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STABILITY OF THE SURJECTIVITY OF ENDOMORPHISMS AND ISOMETRIES OF $\mathcal{B}(H)$

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Abstract. We determine the largest positive number c with the property that whenever Φ, Ψ are endomorphisms, respectively unital isometries of the algebra of all bounded linear operators acting on a separable Hilbert space, $\|\Phi(A) - \Psi(A)\| < c\|A\|$ holds for every nonzero A and Φ is surjective, then so is Ψ. It turns out that in the first case we have $c = 1$, while in the second one $c = 2$.

Introduction

In the theory of Banach algebras the following statement is of fundamental importance. If a, b belong to a unital Banach algebra $\mathcal{A}$, a is invertible and $\|a - b\| < 1/\|a^{-1}\|$, then b is invertible as well. Roughly speaking, elements close enough to an invertible element of $\mathcal{A}$ are also invertible. It seems to be an exciting question to find the limit of the problem, i.e. to determine how far one can go from an invertible element to still have this automatic invertibility property. Of course, as the example $\|1 - 0\| = 1$ shows, without any further restriction nothing new can be asserted. But if we have some more information on a and b, they belong to a subset $\mathcal{A}'$ of $\mathcal{A}$, then it is reasonable to expect some refinements in the above statement.

In this paper we treat this question in the case of two particular but very important subsets of the Banach algebra of all bounded linear transformations on the Banach space $\mathcal{B}(H)$ of all continuous linear operators acting on the separable Hilbert space H. These are the semigroups of endomorphisms and isometries of $\mathcal{B}(H)$. The role that these classes of transformations play in the study of algebraic and geometrical properties of operator algebras needs no further explanation.

Our aim is to determine the largest neighbourhood of an isomorphism, respectively unital surjective isometry of $\mathcal{B}(H)$ in which neighbourhood the endomorphisms, respectively unital isometries are all invertible, i.e. they are bijective. In fact, in our case the surjectivity is the main point.

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The study of automatic surjectivity of transformations on Banach algebras has nice and important applications. For example, in our paper [Mol], on the basis of such a result we have easily obtained that the groups of isomorphisms as well as surjective isometries of $\mathcal{B}(H)$ are topologically reflexive in the sense of Loginov and Shul′man. This former result [Mol, Theorem 1] states that if $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ is a Jordan homomorphism (a linear mapping with the property that $\Phi(A^2) = \Phi(A)^2$ $(A \in \mathcal{B}(H))$) whose range is supposed to contain merely two operators with extremal ranges, one of which has rank one and the other one has dense range, then Φ is bijective.

Our problem presented in this paper can be viewed also from a slightly different aspect by saying that we are interested in examining the stability of the surjectivity property of endomorphisms and isometries of $\mathcal{B}(H)$.

The first possibility that we have is to determine the largest positive number c having the property that whenever $\Phi, \Psi : \mathcal{B}(H) \to \mathcal{B}(H)$ are endomorphisms (resp. unital isometries), $\|\Phi - \Psi\| < c$ and Φ is surjective, then so is Ψ. However, instead of considering the operator norm we offer the finer possibility of pointwise distance. This is to determine the largest positive number c with the property that for any endomorphisms (resp. unital isometries) $\Phi, \Psi : \mathcal{B}(H) \to \mathcal{B}(H)$ if $\|\Phi(A) - \Psi(A)\| < c\|A\|$ $(0 \neq A \in \mathcal{B}(H))$ and Φ is surjective, then so is Ψ. It is easy to see that the solution of the second problem also provides the solution of the first one.

Finally, let us fix the notation and terminology. In what follows, all projections are supposed to be self-adjoint. $\mathcal{C}(H)$ and $\mathcal{F}(H)$ stand for the ideals of all compact and finite rank operators, respectively. By a homomorphism we mean a linear and multiplicative (but not necessarily *-preserving) mapping. A transformation $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ is said to be unital if it maps the identity into the identity, i.e. $\Phi(I) = I$.

Results

We begin with the easier problem of endomorphisms. It is well-known that every automorphism Φ of a C^*-algebra is continuous, its norm is equal to the norm of its inverse and hence we have $\|\Phi\| \geq 1$. In fact, there are automorphisms of arbitrarily large norm. Therefore, the general theory of Banach algebras gives us that if Ψ is a continuous endomorphism of the same C^*-algebra and $\|\Phi - \Psi\| < 1/\|\Phi^{-1}\|$ (≤ 1), then Ψ is bijective. However, in the case of $\mathcal{B}(H)$ we have the following result.

Theorem 1. *Let H be a separable infinite dimensional Hilbert space. Let $\Phi, \Psi : \mathcal{B}(H) \to \mathcal{B}(H)$ be homomorphisms. If Φ is surjective and*

$$\|\Phi(A) - \Psi(A)\| < \|A\|$$

holds true for every nonzero operator $A \in \mathcal{B}(H)$, then Ψ is also surjective.

Remark. The statement of this theorem cannot be strengthened in the way that one allows also equality in the strict inequality above. In fact, in this case the identity and the zero mapping on $\mathcal{B}(H)$ could serve as a counterexample.

Proof of the theorem. First observe that Φ is necessarily injective. This follows from the well-known facts that every homomorphism of $\mathcal{B}(H)$ is continuous, the only nontrivial closed ideal of $\mathcal{B}(H)$ is $\mathcal{C}(H)$ and that the Calkin algebra $\mathcal{B}(H)/\mathcal{C}(H)$ cannot be mapped into $\mathcal{B}(H)$ by an injective homomorphism (cf. [Mol, proof of

Theorem 1]). The form of the automorphisms of $\mathcal{B}(H)$ is also well-known [Che]. Namely, for every automorphism Φ of $\mathcal{B}(H)$ there is an invertible operator $T \in \mathcal{B}(H)$ for which

$$\Phi(A) = TAT^{-1} \qquad (A \in \mathcal{B}(H)).$$

Obviously, Φ maps rank-one operators into rank-one operators. Let $P \in \mathcal{B}(H)$ be a rank-one projection and consider the idempotents $Q = \Phi(P)$, $Q' = \Psi(P)$. Since Q is also rank-one and $\|Q - Q'\| < 1$, it follows quite easily that Q' has rank one as well. Indeed, if the rank of Q' is at least 2, then there is a unit vector x in the range of Q' which is orthogonal to the range of Q. We then obtain

$$1 = |\langle Qx, x\rangle - \langle Q'x, x\rangle| \leq \|Q - Q'\| < 1$$

which is a contradiction. Consequently, the range of Ψ contains a rank-one operator. Moreover, since

$$\|I - \Psi(I)\| = \|\Phi(I) - \Psi(I)\| < 1,$$

we infer that the range of Ψ contains a surjective operator as well. Our theorem [Mol, Theorem 1] now applies to get the conclusion of the statement. □

Our second result gives the solution of the problem for the case of unital isometries.

Theorem 2. *Let H be a separable infinite dimensional Hilbert space. Let $\Phi, \Psi : \mathcal{B}(H) \to \mathcal{B}(H)$ be unital isometries. If Φ is surjective and*

$$\|\Phi(A) - \Psi(A)\| < 2\|A\| \tag{1}$$

holds true for every nonzero operator $A \in \mathcal{B}(H)$, then Ψ is also surjective.

Remark. Just as in the case of our previous theorem, we give an example to demonstrate that this statement also cannot be made stronger by allowing equality in the strict inequality (1). What we need is only a non-surjective unital isometry, for example

$$\Psi : A \longmapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix},$$

where we have used the natural identification of $\mathcal{B}(H)$ and $\mathcal{B}(H \oplus H)$.

Proof of the theorem. We divide the proof into several steps.

First observe that we may assume that Φ is the identity on $\mathcal{B}(H)$. Indeed, if we replace A by $\Phi^{-1}(A)$ in (1), then we have

$$\|A - \Psi(\Phi^{-1}(A))\| < 2\|\Phi^{-1}(A)\| = 2\|A\| \qquad (0 \neq A \in \mathcal{B}(H)).$$

So, in what follows we do suppose that Φ is the identity and then we have the inequality

$$\|A - \Psi(A)\| < 2\|A\| \tag{2}$$

for every $0 \neq A \in \mathcal{B}(H)$.

Since Ψ is a unital mapping between C^*-algebras with norm 1, a well-known theorem of Russo and Dye [RD, Corollary 1] implies that Ψ is positivity preserving (i.e. $\Psi(A) \geq 0$ whenever $A \geq 0$).

Step 1. If $P \in \mathcal{B}(H)$ is a projection, then we have

$$\|P - \Psi(P)\| < 1.$$

Indeed, replacing P by $P-\frac{1}{2}I$ in (2), we obtain $\|P-\Psi(P)\|<2\|P-\frac{1}{2}I\|=1$.

Step 2. If P is a rank-one projection, then there are a rank-one projection Q and a positive operator A with $\|A\|<1$ such that $QA=AQ=0$ and $\Psi(P)=Q+A$.

Let $\|P-\Psi(P)\|<c<1$ and $c\le\lambda<1$. Denote by E the spectral measure of $\Psi(P)$ on $[0,1]$ (note that $\Psi(P)$ is positive with norm 1). Let

$$A_\lambda=\int_{[0,\lambda[}\omega\,dE(\omega),\qquad Q_\lambda=E[\lambda,1].$$

Obviously, $A_\lambda+\lambda Q_\lambda\overset{\lambda\to1}{\longrightarrow}\Psi(P)$ in the operator norm topology. Hence, there is a $c\le\lambda_0<1$ such that

$$\|P-(\lambda Q_\lambda+A_\lambda)\|<c\qquad(\lambda_0\le\lambda<1).$$

Multiplying by Q_λ from both sides, we obtain

$$\|Q_\lambda PQ_\lambda-\lambda Q_\lambda\|<c\qquad(\lambda_0\le\lambda<1). \tag{3}$$

Plainly, $rng\,Q_\lambda\supset rng\,Q_\lambda PQ_\lambda$. Let $\lambda_0\le\lambda<1$ and suppose that this inclusion is proper. In this case we can choose a unit vector x such that $Q_\lambda x=x$, $x\perp rng\,Q_\lambda PQ_\lambda$. From (3) it follows that

$$\lambda=|\langle Q_\lambda PQ_\lambda x,x\rangle-\lambda\langle Q_\lambda x,x\rangle|<c.$$

Since this is a contradiction, we infer that

$$rng\,Q_\lambda=rng\,Q_\lambda PQ_\lambda\qquad(\lambda_0\le\lambda<1).$$

This means that the range of Q_λ is at most one dimensional. Since $\Psi(P)$ is positive and its norm is 1, $1\in\sigma(\Psi(P))$. Therefore, $Q_\lambda\neq0$ and we obtain that Q_λ is rank-one for every $\lambda_0\le\lambda<1$. Since (Q_λ) is monotone decreasing as $\lambda\uparrow1$, we infer that $Q_\lambda=Q_\mu$ for every $\lambda,\mu\ge\lambda_0$. Hence, we obtain that $E(1)$ is a rank-one projection and $E[\lambda_0,1[=0$. These result in

$$\Psi(P)=A_{\lambda_0}+E(1)$$

which completes the proof of this step.

Step 3. Let P,P' be rank-one projections with $PP'=P'P=0$. Consider the corresponding resolutions

$$\Psi(P)=Q+A,\qquad\Psi(P')=Q'+A'$$

given in Step 2. Then we have $QQ'=Q'Q=0$, $QA'=A'Q=0$ and $Q'A=AQ'=0$.

Since Ψ is an isometry, we have

$$\|Q+A+Q'+A'\|=1. \tag{4}$$

If x is a unit vector in the range of Q, then we obtain from (4) that

$$\langle Qx,x\rangle+\langle Ax,x\rangle+\langle Q'x,x\rangle+\langle A'x,x\rangle\le1.$$

But $\langle Qx,x\rangle=1$ and the other terms are all nonnegative. Hence, it follows that

$$\langle Q'Qy,Qy\rangle=0,\quad\langle A'Qy,Qy\rangle=0\quad(y\in H).$$

This plainly implies that $(Q'Q)^*(Q'Q)=QQ'Q=0$ and $(\sqrt{A'}Q)^*\sqrt{A'}Q=QA'Q=0$. Thus we have $Q'Q=0$, $\sqrt{A'}Q=0$, $A'Q=0$. Referring to the symmetry of the role played by the involved operators, the proof is complete.

Step 4. Let P be a projection with rank n ($1 < n \in \mathbb{N}$). Let $P_1, ..., P_n$ be pairwise orthogonal rank-one projections with resolutions $\Psi(P_i) = Q_i + A_i$ ($i = 1, ..., n$) given in Step 2, $P = P_1 + ... + P_n$ and let $Q = Q_1 + ... + Q_n$, $A = A_1 + ... + A_n$. Then we have $\Psi(P) = Q + A$, Q has rank n, A is a positive operator with $\|A\| < 1$ and $QA = AQ = 0$.

The statements that Q has rank n, A is positive and $QA = AQ = 0$ follow from the previous steps. The only thing that we have to prove is that $\|A\| < 1$. Since $0 \leq A \leq Q + A = \Psi(P)$ and $\|\Psi(P)\| = 1$, we have $\|A\| \leq 1$. Suppose that $1 \in \sigma(A)$. We use an argument similar to that followed in Step 2. Let $\|P - (Q + A)\| = \|P - \Psi(P)\| < c < 1$. If E denotes the spectral measure of A, then there is a $c \leq \lambda < 1$ and a $0 < \mu < c$ such that for the operators

$$A_{\mu\lambda} = \int_{[\mu,\lambda[} \omega \, dE(\omega), \quad Q_\lambda = E[\lambda, 1] \neq 0$$

we have

$$\|P - (Q + \lambda Q_\lambda + A_{\mu\lambda})\| < c. \tag{5}$$

We know that $QA = AQ = 0$. Using the properties of the spectral measure and spectral integral we can infer that $QE(B) = E(B)Q = 0$ holds for every Borel set B of $[0, 1]$ which is bounded away from 0. Consequently, we have $QQ_\lambda = Q_\lambda Q = 0$ and $QA_{\mu\lambda} = A_{\mu\lambda}Q = 0$. Let $\tilde{Q} = Q + Q_\lambda$. From (5) we infer that

$$\|\tilde{Q}P\tilde{Q} - (Q + \lambda Q_\lambda)\| = \|\tilde{Q}(P - (Q + \lambda Q_\lambda + A_{\mu\lambda}))\tilde{Q}\| < c. \tag{6}$$

Since the range of $\tilde{Q}P\tilde{Q}$ is at most n dimensional and, by $Q_\lambda \neq 0$, $rng\, \tilde{Q}$ is at least $n+1$ dimensional, we can choose a unit vector from the range of $\tilde{Q}$ such that $x \perp rng\, \tilde{Q}P\tilde{Q}$. Then using (6) we obtain

$$\|Qx\|^2 + \lambda\|Q_\lambda x\|^2 = |\langle \tilde{Q}P\tilde{Q}x, x\rangle - \langle (Q + \lambda Q_\lambda)x, x\rangle| < c. \tag{7}$$

But we have $\|Qx\|^2 + \|Q_\lambda x\|^2 = \|\tilde{Q}x\|^2 = 1$. Hence, it follows from (7) that $\lambda < c$ which is a contradiction. This completes the proof of Step 4.

If $P \in \mathcal{B}(H)$ is a finite rank projection, then we can define

$$F(P) = \lim_{n\to\infty} \Psi(P)^n$$

since, by Step 4, the sequence on the right side converges in the operator norm topology. Moreover, $F(P)$ is a projection with the same rank as P has and for arbitrary finite rank projections P, P' with $PP' = P'P = 0$ we have $F(P)F(P') = F(P')F(P) = 0$ and $F(P + P') = F(P) + F(P')$.

Let us call any maximal family of pairwise orthogonal rank-one projections in $\mathcal{B}(H)$ a projection base. In what follows our aim is to prove that $\sum_n F(P_n) = \sum_n F_n(P'_n)$ holds true for arbitrary projection bases $(P_n), (P'_n)$ in $\mathcal{B}(H)$. To this end, we extend F to the set $\mathcal{SF}(H)$ of all self-adjoint elements of $\mathcal{F}(H)$ in the following way. Let

$$\tilde{F}(T) = \sum_{i=1}^{n} \lambda_i F(P_i)$$

where $T = \sum_{i=1}^{n} \lambda_i P_i$, the P_is are pairwise orthogonal finite-rank projections and $\lambda_i \in \mathbb{R}$ ($i = 1, ..., n$). It is not hard to verify that $\tilde{F}$ is well-defined.

Step 5. $\tilde{F}$ is additive.

Let $T, T' \in \mathcal{SF}(H)$ be with spectral resolutions $T = \sum_{i=1}^{n} \lambda_i P_i$ and $T' = \sum_{j=1}^{m} \lambda'_j P'_j$. We intend to apply a fundamental theorem of quantum logics on the form of σ-homomorphisms between lattices of projections. This result is due to Jajte and Paszkiewicz [Dvu, Theorem 5.2.13] and has the following corollary. If K_1, K_2 are separable Hilbert spaces, $\dim K_1 \geq 3$ and G is a σ-orthoadditive, projection-valued mapping on the lattice of projections in $\mathcal{B}(K_1)$, then it is the restriction of a Jordan *-homomorphism of $\mathcal{B}(K_1)$. So, let us consider an at least 3 dimensional subspace H_0 of H which contains the ranges of the projections P_i, P'_j. Using the natural embedding of $\mathcal{B}(H_0)$ into $\mathcal{B}(H)$, we can infer that there is a Jordan *-homomorphism on the self-adjoint part of the image of this embedding which there coincides with $\tilde{F}$. In particular, we have $\tilde{F}(T + T') = \tilde{F}(T) + \tilde{F}(T')$.

It now follows that $\tilde{F}$ is a real-linear mapping on $\mathcal{SF}(H)$. Moreover, it is straightforward from the definition of $\tilde{F}$ that $\tilde{F}(T^2) = \tilde{F}(T)^2$ holds true for every $T \in \mathcal{SF}(H)$. Linearizing this equation, i.e. replacing T by $T + S$ (S is also self-adjoint), we arrive at

$$\tilde{F}(TS + ST) = \tilde{F}(T)\tilde{F}(S) + \tilde{F}(S)\tilde{F}(T).$$

If we define

$$\tilde{J}(T + iS) = \tilde{F}(T) + i\tilde{F}(S) \qquad (T, S \in \mathcal{SF}(H)),$$

then it is easy to check that $\tilde{J}$ is a Jordan *-homomorphism on $\mathcal{F}(H)$. Moreover, $\tilde{J}$ is continuous which follows from the fact that $\tilde{F}$ is isometric. Therefore, $\tilde{J}$ has a continuous extension J onto $\mathcal{C}(H)$. It is apparent that this J is a Jordan *-homomorphism on $\mathcal{C}(H)$.

Step 6. For arbitrary projection bases $(P_n), (P'_n)$ in $\mathcal{B}(H)$ we have $\sum_n F(P_n) = \sum_n F(P'_n)$.

[Sto, Lemma 3.1] states that every Jordan *-homomorphism of a C^*-algebra $\mathcal{A}$ into a von Neumann algebra can be extended to an ultra-weakly continuous Jordan *-homomorphism defined on the second dual $\mathcal{A}^{**}$. Since the second dual of $\mathcal{C}(H)$ is $\mathcal{B}(H)$, then we have an ultra-weakly continuous Jordan *-homomorphism $J' : \mathcal{B}(H) \to \mathcal{B}(H)$ that extends J. Apparently, J' also extends F. Using the fact that the weak and the ultra-weak topologies coincide on every bounded subset of $\mathcal{B}(H)$ and $\sum_n P_n = \sum_n P'_n$, we obtain the statement.

In what follows, denote by E the projection $\sum_n F(P_n)$, where (P_n) is any projection base in $\mathcal{B}(H)$.

Step 7. We have $E = I$.

Suppose, on the contrary, that $E < I$. Let $x \in H$ be a unit vector which is orthogonal to the range of E. Consider the rank-one projection $P = x \otimes x$ and let $Q = F(P)$. Since $Q \leq E$, we have $PQ = QP = 0$. Moreover, we know that $\|P - \Psi(P)\| < 1$ and $\Psi(P)F(P) = F(P)\Psi(P) = F(P) = Q$ (see Step 2); hence it follows that $\|QPQ - Q\| < 1$. Taking the relation $PQP = 0$ into account, we obtain a contradiction. This gives the proof of this step.

Step 7 has an important corollary. Let P be a rank-one projection. If we extend it to a projection base (P_n), then for the positive operator A in the resolution of $\Psi(P)$ we obtain that $AF(P_n) = F(P_n)A = 0$ for every $n \in \mathbb{N}$. Our previous step implies that $A = 0$ and thus we have $\Psi(P) = F(P) = J(P)$.

Step 8. Ψ is a Jordan *-homomorphism on $\mathcal{B}(H)$.

We first prove that if (P_n) is any sequence of pairwise orthogonal rank-one projections and $P = \sum_n P_n$, then $\Psi(P) = \sum_n \Psi(P_n)$. Indeed, the series on the right side is easily seen to be weakly convergent. Moreover, by the positivity preserving property of Ψ, we have $\Psi(P_n) \leq \Psi(P)$ for every n. Let us extend (P_n) by (Q_n) to a projection base. Considering the inequalities

$$\sum_n \Psi(P_n) \leq \Psi(P), \quad \sum_n \Psi(Q_n) \leq \Psi(I - P)$$

and $\Psi(I) = I$, by Step 7 we obtain then there are in fact equalities in the inequalities above. Hence, Ψ maps projections into projections. Now, a standard argument shows that Ψ is a Jordan *-homomorphism. Let $P, P' \in \mathcal{B}(H)$ be orthogonal projections. Since in this case $\Psi(P) + \Psi(P') = \Psi(P + P')$ is also a projection, we have $\Psi(P)\Psi(P') + \Psi(P')\Psi(P) = 0$ which further implies that $\Psi(P)$ and $\Psi(P')$ are orthogonal to each other. Hence, if $P_1, ..., P_n$ are pairwise orthogonal projections and $\lambda_1, ..., \lambda_n \in \mathbb{C}$, then we have

$$(\Psi(\sum_{i=1}^{n} \lambda_i P_i))^2 = (\sum_{i=1}^{n} \lambda_i \Psi(P_i))^2 = \sum_{i=1}^{n} \lambda_i^2 \Psi(P_i) = \Psi(\sum_{i=1}^{n} \lambda_i^2 P_i).$$

Therefore, using the continuity of Ψ and the spectral theorem of self-adjoint operators, we obtain that for every self-adjoint $T \in \mathcal{B}(H)$, the operator $\Psi(T)$ is self-adjoint and $\Psi(T^2) = \Psi(T)^2$ holds true. We can now argue in the same way as we did after the proof of Step 5. That is, a linearization of this last equation together with the linearity of Ψ gives us that Ψ is a Jordan *-homomorphism.

Step 9. Ψ is surjective.

Since Ψ is a Jordan homomorphism and $\|P - \Psi(P)\| < 1$ holds for every projection $P \in \mathcal{B}(H)$, we can argue as in the proof of Theorem 1, i.e. referring to [Mol, Theorem 1] we obtain the surjectivity of Ψ.

The proof of Theorem 2 is complete. □

Remark. Concerning our Theorem 2, some questions arise in a natural manner. First, one can ask whether the statement could be generalized to the case of arbitrary Banach spaces. It is very easy to see that this question can be answered in the negative.

Example 1. Let K be a separable infinite dimensional Hilbert space and let (e_n) be a complete orthonormal sequence in K. Define a non-surjective isometry $T : K \to K$ by

$$T = e_1 \otimes e_1 + \sum_{n=2}^{\infty} e_{n+1} \otimes e_n.$$

Then I and T have a nonzero common fixed point and we can verify that the inequality

$$\|x - Tx\| < 2\|x\|$$

holds true for every nonzero $x \in K$. Indeed, if x is a unit vector such that $\|x - Tx\| = 2$, then taking the fact $\|x\| = \|Tx\| = 1$ into account, we obtain that $Tx = -x$. But this implies $x = 0$ which is a contradiction.

Although we have the above negative answer, we feel that there is some hope to achieve positive results in the C^*-algebra setting. We would greatly appreciate any result in this direction.

Another question concerns the assumption of unitality of the involved isometries. This is a folk result and is, in fact, a consequence of Kadison's fundamental theorem on the structure of surjective isometries of a C^*-algebra [Kad], that for every surjective isometry Φ of $\mathcal{B}(H)$, there are unitaries U, V and antiunitaries (conjugate-linear surjective isometries) U', V' on H such that Φ is either of the form

$$\Phi(A) = UAV \qquad (A \in \mathcal{B}(H))$$

or of the form

$$\Phi(A) = U'A^*V' \qquad (A \in \mathcal{B}(H)).$$

Hence, it seems natural to study the question of unitality at least for the mappings occurring in the following example.

Example 2. Let $U, V \in \mathcal{B}(H)$ be isometries and define the mapping $\Omega(A) : \mathcal{B}(H) \to \mathcal{B}(H)$ by

$$\Omega(A) = UAV^* \qquad (A \in \mathcal{B}(H)). \tag{8}$$

We assert that if

$$\|A - \Omega(A)\| < 2\|A\| \qquad (0 \neq A \in \mathcal{B}(H)), \tag{9}$$

then Ω is surjective. To see this, first suppose that neither U nor V is surjective. Let $x, y \in H$ be unit vectors which are orthogonal to the range of U and V, respectively. Define orthonormal sequences $x_n = U^n x$, $y_n = V^n y$ $(n \in \mathbb{N})$. Furthermore, define a numerical sequence (λ_n) by $\lambda_1 = -1, \lambda_2 = 1, \lambda_3 = \lambda_4 = ... = 0$. If

$$A = \sum_n \lambda_n x_n \otimes y_n,$$

then $\|A\| = 1$ and we have

$$\|A - UAV^*\| = \|\sum_n \lambda_n x_n \otimes y_n - \sum_n \lambda_n x_{n+1} \otimes y_{n+1}\| = 2.$$

Consequently, either U or V, say V, is surjective. If we replace A in (9) by V, then we have $\|V - U\| < 2$. We then have $\|I - V^*U\| < 2$. Since the spectrum of a non-surjective isometry is the whole unit disc, its distance to I is 2. It follows that the isometry V^*U is surjective which proves the surjectivity of U.

Since not every isometry of $\mathcal{B}(H)$ is in the form (8), we leave the question of whether the condition $\Phi(I) = \Psi(I) = I$ can be omitted from the statement of Theorem 2 as an open problem.

Remark. To conclude the paper, we make some remarks on an interesting question raised by the referee of the first version of the manuscript. This reads as follows. Taking Kadison's theorem into account, if U, V are unitaries on H and $\|UAV - A\| < 2\|A\|$ holds true for every nonzero $A \in \mathcal{B}(H)$, then what can one say about U and V? Here we treat only the unital case, that is when $V = U^*$. It is quite easy to see that in this case we necessarily have

$$\langle Ux, x\rangle \neq 0 \qquad (0 \neq x \in H). \tag{10}$$

Indeed, if $x \in H$, $\|x\| = 1$ and $\langle Ux, x\rangle = 0$, then for the operator $A = -Ux \otimes x + U^2x \otimes Ux$ we obtain that $\|A\|^2 = \|A^*A\| = 1$ and that $\|(UAU^* - A)(Ux)\| = 2$, from which it follows that $\|UAU^* - A\| = 2$. We now assert that on $\mathcal{C}(H)$, the condition (10) is sufficient as well, which means that in this case we have $\|UAU^* - A\| < 2\|A\|$ $(0 \neq A \in \mathcal{C}(H))$. To see this, suppose on the contrary that $\|UAU^* - A\| = 2$ holds true for some $A \in \mathcal{C}(H)$ with $\|A\| = 1$. Since for any compact operator T on H there is a unit vector $x_0 \in H$ for which $\|Tx_0\| = \|T\|$, we can choose a unit vector $x \in H$ such that $\|UAU^*x - Ax\| = 2$. Due to the fact that both terms here are of norm not greater than 1, we infer that $UAU^*x = -Ax$. Since, by the Cauchy-Schwarz inequality, we have $1 = \|Ax\|^2 = |\langle A^*Ax, x\rangle| \leq \|A^*Ax\|\|x\| \leq 1$, it follows that A^*Ax and x are linearly dependent. But A^*A is a positive operator and, consequently, we have $A^*Ax = x$. Using $AU^*x = -U^*Ax$, we compute

$$\begin{aligned}\langle UAx, Ax\rangle &= \langle Ax, U^*Ax\rangle = -\langle Ax, AU^*x\rangle \\ &= -\langle A^*Ax, U^*x\rangle = -\langle x, U^*x\rangle = -\langle Ux, x\rangle.\end{aligned}$$

We know that $\|Ax\| = 1 = \|x\|$. Since the numerical range of any operator is convex, we get that 0 belongs to the numerical range of U; i.e. there is a unit vector y in H for which $\langle Uy, y\rangle = 0$. But this is a contradiction.

To give the complete answer of the problem even in the unital case, we would need a much finer approach. We leave this problem as the final open question of this paper.

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