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THE ORLICZ-PETTIS THEOREM FOR TOPOLOGICAL RIESZ SPACES

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ABSTRACT. A finitely additive vector measure from a σ-ring to a Riesz space is countably additive (exhaustive) for all Hausdorff Lebesgue topologies on the range space, or for none of them. In particular, subseries convergent series are the same for all Hausdorff Lebesgue topologies on a Riesz space.

Our terminology concerning locally solid topological Riesz spaces is that of [1]. In particular, such a space as well as its topology is said to be *Lebesgue* if every decreasing net with infimum 0 is topologically convergent to 0.

A finitely additive measure $\mathbf{m} : \mathcal{R} \to X$, where $\mathcal{R}$ is a ring of sets and X is a topological vector space, is called *exhaustive* (strongly bounded) if $\mathbf{m}(A_n) \to 0$ for each disjoint sequence (A_n) in $\mathcal{R}$.

Let $\mathcal{P}$ denote the power set of $\mathbb{N}$. We recall that $\mathcal{P}$ becomes a complete metric space when equipped with the metric $d(M, N) = \nu(M \triangle N)$ induced by the measure $\nu(N) = \sum_{n\in N} 2^{-n}$. Proposition 1 below is standard and goes back to Saks (see e.g. [4], Proof of Thm. III.7.2 and notes on p. 234); for Proposition 2, see [3, Lemma].

Proposition 1. *If $\mathbf{m} : \mathcal{P} \to X$ is a finitely additive measure, then $\mathbf{m}$ is countably additive iff $\mathbf{m}$ is continuous iff $\mathbf{m}$ has a point of continuity.*

Proposition 2. *Let X be a metrizable topological vector space. Then a finitely additive measure $\mathbf{m} : \mathcal{P} \to X$ is exhaustive iff every disjoint sequence (N_k) in $\mathcal{P}$ has a subsequence (M_k) such that $\mathbf{m}$ is countably additive on the σ-algebra in $\mathbb{N}$ generated by (M_k).*

Theorem. *Let τ_1 and τ_2 be two Hausdorff Lebesgue topologies on a Riesz space X, $\mathcal{R}$ a σ-ring of sets, and $\mathbf{m} : \mathcal{R} \to X$ a finitely additive measure. If $\mathbf{m}$ is τ_1-countably additive (resp. exhaustive), then it is τ_2-countably additive (resp. exhaustive).*

Proof. Clearly, we may assume $\mathcal{R} = \mathcal{P}$. We proceed by a series of reductions.

(1) In view of [1, Thm. 11.10], both τ_1 and τ_2 admit Hausdorff Lebesgue extensions to the Dedekind completion of X. Thus, we may assume that X is Dedekind complete.

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(2) In order to prove that $\mathbf{m}$ is τ_2-countably additive (resp. exhaustive), it suffices to show it has this property with respect to each τ_2-continuous monotone F-seminorm on X. Let p be any of such F-seminorms, and let X_p be its carrier, that is, the band obtained as the disjoint complement to the null-band $p^{-1}(0)$ of p. We equip X_p with the metric Lebesgue topology determined by p and have to show that the composition of $\mathbf{m}$ with the band projection onto X_p is countably additive (resp. exhaustive). Thus, we can assume in what follows that τ_2 is metrizable and, consequently, X has the countable sup property [1, 17.8, Remark]. That is, whenever $A \subset X$ and $x = \sup A$ exists, there is a countable subset A' of A with $\sup A' = x$.

(3) Let B be the band generated by the sequence $u_n = \mathbf{m}(\{n\})$. Note that if $\mathbf{m}$ is τ_1-countably additive, then B contains the range of $\mathbf{m}$. On the other hand, in the exhaustive case, the assertion will follow easily once it is shown that $u_n \to 0$ (τ_2). Thus, replacing $\mathbf{m}$ by its composition with the band projection onto B, we can assume that $X = B$. Then, as X has also the countable sup property, any disjoint system in X is at most countable.

(4) Now, by [1, Thm. 12.4], X admits a metrizable Lebesgue topology weaker than τ_1. Therefore, we may assume that also τ_1 is metrizable. So, by Proposition 2, only the countably additive case has to be considered in what follows.

(5) Thus, it remains to prove that $\mathbf{m}$ is τ_2-countably additive, or τ_2-continuous, when both τ_1 and τ_2 are metric, and X is generated by the sequence (u_n). Denote $v_k = k\sum_{n=1}^{k} |u_n|$, and observe that $x = \sup_k(x \wedge v_k)$ for every $x \in X_+$. Define maps $\mathbf{m}_k : \mathcal{P} \to X$ by $\mathbf{m}_k(N) = \big(\mathbf{m}(N) \wedge v_k\big) \vee (-v_k)$, $k = 1, 2, \ldots$. Each $\mathbf{m}_k$ is τ_2-continuous because τ_1 and τ_2 coincide on order intervals [1, Thm. 12.9]. Moreover, as τ_2 is Lebesgue, $\mathbf{m}(N) = \tau_2\text{-}\lim_k \mathbf{m}_k(N)$ for every $N \in \mathcal{P}$. Therefore, the map $\mathbf{m} : \mathcal{P} \to (X, \tau_2)$ is of the first Baire class and, consequently, has a point of continuity (see [6], Remark 5 on p. 397). Apply Proposition 1. □

Remark. The theorem remains valid if τ_1 is any locally solid topology on X such that the topology $\inf\{\tau_1, \tau_2\}$ is Hausdorff. It is so, for instance, if τ_1 is Fatou (by [1, Thm. 12.7]) or merely σ-Fatou (by [7, Thm. 2.7 and Cor. 3.12]), or metrizable (by [2, Thms. 3.2 and 3.5]).

For a survey of Orlicz-Pettis type theorems, see [5].

Added in proof

It is important to point out that measures $\mathbf{m}$ as in the Theorem need not be order bounded. We are grateful to Z. Lipecki for the following simple example of a countably additive measure $\mathbf{m} : \mathcal{P} \to L_2[0,1]$ which is not order bounded: $\mathbf{m}(N) = \sum_{n=1}^{\infty} n^{-1} 1_N(n) r_n$, where (r_n) is the Rademacher sequence.

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