A GENERALIZATION OF BANCHOFF’S TRIPLE POINT THEOREM

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Abstract. Consider an immersion of a surface into $S^3$. Banchoff’s theorem states that the parity of the number of triple points and the parity of the Euler characteristic of the surface coincide. Here we generalize this theorem to codimension 1 immersions of arbitrary even dimensional manifolds in spheres. The proof is an analogue of a proof of Banchoff’s theorem circulated in preprint form due to R. Fenn and P. Taylor in 1977.

Let us consider a codimension 1 smooth generic (i.e. self-transverse) immersion $f$ of a closed manifold $M^n$ in the sphere $S^{n+1}$. Let us recall how a neighborhood of an $i$-tuple point (in $R^{n+1} \subset S^{n+1}$) looks like in such a self-transverse immersion. Consider the coordinate hyperplanes in $R^i$ and take the direct product of this configuration with $R^{n+1-i}$. What is obtained is diffeomorphic to the neighborhood of an $i$-tuple point in the image of $f$.

For any natural number $i$, $1 \leq i \leq n+1$, let us denote by $\tilde{\Delta}_i$ the set of $i$-tuple points in $S^{n+1}$, i.e.

$$\tilde{\Delta}_i = \{ y \in S^{n+1} \mid f^{-1}(y) \text{ consists of } i \text{ different points} \}.$$

As is well known, $\dim \tilde{\Delta}_i = n + 1 - i$, and $\bigcup_{r=i}^{\infty} \tilde{\Delta}_r$ is an immersed manifold (although it is not in general position, i.e. it is the image of a non-self-transverse immersion). Let $\Delta_i$ be a closed manifold such that $\bigcup_{r=i}^{\infty} \tilde{\Delta}_r$ is the image of an immersion of $\Delta_i$ in $S^{n+1}$.

Remark. Of course, many different manifolds can be immersed into $S^{n+1}$ so that their images are $\bigcup_{r=i}^{\infty} \tilde{\Delta}_r$. For example if a possible $\Delta_i$ is given, then any of its finite coverings serves as well. We make the choice of $\Delta_i$ explicit by assuming that the $i$-tuple points of $f$ are non-multiple points of the immersion $\Delta_i \hookrightarrow S^{n+1}$.

We shall call the manifold $\Delta_i$ the $i$-tuple manifold of $f$. Our theorem claims that for $n$ even the sum of the Euler characteristics of $i$-tuple manifolds is even. (For $n = 2$ this is exactly Banchoff’s theorem.)

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**Theorem.** If \( n > 0 \) is even, then
\[
\sum_{i=1}^{n+1} \chi(\Delta_i) \equiv 0 \mod 2.
\]

The following proof is an analogue of the proof in [FT] for Banchoff’s triple point theorem.

**Proof.** Since \( n \) is even, we can omit the terms corresponding to even \( i \)'s, because in those cases the dimension of \( \Delta_i \) is odd. Now let us triangulate the *image* of \( f \) in such a way that for any \( i \) the set of points of multiplicity \( i \) or higher forms a subcomplex of \( f(M) \).

Let \( \alpha_r^i \) denote the number of \( i \)-dimensional simplexes whose interiors lie in \( \tilde{\Delta}_r \), and let
\[
\beta_r = \alpha_r^0 - \alpha_r^1 + \ldots \pm \alpha_r^{n+1-r}.
\]
Observe that \( \beta_r \) is not the Euler characteristic of any complex. However, we have that
\[
\chi(\Delta_i) = \sum_{r=1}^{n+1} \left( \binom{r}{i} \right) \beta_r.
\]

The coefficient \( \left( \binom{r}{i} \right) \) counts the multiplicity of the self-intersection of \( \Delta_i \) at \( \tilde{\Delta}_r \). So
\[
\sum_{i=1}^{n+1} \chi(\Delta_i) = \sum_{i=1}^{n+1} \sum_{r=i}^{n+1} \left( \binom{r}{i} \right) \beta_r,
\]
where \( * \) indicates that the sum is taken only for odd \( i \)'s. After changing the order of the summations we get:
\[
\sum_{r=1}^{n+1} \left( \sum_{i=1}^{r} \left( \binom{r}{i} \right) \beta_r = \sum_{r=1}^{n+1} 2^{r-1} \beta_r \equiv \beta_1 \mod 2. \tag{1}
\]

Now let us paint the complement of \( f(M) \) in \( S^{n+1} \) in two colors in a chessboard-style, i.e. let any two neighboring domains have different colors (where “neighboring” means that they are separated by a component of \( \tilde{\Delta}_1 \)). This is possible, since \( H_n(S^{n+1}; \mathbb{Z}_2) = 0 \).

Let \( N \) be the boundary of an \( \varepsilon \)-neighborhood of \( f(M) \) in the black subset of \( S^{n+1} \). Notice that from the given triangulation of \( f(M) \) we can construct a triangulation of \( N \) by pushing the simplexes from \( f(M) \) to \( N \) in a reasonable way. Simplexes in \( \tilde{\Delta}_1 \) will have \( 2^{r-1} \) counterparts in \( N \) (\( i \) hyperplanes divide the Euclidean \( n \)-space into \( 2^i \) parts, half of which are black). Thus:
\[
\chi(N) = \sum_{i=1}^{n+1} 2^{i-1} \beta_i \equiv \beta_1 \mod 2.
\]
But \( \chi(N) \) is even, because \( N \) is embedded in codimension 1 (and \( n > 0 \)), so the proof is complete.

**Remark 1.** As is clear from the proof, the space \( S^{n+1} \) can be replaced by any manifold such that its \( n \)th \( \mathbb{Z}_2 \)-homology group is 0.
Remark 2. The above proof does not work for $n$ odd, since the sum $\sum_{i=1}^{r} \binom{r}{i}$ (where the star this time means summation for even $i$'s) equals to $2^{r-1} - 1$, so the sum in formula (1) gives $\sum_{r=2}^{n+1} \beta_r$ (which is clearly the Euler characteristic of the complex $f(M)$).

The figure 8 immersion of the circle in the plane shows that the statement of the theorem is false for $n = 1$. A theorem of Freedman [F] (and its generalization to unoriented 3-manifolds given in [A]) shows that it is true for $n = 3$. We do not know whether it is true or not for $n > 3$.

Remark 3. If we consider only oriented $n$-manifolds and their codimension 1 immersions in $S^{n+1}$, and the $n$th stable homotopy group of spheres has no 2-primary torsion, then the Euler characteristics of the $i$-tuple manifolds are all even, for any $i$. (Indeed, for any $i$, $\chi(\Delta_i) \mod 2$ defines a homomorphism from the stable homotopy group $\pi_{n+N}(S^N)$, $N >> n$ to $\mathbb{Z}_2$.)

In particular the statement of the theorem is true for $n = 5$ or $n = 13$ for oriented manifolds.

Remark 4. If the dimension $n = 4$, then more is true than is stated in the theorem, namely all $\chi(\Delta_i)$'s are even, since the stable homotopy group $\pi_{5}(RP^{\infty})$ vanishes (see [L]), and this group is isomorphic to the cobordism group of immersions of 4-manifolds into $R^5$.

REFERENCES


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