*

*Remark.* Corollary 1.2 shows that a local cyclicity of  $Q_\lambda$  on  $\psi$  is bounded by the Bautin index  $d$ , which plays the role of the “degree” of this curve with respect to the family  $Q_\lambda$ .

Representing  $\psi(t)$  as a series  $\psi(t) = \sum_{k=1}^\infty a_k t^k$ ,  $a_k \in \mathbb{C}^n$ , and choosing properly coefficients  $a_k$ , one can obtain (even for the linear family  $Q_\lambda$  of linear functions) any prescribed value of  $d$ , and for a given  $d$  any prescribed value of the constant  $c$ . Hence to get more explicit bounds, one has to use specific properties of  $\psi$  and  $Q$ . We do this in the next section.

Now we show that the maximum of multiplicities of  $Q_{\lambda/\psi}$  at zero bounds also the Bautin index.

**Theorem 1.3.** *Let  $\psi : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^n, 0)$  be an analytic curve, and let  $Q_\lambda : (\mathbb{C}^n, 0) \rightarrow \mathbb{C}$  be an analytic family linear in  $\lambda \in \mathbb{C}^m$ . If for any  $\lambda \in \mathbb{C}^m$  either  $f_\lambda(t) = Q_\lambda(\psi(t)) \equiv 0$ , or the multiplicity of zero of  $f_\lambda(t)$  at the origin does not exceed  $\mu$ , then the Bautin index  $d$  of  $f_\lambda$  does not exceed  $\mu$ .*

*Proof.* Write, as above,

$$f_\lambda(t) = \sum_{k=0}^\infty v_k(\lambda)t^k.$$

Now the assumption on multiplicities can be restated as follows: if  $v_0(\lambda) = v_1(\lambda) = \dots = v_\mu(\lambda) = 0$ , then  $f_\lambda(t) \equiv 0$ , i.e. the rest of  $v_j(\lambda)$  also vanishes. Thus, each  $v_j$ ,  $j > \mu$ , vanishes on  $L_\mu$ , and hence the sequence  $L_0 \supseteq L_1 \supseteq \dots$  stabilizes not later than on  $L_\mu$ .

**Corollary 1.4.** *Maximal cyclicity of  $Q_\lambda/\psi$  at zero does not exceed maximal multiplicity.*

Yet another reformulation of the same property is as follows:

**Corollary 1.5.** *If  $N$  zeroes of  $Q_\lambda/\psi$  can bifurcate from the origin, then for some values of  $\lambda$  the multiplicity of zero of  $Q_\lambda/\psi$  at the origin is finite and at least  $N$ .*

Although this result is implied by the previous ones, the following explanation can clarify the situation: Assume that  $N$  zeroes can bifurcate from the origin for  $f_\lambda(t)$ , as  $\lambda$  changes. Then, according to the result of [12] stated in the proof of Theorem 1, the following inequality cannot be true for all  $\lambda$  and  $j \geq N$ :

$$|v_j(\lambda)|R_1^j \leq \alpha \max_{i=0, \dots, N-1} |v_i(\lambda)|R_1^i.$$

Hence one of the linear functions  $v_j(\lambda)$ ,  $j \geq N$ , does not vanish identically on  $L_{N-1} = \{v_0(\lambda) = \dots = v_{N-1}(\lambda) = 0\}$ , and therefore we can find  $\lambda_0 \in L_{N-1}$ , such that some  $v_j(\lambda_0) \neq 0$ ,  $j \geq N$ . Then the multiplicity of zero of  $f_{\lambda_0}(t)$  at  $0 \in \mathbb{C}$  is finite and at least  $N$ .

2. CYCLICITY OF POLYNOMIALS ON TRAJECTORIES OF POLYNOMIAL VECTOR FIELDS

Consider a system of polynomial differential equations

$$(5) \quad \dot{x} = P(x), \quad x \in \mathbb{C}^n,$$

where  $P(x)$  is a polynomial vector-field of degree  $p$  in  $x = (x_1, \dots, x_n) \in \mathbb{C}^n$ . We write  $P(x)$  as  $\sum_{|\alpha| \leq p} \gamma_\alpha x^\alpha$ , where  $\alpha \in \mathbb{N}^n$  is a multi-index, and  $\gamma_\alpha \in \mathbb{C}^n$ .

Let  $Q_\lambda(x)$  be a polynomial of degree  $q$ ,  $Q_\lambda(x) = \sum_{|\beta| \leq q} \lambda_\beta x^\beta$ ,  $\lambda_\beta \in \mathbb{C}$ . We assume that  $P(0) \neq 0$ . Then there exists a regular trajectory  $x = \psi(t)$  of the system (5), with  $\psi(0) = 0$ .

The following result is obtained in [6]: the multiplicity of  $Q_\lambda/\psi$  does not exceed  $[pq(p+q)]^{2^{n-2}}$ , if  $Q_\lambda$  does not vanish identically on  $\psi$ . As an immediate consequence we obtain

**Corollary 2.1.** *The cyclicity of the restriction  $Q_\lambda/\psi$  at the origin does not exceed  $[pq(p+q)]^{2^{n-2}}$ .*

Thus Gabrielov’s bound on multiplicities produces an explicit bound for the Bautin index, and hence for the cyclicity of the restriction of polynomials to the trajectories of polynomial vector fields. Since the constants  $R_1, K_1, \rho$  can be easily estimated in this special case, Theorem 1.1 would produce a completely effective bound on the number of zeroes of  $f_\lambda(t) = Q_\lambda(\psi(t))$  on any disk, if we could compute explicitly the constant  $c$ , associated with the series  $f_\lambda(t)$ , as in section 1. However, this computation seems to be difficult, so in this paper we restrict ourselves to cyclicity bounds, which require knowledge of the Bautin index  $d$  only. (Notice that in one important situation, where the coefficients of the series involved are integers, knowing  $d$  implies an effective bound on  $c$ . I would like to thank S. Yakovenko for this remark).

Let us conclude this section with a simple example. Consider a diagonal linear system

$$(6) \quad \dot{x} = Ax, \quad x \in \mathbb{C}^n,$$

with  $A = \text{diag}(\mu_1, \dots, \mu_n)$ . Consider a trajectory  $\psi$  of (6), starting at  $(1, \dots, 1)$ , and a restriction of a linear polynomial  $Q_\lambda(x) = \lambda_1 x_1 + \dots + \lambda_n x_n$  on  $\psi$ . We get  $f_\lambda(t) = \sum_{k=0}^\infty v_k(\lambda)t^k$ , with

$$\begin{aligned} v_0(\lambda) &= \lambda_1 + \dots + \lambda_n, \\ v_1(\lambda) &= \lambda_1 \mu_1 + \dots + \lambda_n \mu_n, \\ v_2(\lambda) &= \frac{1}{2!}(\lambda_1 \mu_1^2 + \dots + \lambda_n \mu_n^2), \\ &\vdots \\ v_i(\lambda) &= \frac{1}{i!}(\lambda_1 \mu_1^i + \dots + \lambda_n \mu_n^i). \end{aligned}$$

Assuming that all the  $\mu_j$  are different from one another, we get that the system  $v_0(\lambda) = \dots = v_{n-1}(\lambda) = 0$  is nondegenerate, and hence  $L_{n-1} = L = 0 \in \mathbb{C}^n$ . Hence the Bautin index of  $f_\lambda(t)$  is  $n - 1$ . Of course, also the constant  $c$  can be easily computed in this case.

**Corollary 2.2.** *For a given  $\mu_1, \dots, \mu_n, \mu_i \neq \mu_j$ , there is a neighborhood of the origin in  $\mathbb{C}$ , such that for any  $\lambda \in \mathbb{C}^n$ ,  $f_\lambda(t) = \lambda_1 e^{\mu_1 t} + \dots + \lambda_n e^{\mu_n t}$  has at most  $n - 1$  zeroes in this neighborhood.*

This is a (weaker) version of the result of W. K. Hayman ([8]). Some additional results, concerning exponential polynomials, can be obtained by the above methods. They will appear separately.

### 3. MULTIPLICITY AND CYCLICITY: A GENERAL CASE

Let  $F : (\mathbb{C}^{m+1}, 0) \rightarrow (\mathbb{C}, 0)$  be an analytic function. For  $\lambda \in \mathbb{C}^m$ , consider  $f_\lambda(t) = F(\lambda, t)$ . Assume that for any  $\lambda \in \mathbb{C}^m$  sufficiently small, either  $f_\lambda(t) \equiv 0$ , or the multiplicity of zero of  $f_\lambda(t)$  at  $0 \in \mathbb{C}$  is at most  $N$ . Let

$$f_\lambda(t) = \sum_{k=0}^{\infty} a_k(\lambda)t^k,$$

where  $a_k(\lambda)$  are analytic functions in a neighborhood of zero in  $\mathbb{C}^m$ . Let  $I_N$  be the ideal in the ring of germs of analytic functions at  $0 \in \mathbb{C}^m$ , generated by  $a_0(\lambda), \dots, a_N(\lambda)$ .

**Theorem 3.1.** *Assume that the ideal  $I_N$  is a radical one (i.e. if  $g^s \in I_N$  for some  $s$ , then  $g \in I_N$ ). Then the cyclicity of  $f_\lambda(t)$  at the origin is at most  $N$ .*

*Proof.* An assumption on multiplicity means that, if for some  $\lambda \in \mathbb{C}^m$ ,  $a_0(\lambda) = a_1(\lambda) = \dots = a_N(\lambda) = 0$ , then any  $a_j(\lambda) = 0$ ,  $j > N$ . In other words, each  $a_j(\lambda)$  vanishes on the set of zeroes  $Y_N$  of the ideal  $I_N$ . By Hilbert Nullstellensatz, there is an integer  $s_j$  such that  $a_j^{s_j} \in I_N$ , and since  $I_N$  is a radical ideal,  $a_j \in I_N$ . Therefore the first  $N$  coefficients  $a_0(\lambda), \dots, a_N(\lambda)$  generate the Bautin ideal  $I = \{a_0(\lambda), \dots, a_j(\lambda), \dots\}$ . Now one of the main results of [4] is that in this case the cyclicity of  $f_\lambda(t)$  is at most  $N$ .

*Remark 1.* The conclusion of Theorem 3.1 is trivial, of course, if  $f_0(t) \not\equiv 0$ .

*Remark 2.* The results of sections 1 and 2 above follow from Theorem 3.1, since in the case of  $a_j(\lambda)$  linear in  $\lambda$ , any ideal  $I_N$  is radical. However, a proof in this special case, given above, is much more transparent than the general proof given in [4].

The following example, suggested by A. Gabrielov, shows that the assumption of radicality of  $I_N$  cannot be omitted: consider the family  $f_\lambda(t) = \lambda(t^{10} - \lambda)$ ,  $\lambda \in \mathbb{C}$ . Here  $f_0(t) \equiv 0$ , and for  $\lambda \neq 0$  ten zeroes bifurcate from the origin. Hence the cyclicity is 10. On the other hand, the maximal (finite) multiplicity of the zero of  $f_\lambda$  at  $0 \in \mathbb{C}$  is zero.

In this example  $a_0(\lambda) = -\lambda^2$ ,  $a_{10} = \lambda$ , and the rest of the  $a_j$  are zeroes. The ideal  $I_0 = \{\lambda^2\}$  is not radical:  $a_{10}^2 \in I_0$ , but  $a_{10}$  does not belong to this ideal.

## REFERENCES

1. N. N. Bautin, *On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type*, Transl. AMS, Series I, **5** (1962), 396–413. MR **15**:527h
2. C. Chicone and M. Jacobs, *Bifurcations of critical periods for plane vector fields*, Trans. Amer. Math. Soc. **312**, N2 (1989), 433–486. MR **89h**:58139
3. J.-P. Francoise and C. C. Pugh, *Keeping track of limit cycles*, J. Diff. Equations **65** (1986), 139–157. MR **88a**:58162
4. J.-P. Francoise and Y. Yomdin, *Bernstein inequality and applications to analytic geometry and differential equations*, to appear.
5. A. Gabrielov, *Projections of semi-analytic sets*, Funct. Anal. Appl. **2** (4) (1968), 282–291. MR **39**:7137
6. A. Gabrielov, *Multiplicities of zeroes of polynomials on trajectories of polynomial vector fields and bounds on degree of nonholonomy*, Math. Res. Lett. **2** (1995), 437–451. MR **97c**:14055
7. A. Gabrielov, *Multiplicities of Pfaffian intersections and the Lojasiewicz inequality*, Selecta Mathematica, New Series **1** (1) (1995), 113–127. MR **96d**:32007
8. W. K. Hayman, *Differential inequalities and local valency*, Pacific J. of Math. **44** (1) (1973), 117–137. MR **47**:5240
9. Yu. Il'yashenko and S. Yakovenko, *Counting real zeroes of analytic functions, satisfying linear ordinary differential equations*, J. Diff. Equations **126** (1) (1996), 87–105. MR **97a**:34010
10. A. G. Khovanski, *Fewnomials*, AMS Publ., Providence, RI, 1991. MR **92h**:14039
11. Yu. V. Nesterenko, *Estimates for the number of zeroes of certain functions*, in Transcendence Theory, Alan Baker Ed., Cambridge Univ. Press (1988). MR **90d**:11085
12. R. Roussarie, *A note on finite cyclicity and Hilbert's 16th problem*, LNM **1331**, Springer, New York and Berlin (1988), 161–168. MR **90b**:58227
13. N. Roytvarf and Y. Yomdin, *Bernstein classes*, to appear.

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