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NOTE ON COMPACT SETS OF COMPACT OPERATORS ON A REFLEXIVE AND SEPARABLE BANACH SPACE

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Abstract. We give a criterion for a subset of the space of compact linear operators from a separable and reflexive Banach X into a Banach space Y to be compact.

1. Introduction

Let us consider (real or complex) Banach spaces X, Y. Then, with its usual norm, the space $\mathcal{K}(X,Y)$ of all compact linear operators $T : X \to Y$ is also a Banach space. K. Vala has discussed [3] the question of compactness for $K \subset \mathcal{K}(X,Y)$ in a general context. In this note, assuming X is reflexive and separable, we give another type of criterion for $K \subset \mathcal{K}(X,Y)$ to be compact.

Throughout this work, $\|\cdot\|$ will denote both the norm on the Banach space X and on the Banach space Y, $B_X = \{x \in X : \|x\| \leq 1\}$ and, for a sequence $\{x_n\} \subset X, x \in X$, $x_n \xrightarrow{w} x$ will indicate $\{x_n\}$ converges weakly to x. A linear operator $T : X \to Y$ is compact if, for each bounded sequence $\{x_n\} \subset X$, there is a converging subsequence $\{Tx_{n(k)}\} \subset Y$. As usual, if M is a metric space, then $A \subset M$ is said to be relatively compact if its closure is a compact set.

Our main tool will be the following version of Arzela-Ascoli's criterion for compactness [1, p. 137].

Theorem. *Suppose F is a Banach space and E is a compact metric space. In order that a subset H of the Banach space $C(E,F)$ of continuous functions of E into E be relatively compact, necessary and sufficient conditions are that H be equicontinuous and that for each $x \in E$, the set $H(x)$ of all $f(x)$ such that $f \in H$ be totally bounded in F.*

2. The criterion

Motivated by Arzela-Ascoli's criterion, we introduce the next definitions. Let $K \subset \mathcal{K}(X,Y)$. (a) We say K is *pointwise relatively compact* if, for each $x \in B_X$, the set $\{Tx : T \in K\} \subset X$ is relatively compact. (b) We say K is *uniformly w-continuous*, if given $\epsilon > 0$ and a sequence $\{x_n\} \subset X$ converging weakly to 0, there is an index N such that $\|Tx_n\| \leq \epsilon,\ N \leq n,\ T \in K$.

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Theorem 1. *Let X be a reflexive and separable Banach space. Then, $K \subset \mathcal{K}(X, Y)$ is relatively compact if, and only if, K is pointwise relatively compact and uniformly w-continuous.*

Proof. Since X is a reflexive and separable Banach space, it is well known that $B \equiv B_X$ is compact and metrizable (for some metric d) with respect to the weak topology. Given $T \in K(X, Y)$, let us define $\phi(T) = T_B$, where T_B is the restriction of T to B. Now T_B is in $C(B, Y)$, for if $x_n \xrightarrow{w} x$, then $Tx_n \to Tx$ since T is compact [2, p. 107]. Hence we have $\phi : K(X, Y) \to C(B, Y)$ and moreover ϕ is a linear isometry.

Suppose first that K is relatively compact. Then by Arzela-Ascoli's theorem, it follows that K is pointwise relatively compact and equicontinuous. In particular, if $x_n \xrightarrow{w} 0$, then we can find some $N \in \mathbb{N}$ such that $\|Tx_n\| = \|Tx_n - T0\| < \epsilon$.

Now, assuming $K \subset \mathcal{K}(X, Y)$ is pointwise relatively compact and uniformly w-continuous, we will establish that K is relatively compact. For this, by Arzela-Ascoli's criterion, it is only left to show that the family B is equicontinuous. Assume this is not so. Thus, we can find an $\epsilon > 0$ and sequences $\{x_n\}, \{z_n\} \subset B$, $\{T_n\} \subset K(X, Y)$, satisfying

$$d(x_n, z_n) \leq \frac{1}{n}, \quad \epsilon \leq \|T_n x_n - T_n z_n\|. \tag{1}$$

By the compactness of B, we will suppose both sequences $\{x_n\}$ and $\{z_n\}$ are convergent. It follows from above that they must converge to the same limit. Hence, $\{x_n - z_n\}$ converges weakly to 0. Now, since K is uniformly w-continuous, there is some index N such that $\|T(x_n - z_n)\| \leq \epsilon$, $N \leq n$. This contradicts (1). $\square$

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