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SMORODINSKY'S CONJECTURE ON RANK-ONE MIXING

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Abstract. We prove Smorodinsky's conjecture: the rank-one transformation, obtained by adding staircases whose heights increase consecutively by one, is mixing.

0. Introduction

The first rank-one mixing transformation was constructed by Ornstein [O] using "random" spacers on each column. We refer to [F1] for a description of rank-one constructions. Recently, the first rank-one mixing transformation was constructed with an explicit formula for adding spacers [AF]. In [AF] a method for adding staircases was given which produced mixing. However, Smorodinsky's conjecture remained open. M. Smorodinsky conjectured that by adding staircases whose heights increase consecutively by one, the resulting transformation (classical staircase construction) is mixing.

In this paper, we will prove that an infinite staircase construction, whose sequence r_n of cuts and h_n of heights satisfy the condition $\lim_{n\to\infty} \frac{r_n^2}{h_n} = 0$, is mixing. Thus Smorodinsky's conjecture follows as a corollary.

1. Staircase constructions

A rank-one transformation T is called a **staircase construction** if there exists a sequence $(r_n)_{n=1}^{\infty}$ of natural numbers such that each column C_{n+1} is obtained by cutting C_n into r_n subcolumns of equal width, placing $i-1$ spacers on the i^{th} subcolumn for $1 \leq i \leq r_n$, and then stacking the $(i+1)^{st}$ subcolumn on top of the i^{th} subcolumn for $1 \leq i \leq r_n$. Denote $T = T_{(r_n)}$. Let h_n be the height of column C_n for $n \geq 1$. From now on, we assume the sequence (h_n) is derived from the sequence (r_n) in this manner. If the sequence r_n is bounded then T is called a **finite staircase construction**. The **classical staircase construction** is given by $r_n = n$. If $r_n \to \infty$, we call T an **infinite staircase construction**. In this case T may not be finite measure preserving since we may be adding measure too quickly. Assume T is finite measure preserving. The following question remains open:

Question. *Is every infinite staircase construction mixing?*

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Recently, much has been proved about rank-one mixing transformations. They are mixing of order 3 [Ka], and by the same argument, mixing of all orders. King [Ki] proved they have minimal self-joinings of all orders. In [R], Ryzhikov proved finite rank mixing transformations are mixing of all orders. In [Kl], it is proved that the classical staircase construction has singular spectrum, and in [KR] it is proved that any rank-one transformation satisfying $\sum_{n=1}^{\infty}(1/r_n^2) = \infty$ has singular spectrum, where r_n is still the number of cuts at the n^{th} stage.

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2. Mixing staircase constructions

Staircase constructions are known to be weakly mixing (see [F2] for infinite staircase constructions and [AF] for finite staircase constructions), hence totally ergodic. This means each power T^j is ergodic. This is a necessary fact and is used repeatedly. Now we formulate von Neumann's mean ergodic theorem to suit our needs. Note

$$\begin{aligned}
|\frac{1}{N}\sum_{i=0}^{N-1}\mu(T^{-i}A\cap B)-\mu(A)\mu(B)| &= |\frac{1}{N}\sum_{i=0}^{N-1}\mu(A\cap T^{i}B)-\mu(A)\mu(B)| \\
&\le \int_A |\frac{1}{N}\sum_{i=0}^{N-1}\mathcal{X}_B(T^{-i}x)-\mu(B)|d\mu \\
&\le ||\frac{1}{N}\sum_{i=0}^{N-1}\mathcal{X}_B(T^{-i}x)-\mu(B)||_1.
\end{aligned}$$

Hence T ergodic implies

$$|\frac{1}{N}\sum_{i=0}^{N-1}\mu(T^{-i}A\cap B)-\mu(A)\mu(B)| \to 0$$

uniformly over sets A, as $N \to \infty$. This is used in Lemma 2.2 and our main theorem.

Lemma 2.1 is a basic fact about measure preserving transformations which is utilized in Theorem 3.1. We leave the proof to the reader.

In the argument of Lemma 2.2 we establish that ρ_n satisfying $h_n \le \rho_n \le 2h_n$ is a mixing sequence. This is used to prove convergence in mean of the transformations T^{ρ_n} (uniformly over n).

Lemma 2.1. *Let T be a finite measure preserving transformation and let B be any measurable set. For any positive integers R, L and ρ we have*

$$\int |\frac{1}{R}\sum_{i=0}^{R-1}\mathcal{X}_B(T^{-i}x)-\mu(B)|d\mu(x) \le \int |\frac{1}{L}\sum_{i=0}^{L-1}\mathcal{X}_B(T^{-i\rho}x)-\mu(B)|d\mu(x) + \frac{\rho L}{R}.$$

Lemma 2.2. *Let T be an infinite staircase construction and let B be a union of levels in some column. If ℓ_n and ρ_n are sequences of positive integers which converge* *to infinity and such that* $h_n \le \rho_n \le 2h_n$, *then*

$$\lim_{n\to\infty} \int |\frac{1}{\ell_n} \sum_{i=0}^{\ell_n-1} \mathcal{X}_B(T^{-i\rho_n}x) - \mu(B)| d\mu(x) = 0.$$

Proof. Let $\epsilon > 0$. Let j be a positive integer such that $i\rho_n = jh_n + t$ where $0 \le t \le h_n$. Consider the disjoint union $C_n = D_1 \cup D_2$ where D_1 is the union of the top t levels of C_n. Let $B_1 = B \cap D_1$ and $B_2 = B \cap D_2$. Let

$B_1' = B_1 \setminus \{\text{bottom } (j+1)r_n \text{ levels of } D_1\} \setminus \{(j+1) \text{ rightmost subcolumns of } C_n\}.$

We have not thrown away much:

$$\mu(B_1' \triangle B_1) \le \frac{(j+1)r_n}{h_n} + \frac{j+1}{r_n} \to 0 \quad \text{as} \quad n \to \infty.$$

For each level I of C_n let $I' = I \cap B_1'$. Each I' passes completely through the staircase on C_n, $(j+1)$ times, under $T^{i\rho_n}$. Thus $T^{i\rho_n}I'$ intersects $(r_n - j - 1)$ levels of C_n with precisely $\frac{\mu(I)}{r_n}$ measure (on each level of intersection). Moreover the levels for which $T^{i\rho_n}I'$ intersect appear in an arithmetic progression in C_n. Let I^* denote the first (top) of such levels, and let $B_1^* = \bigcup_{I' \subset B_1'} I^*$. Thus

$$\mu(T^{i\rho_n}B_1' \cap B) = \frac{1}{r_n} \sum_{\eta=0}^{r_n-j-1} \mu(T^{-\eta(j+1)}B_1^* \cap B).$$

Similarly we define $B_2' \subset B_2$ and B_2^* so that

$$\mu(T^{i\rho_n}B_2' \cap B) = \frac{1}{r_n} \sum_{\eta=0}^{r_n-j-1} \mu(T^{-\eta j}B_2^* \cap B).$$

Since T^{-j} and $T^{-(j+1)}$ are ergodic we get $\mu(T^{i\rho_n}B \cap B) \to \mu(B)^2$ as $n \to \infty$. Hence for $i_1 \ne i_2$, we have

$$\mu(T^{i_1\rho_n}B \cap T^{i_2\rho_n}B) \to \mu(B)^2 \quad \text{as} \quad n \to \infty.$$

By a technique of Blum-Hansen [BH], there exists a positive integer L such that for sufficiently large n,

$$\int |\frac{1}{L} \sum_{i=0}^{L-1} \mathcal{X}_B(T^{-i\rho_n}x) - \mu(B)| d\mu(x) < \epsilon.$$

Therefore

$$\limsup_{n\to\infty} \int |\frac{1}{\ell_n} \sum_{i=0}^{\ell_n-1} \mathcal{X}_B(T^{-i\rho_n}x) - \mu(B)| d\mu(x) \le \epsilon.$$

□

Before proceeding with the main theorem, we show that any staircase construction satisfies

(H) $$\frac{h_{p-1}^2}{h_p} \to \infty \qquad \text{as} \quad p \to \infty.$$

First we write

$$\frac{h_{p-1}^2}{h_p} = \frac{r_{p-1}h_{p-1}^2}{r_{p-1}h_p} = (\frac{r_{p-1}h_{p-1}}{h_p})(\frac{h_{p-1}}{r_{p-1}}).$$

While the first factor $\frac{r_{p-1}h_{p-1}}{h_p} \to 1$, our second factor $\frac{h_{p-1}}{r_{p-1}} \to \infty$ as $p \to \infty$ since T is finite measure preserving.

Theorem 2.3. *Let r_n be a divergent sequence of positive integers. The staircase construction $T = T_{(r_n)}$ is mixing, if $\lim_{n\to\infty} \frac{r_n^2}{h_n} = 0$.*

Proof. Let m_n be a sequence of positive integers such that $h_n \le m_n < h_{n+1}$. We may choose positive integers k_n and t_n so that

$$m_n = k_n h_n + t_n$$

where $1 \le k_n \le r_n$ and $0 \le t_n < h_n$.

Let A and B be sets which form a union of levels from some column. Since we let $n \to \infty$ we may assume each of A or B is a union of levels in C_n. We need only show that $\mu(T^{m_n}A \cap B) - \mu(A)\mu(B) \to 0$ as $n \to \infty$. We will consider the mixing in three different parts of the column C_n. Partition C_n into the r_n subcolumns of equal width. Let D_1 be the set consisting of the (k_n+1) rightmost subcolumns. Let $D_2 = \{\text{union of the top } t_n \text{ levels of } C_n\} \cap D_1^c$. Finally let D_3 be the remaining part of C_n. Thus D_1, D_2 and D_3 are disjoint, and $C_n = \bigcup_{i=1}^3 D_i$. Denote $A_i = A \cap D_i$ for $i = 1, 2, 3$. Figure 2.4 shows the partition of C_n into the three sets D_1, D_2 and D_3 where $r_n = 18$, $k_n = 4$ and $t_n = 3$.

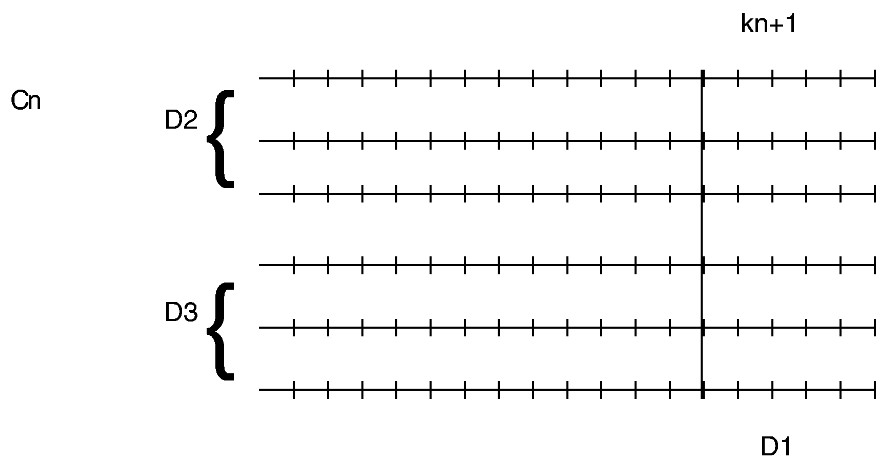

FIGURE 2.4

Mixing on $\mathbf{D_1}$. The set D_1 sits as whole levels in the top $((k_n+1)h_n + r_n^2)$ levels of C_{n+1}. Let $\bar{D}_1 = D_1 - \{\text{bottom } (h_n + r_{n+1}) \text{ levels in } D_1\} - \{\text{rightmost subcolumn of } C_{n+1}\}$. Thus each level of C_{n+1}, which is in $\bar{D}_1$, gets pushed completely through the staircase on C_{n+1} (under T^{m_n}). Let $\bar{A}_1 = A_1 \cap \bar{D}_1$. Then

$$\mu(\bar{A}_1) \ge \mu(A_1) - \frac{h_n + r_{n+1}}{h_{n+1}} - \frac{1}{r_{n+1}} \ge \mu(A_1) - \frac{1}{r_n} - \frac{r_{n+1}}{h_{n+1}} - \frac{1}{r_{n+1}}.$$

Hence $\mu(A_1 \triangle \bar{A}_1) \to 0$ as $n \to \infty$.

Let $I \subset \bar{A}_1$ be a level of C_{n+1}. $T^{m_n}I$ intersects $(r_{n+1} - 1)$ consecutive levels of C_{n+1} with $\frac{\mu(I)}{r_{n+1}}$ measure (on each level in the intersection). Let I^* denote the first(top) of such levels, and let $A^* = \bigcup_{I\subset \bar{A}_1} I^*$. Thus we have

$$|\mu(T^{m_n}\bar{A}_1 \cap B) - \mu(\bar{A}_1)\mu(B)| = |\frac{\mu(I)}{r_{n+1}} \sum_{i=0}^{r_{n+1}-2} \mu(T^{-i}A^* \cap B) - \mu(A^*)\mu(B)|.$$

Since T is ergodic and $r_n \to \infty$, the second expression tends to 0 as $n \to \infty$. Therefore we have that $|\mu(T^{m_n}A_1 \cap B) - \mu(A_1)\mu(B)| \to 0$ as $n \to \infty$.

Mixing on $\mathbf{D_2}$. Let $D^* = \{\text{levels } \mathcal{L} \text{ of } \ C_n \ \mid \mathcal{L} \cap D_2 \neq \emptyset\}$. Let $A' = A_2 - \{\text{bottom } \ r_n^2 \text{ levels of } \ D^*\}$. We have not thrown away much:

$$\mu(A') \geq \mu(A_2) - \frac{r_n^2}{h_n}.$$

For each level I of C_n let $I' = I \cap A'$. Each $I' \subset A'$ passes completely through the staircase on C_n, $(k_n + 1)$ times, under T^{m_n}. Thus $T^{m_n}I'$ intersects $(r_n - k - 1)$ levels of C_n with precisely $\frac{\mu(I)}{r_n}$ measure (on each level of intersection). Moreover the levels for which $T^{m_n}I'$ intersect, appear in an arithmetic progression in C_n. Let I^* denote the first (top) of such levels, and let $A^* = \bigcup_{I'\subset A'} I^*$. (Note that A^* is a union of whole levels in C_n.) Thus $\mu(A') = (\frac{r_n-k_n-1}{r_n})\mu(A^*)$ and

$$\begin{aligned}\mu(T^{m_n}A' \cap B) &= \frac{1}{r_n} \sum_{i=0}^{r_n-k_n-2} \mu(T^{-i(k_n+1)}A^* \cap B) \\ &= \frac{r_n - k_n - 1}{r_n} (\frac{1}{r_n - k_n - 1} \sum_{i=0}^{r_n-k_n-2} \mu(T^{-i(k_n+1)}A^* \cap B)).\end{aligned}$$

Hence

$$\begin{aligned}&\mu(T^{m_n}A' \cap B) - \mu(A')\mu(B) \\ &= \frac{r_n - k_n - 1}{r_n} (\frac{1}{r_n - k_n - 1} \sum_{i=0}^{r_n-k_n-2} \mu(T^{-i(k_n+1)}A^* \cap B) - \mu(A^*)\mu(B)).\end{aligned}$$

The previous expression will converge to zero, if

$$\int |\frac{1}{r_n - k_n - 1} \sum_{i=0}^{r_n-k_n-2} \mathcal{X}_B(T^{-i(k_n+1)}x) - \mu(B)| d\mu(x) \to 0.$$

Since $\mu(A') = (\frac{r_n-k_n-1}{r_n})\mu(A^*)$ we may assume $\frac{k_n}{r_n}$ is bounded away from 1. So if we choose p so that $h_{p-1} \leq (k_n + 1) \leq h_p$, then property (H) implies $\frac{(r_n-k_n-2)(k_n+1)}{h_p} \to \infty$ as $n \to \infty$.

Now choose $k_n' = \inf\{i \in \mathbb{Z}^+ : i(k_n + 1) \geq h_p\}$. Thus $\frac{r_n-k_n-2}{k_n'} \to \infty$ as $n \to \infty$. By Lemma 2.2 we can choose $\ell_n \to \infty$ so that

$$\int |\frac{1}{\ell_n} \sum_{i=0}^{\ell_n-1} \mathcal{X}_B(T^{-i(k_n'(k_n+1))}x) - \mu(B)| d\mu(x) \to 0 \quad \text{as} \ \ n \to \infty$$

and $\frac{r_n-k_n-2}{\ell_n k_n'} \to \infty$. Therefore by Lemma 2.1

$$\int |\frac{1}{r_n - k_n - 1} \sum_{i=0}^{r_n-k_n-2} \mathcal{X}_B(T^{-i(k_n+1)}x) - \mu(B)| d\mu(x) \to 0 \quad \text{as} \ \ n \to \infty.$$

Hence T is mixing on D_2.

Mixing on $\mathbf{D_3}$. This can be handled in the same manner as D_2 with $(k_n + 1)$ replaced by k_n.

Therefore T is mixing. $\square$

Corollary. *The classical staircase construction is mixing.*

Proof. For the classical staircase construction our sequence of cuts $r_n = n$. Thus $h_n \geq n!$. Therefore $\frac{r_n^2}{h_n} \leq \frac{n^2}{n!} \to 0$. $\square$

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