ENVELOPING SEMIGROUPS AND MAPPINGS
ONTO THE TWO-SHIFT

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Abstract. Enveloping semigroups of topological actions of semigroups $G$ on compact spaces are studied. For zero dimensional spaces, and under modest conditions on $G$, the enveloping semigroup is shown to be the Stone-Čech compactification if and only if some Cartesian product has the two-shift as a factor. Examples are discussed showing that, unlike in the measure theory case, positive entropy does not imply the existence of such a factor even if the Cartesian product has large entropy.

Let $G$ be a discrete multiplicative semigroup with at least one element $g_0$ such that $g \to g_0 g$ is injective. Let $X$ be a compact Hausdorff space. We say $\sigma : G \times X \to X$, written $(g,x) \to \sigma_g(x)$, is an action of $G$ on $X$, and we say $(X,G,\sigma)$ is a $G$-flow, if $\sigma_{gh} = \sigma_g \circ \sigma_h$ for all $g,h \in G$. We sometimes refer to the flow $X$ when $G$ and $\sigma$ are understood. We will assume that if $g_1 \neq g_2$ then there is at least one $x \in X$ such that $\sigma_{g_1}(x) \neq \sigma_{g_2}(x)$. Given a $G$-flow $X$ let $E = E_X$ be the closure of $\{ \sigma_g : g \in G \}$ in $X^X$ with the product topology. Clearly $E$ is compact and it can easily be shown that $E$ is a subsemigroup of $X^X$ where $(\xi \eta)(x) = \xi(\eta(x))$ for $\xi, \eta \in X^X$ and $x \in X$. $E$ is called the enveloping semigroup of $X$ (the enveloping semigroup was first introduced by Ellis in [3]. Other sources in the literature dealing with enveloping semigroups include [1], [4] and [5]). Let $G_E = \{ \sigma_g : g \in G \}$. Observe $g \to \sigma_g$ is an injection of $G$ onto the dense subset $G_E$ of $E$.

Let $C$ be the set of all continuous real valued functions with domain $X$ and let $B$ be the set of all bounded real valued functions with domain $G$. For $x \in X$ and $f \in C$ let $f_x \in B$ be defined by $f_x(g) = f(\sigma_g(x))$. Let $A = A(X) \subseteq B$ be the smallest uniformly closed algebra containing $\{ f_x : x \in X, f \in C \}$. We slightly abuse notation to use the same symbol $f_x$ for the corresponding function defined on $G_E$. If $x \in X$, and $f \in C$ define $\overline{f_x} : E \to R$ by $\overline{f_x}(\eta) = f(\eta(x))$. Observe that each $\overline{f_x}$ is in $C_R(E)$, the space of continuous real valued functions on $E$. Clearly $\overline{f_x}$ is the continuous extension of $f_x$ from $G_E$ to $E$. It follows that every $\phi \in A$ has a (unique) continuous extension $\overline{\phi} : E \to R$.

**Proposition 1.** $\{ \overline{\phi} : \phi \in A \} = C_R(E)$. In particular, $A = B$ if and only if the embedding $g \to \sigma_g$ is a realization of the Stone-Čech compactification $\beta$ of $G$.

**Proof.** It is clear that $\{ \overline{\phi} : \phi \in A \} \subseteq C_R(E)$ is a uniformly closed subalgebra of $C_R(E)$. If $\xi$ and $\eta$ are distinct elements of $E$ then there is some $x \in X$ with
Clearly $$\xi(x) \neq \eta(x)$$. Choose $$f \in C$$ so that $$f(\xi(x)) \neq f(\eta(x))$$, and thus $$\overline{f}(\xi) \neq \overline{f}(\eta)$$. This shows that $$\{ \overline{\phi} : \phi \in A \}$$ separates points of $$E$$ and so, by the Stone-Weierstrass theorem, $$\{ \overline{\phi} : \phi \in A \} = CR(E)$$.

As is well-known, a realization of the Stone-Čech compactification of the discrete space $$G$$ consists of a compact space $$E$$ and an injective mapping $$g \to \sigma_g$$ of $$G$$ onto a dense subspace $$G_E$$ of $$E$$ such that:

1. The set $$G_E$$ is discrete as a subspace of $$E$$.
2. Every bounded function on $$G_E$$ extends to a continuous function on $$E$$.

In fact (1) follows from (2) and (2) is simply the assertion that $$A = B$$. We have assumed $$g \to \sigma_g$$ is injective, and $$G_E$$ is dense by the definition of $$E$$. This shows that $$E$$ is the Stone-Čech compactification when $$A = B$$. The converse assertion is an immediate consequence of (2).

**Remark.** A similar result holds if $$G$$ has a completely regular topology and the action $$\sigma$$ of $$G$$ on $$X$$ is jointly continuous.

**Proposition 2.** Let $$I$$ be any non-void index set. Let $$X^I$$ be the product space with the product topology and let $$\sigma^I$$ be the product action on $$X^I$$, defined by $$\sigma^I_i(x) = \sigma(x_i)$$ for $$x \in X_i$$ and $$i \in I$$. Then $$A(X^I) = A(X)$$.

**Proof.** Clearly $$A(X) \subseteq A(X^I)$$, so we establish the reverse inclusion. Let $$\pi_i(x) = x_i$$ for $$x \in X^I$$ and $$i \in I$$. Clearly $$\{ f \circ \pi_i : f \in C, \ i \in I \}$$ separates points of $$X^I$$. Given $$f \in CR(X^I)$$ and $$\epsilon > 0$$, it follows from the Stone-Weierstrass theorem that there exist a positive integer $$k$$, functions $$f^1, \ldots, f^k \in C$$, indices $$i_1, \ldots, i_k \in I$$, and a real polynomial $$p$$ of $$k$$ variables such that $$|f(x) - p(f^1(\pi_{i_1}(x)), \ldots, f^k(\pi_{i_k}(x)))| < \epsilon$$ for all $$x \in X^I$$. Examination of the $$\sigma^I$$ action shows that

$$|f(x) - p(f^{i_1}(x_{i_1}), \ldots, f^{i_k}(x_{i_k}))| < \epsilon.$$ 

Since $$A(X)$$ consists of the uniform closure of just such polynomial expressions, this inequality shows $$f \in A(X)$$. 

We will apply the above results to actions on a totally disconnected space. We begin with a general definition of a shift space on a finite set.

**Definition.** Let $$\alpha$$ be a finite set with the discrete topology. Let $$X = \alpha^G$$ be the product space with the product topology. Let $$\sigma$$ be defined by $$(\sigma_g(x))_h = x_{hg}$$. We call $$(X, G, \sigma)$$ the $$G$$-shift on $$\alpha$$.

**Remark.** The mapping $$\sigma$$ defined above is an action, and $$g \neq h \implies \sigma_g \neq \sigma_h$$.

**Proof.** First observe $$\sigma_{gh}(x)_t = x_{tgh} = (\sigma_h(x))_{tg} = (\sigma_g(\sigma_h(x)))_t \ \forall g, h, t \in G$$ and $$x \in X$$. This shows $$\sigma_{gh} = \sigma_g \circ \sigma_h$$. Because $$X$$ has the product topology, evaluation $$x \to x_t$$ is continuous for fixed $$t$$ and $$g$$. But continuity of $$\sigma_g : X \to X$$ is equivalent to continuity of $$x \to (\sigma_g(x))_t = x_{tg}$$ at every $$x \in X$$ and for every $$t \in G$$. This shows each $$\sigma_g$$ is continuous. Finally, we have assumed the existence of an element $$g_0 \in G$$ such that $$g \to g_0g$$ is injective. Given distinct $$g$$ and $$h$$ in $$G$$ let $$x \in X$$ be such that $$x_{g_0} \neq x_{g_0h}$$. Then $$\sigma_g(x) \neq \sigma_h(x)$$ since $$(\sigma_g(x))_{g_0} \neq (\sigma_h(x))_{g_0}$$.

**Proposition 3.** Let $$(X, G, \sigma)$$ be a $$G$$-shift on some finite set $$\alpha$$ of cardinality greater than 1. Then $$A_X = B$$, and so $$E_X$$ is the Stone-Čech compactification of $$G$$. 


Proof. Clearly it is sufficient to show that for any subset $U \subseteq G$, the characteristic function $\chi = \chi_U$ of $U$ is in $A$. We can map $\alpha$ bijectively to $\{0, 1, \ldots, k-1\}$ for some $k > 1$ and for notational simplicity we assume that $\alpha$ in fact equals $\{0, 1, \ldots, k-1\} \subseteq \mathbb{R}$. Choose $g_0 \in G$ so $g \to g_0g$ is injective, and let $V = g_0U$. Define $\tilde{x} \in X$ as the characteristic function of $V$. Define $f : X \to \mathbb{R}$ by $f(x) = x_{g_0}$. Observe $f_\tilde{x}(g) = f(\sigma_g(\tilde{x})) = (\sigma_g(\tilde{x}))_{g_0} = \tilde{x}_{g_0g} = \chi(g)$.

**Definition.** If $(X, G, \sigma)$ and $(Y, G, \tau)$ are flows then a continuous mapping $\Pi : X \to Y$ is called a flow mapping if $\tau_g(\Pi(x)) = \Pi(\sigma_g(x))$ for all $x \in X$ and $g \in G$. If also $\Pi$ is surjective then we call $\Pi$ a factor mapping of $X$ onto $Y$ and we call $Y$ a factor of $X$.

**Theorem 1.** Let $(X, G, \sigma)$ be an action of the discrete semigroup $G$ on the compact totally disconnected Hausdorff space $X$, satisfying $g \neq h \implies \sigma_g \neq \sigma_h$. Assume that there is an element $g_0 \in G$ such that $g \to g_0g$ is injective. Let $A(X)$, $B$ and $E_X$ be as above. Let $(Y, G, \tau)$ be the $G$-shift on the two symbol set $\alpha = \{0, 1\}$. Assume that there is at least one $\tilde{y} \in Y$ such that $\{\tau_g(\tilde{y}) : g \in G\}$ is dense in $Y$ (i.e., $(Y, G, \tau)$ is transitive). Then the following four statements are equivalent:

1. For some finite non-void index set $I$, the shift action $(Y, G, \tau)$ is a factor of the product action $(X^I, G, \sigma^I)$.
2. For some non-void index set $I$, the shift action $(Y, G, \tau)$ is a factor of the product action $(X^I, G, \sigma^I)$.
3. $A = B$.
4. $E_X$, together with the mapping $g \to \sigma_g$, is a realization of the Stone-Čech compactification of $G$.

Proof. Clearly ($1$) $\implies (2)$ and we have seen from Proposition 1 that $(3) \iff (4)$. Also ($2$) $\implies B = A(Y)$ by Proposition 3, but $A(Y) \subseteq A(X^I) = A(X)$ by Proposition 2, and $A(X) \subseteq B$. So ($2$) $\implies (3)$.

We complete the proof by showing $(3) \implies (1)$.

Assume ($3$) holds. Select $\tilde{y}$ such that $\{\sigma_g(\tilde{y}) : g \in G\}$ is dense in $Y$. Since $\tilde{y}$ has domain $G$ and values in $\{0, 1\}$ it is an element of $B$. From ($3$) and by the Stone-Weierstrass theorem, there exist a positive integer $k$, functions $f^1, \ldots, f^k$ in $C$, points $\tilde{x}_1, \ldots, \tilde{x}_k$ in $X$, and a polynomial $p$ in $k$ variables so that

$$|p(f^1_{\tilde{x}_1}(g), \ldots, f^k_{\tilde{x}_k}(g)) - \tilde{y}_g| < \frac{1}{2}$$

for all $g \in G$. Since $X$ is totally disconnected, any two points of $X$ can be distinguished by some continuous function that assumes only two values. Again using the Stone-Weierstrass theorem, we may rechoose $k$, $p$, and the $f^i$, and renumber the $x_i$ so that, in addition to the above inequality, the range $T$ of $p(f^1_{\tilde{x}_1}(g), \ldots, f^k_{\tilde{x}_k}(g))$ is a finite set. Let $\gamma \in (\frac{1}{2}, \frac{3}{2}) \setminus T$ and let $\chi$ be the characteristic function of $[\gamma, \infty)$. Define $\Pi : X^k \to Y$ by $\Pi(x_1, \ldots, x_k)(g) = \chi(p(f^1_{x_1}(g), \ldots, f^k_{x_k}(g)))$. The map $\Pi$ is continuous on $X$ since $\gamma$ is continuous on $T$. Calculate

$$\tau_g(\Pi(x_1, \ldots, x_k))(g) = \Pi(x_1, \ldots, x_k)(gh) = \chi(p(f^1_{x_1}(gh), \ldots, f^k_{x_k}(gh)))$$

$$= \chi(p(f^1(\sigma_{gh}(x_1)), \ldots, f^k(\sigma_{gh}(x_k))))$$

$$= \chi(p(f^1(\sigma_g(\sigma_h(x_1))), \ldots, f^k(\sigma_g(\sigma_h(x_k))))) = \Pi(\sigma^I_h(x_1, \ldots, x_k))(g),$$

so $\Pi$ is a flow mapping. Finally, $\Pi : X^I \to Y$ is surjective since $\Pi(\tilde{x}_1, \ldots, \tilde{x}_k) = \tilde{y}$. 

The hypothesis in Theorem 1 requiring $Y$ to be transitive is impractical as it may not be easily verifiable. In contrast, the transitivity of $Y$ follows from the following modest condition on $G$, when the cardinality of $G$ is infinite, which is easy to check:

**Definition.** $G$ is said to satisfy the Separation Condition if the cardinality $|G|$ of $G$ is infinite and if for all $h,k \in G$, $|\{g \in G : hg = k\}| < |G|$.

**Theorem 2.** If $G$ satisfies the Separation Condition, then the shift space $(Y,G,\tau)$ is transitive.

We first need the lemma:

**Lemma.** Let $G$ be an infinite semigroup and let $G_0$ and $G_1$ be a partition of $G$. Let $y^c$ be the characteristic function of $G_1$. Then $y^c$ has dense $\tau$-orbit if and only if whenever $F_0$ and $F_1$ are disjoint finite subsets of $G$, there is an element $g \in G$ such that $F_0g \subseteq G_0$ and $F_1g \subseteq G_1$.

**Proof (of Lemma).** “if”: Pick $y \in Y$ and let $F$ be a finite subset of $G$. Let $F_i = \{g \in G : yg = i\}, i = 0, 1$. Choose $g \in G$ such that $F_i g \subseteq G_i, i = 0, 1$. Pick $h_i \in F_i$. Then $(\sigma g y^c)h_i = y^c_h = i = yh_i$. This shows that given $y$ and $F$, there is a $g$ such that $\sigma g y^c$ agrees with $y$ on $F$. That is, the orbit of $y^c$ is dense in $Y$.

The converse is proven similarly.

**Proof (of Theorem 2).** We construct sets $G_i^0, i = 0, 1$, inductively, using transfinite induction to account for uncountable $G$. It will be clear from the construction that $G_i^0 \subseteq G_i^1$ if $\eta < \nu$.

For any ordinal $\eta$, we let $\overline{\eta}$ be the set of strict predecessors of $\eta$. Now let $\xi$ be the least ordinal such that $|\overline{\xi}| = |G|$. Let $\mathcal{F}$ consist of all ordered pairs of disjoint finite subsets of $G$. Let $\zeta \rightarrow (F_0^\zeta, F_1^\zeta)$ be a bijective mapping of $\overline{\xi}$ onto $\mathcal{F}$.

Set $G_0^0 = G_1^0 = \emptyset$.

Suppose that for some ordinal $\zeta$ with $|\overline{\zeta}| < |G|$ the sets $G_0^\zeta$ and $G_1^\zeta$ have been defined such that $G_0^\zeta \cap G_1^\zeta = \emptyset$ and $|G_i^\zeta| < |G|, i = 0, 1$. Choose $g_\zeta \in G$ such that the four sets $G_0^\zeta, G_1^\zeta, F_0^{\zeta}g_\zeta, F_1^{\zeta}g_\zeta$ are pairwise disjoint. Such a $g_\zeta$ exists since $|G_0^\zeta \cup G_1^\zeta \cup F_0^{\zeta} \cup F_1^{\zeta}| < |G|$ and since $G$ satisfies the Separation Condition. Set $G_i^{\zeta+1} = G_i^\zeta \cup F_i^{\zeta}g_\zeta$. Then $G_0^{\zeta+1}$ and $G_1^{\zeta+1}$ have been defined, are disjoint, and have cardinality strictly less than that of $G$.

If $\zeta$ is a limit ordinal with $|\overline{\zeta}| < |G|$ and if $G_i^\eta, i = 0, 1, \eta < \zeta$, then set $H_i^{\zeta} = \bigcup_{\eta < \zeta} G_i^\eta$ and choose $g_\zeta$ so that $H_i^{\zeta}, H_i^{\zeta}, F_0^{\zeta}g_\zeta, F_1^{\zeta}g_\zeta$ are pairwise disjoint. Set $G_i^{\zeta} = H_i^{\zeta} \cup F_i^{\zeta}$.

Finally, set $G_0 = \bigcup_{\eta < \zeta} G_0^\eta$ and $G_1 = G \setminus G_0$. Let $y^c$ be the characteristic function of $G_1$. It is now easy to check that if $(F_0^\zeta, F_1^\zeta) \in \mathcal{F}$, then $g_\zeta$ satisfies the condition of the lemma.

We now consider the case $G = (\mathbb{Z}, +)$, the integers under addition. We let $\sigma = \sigma_1$ and then $\sigma^\eta = \sigma_\eta$.

It is well-known that a mixing subshift of finite type has a Cartesian product with the full 2-shift as a factor (see [2]) and so the enveloping semigroup $E$ is the Stone-Čech compactification $\beta$ of $\mathbb{Z}$. The first theorem in [6] implies that no flow “constructed from minimal flows” can have $E = \beta$. Let us be more precise: A bounded function $F : \mathbb{Z} \rightarrow \mathbb{R}$ is called minimal if for every finite $F \subset \mathbb{Z}$ and every
\[ \epsilon > 0 \text{ the set } \{ n \in \mathbb{Z} : |F(n + i) - F(i)| < \epsilon \forall i \in F \} \text{ is syndetic in } \mathbb{Z}. \] Let \( \mathcal{U} \) be the smallest uniformly closed algebra containing every minimal function. It is shown in [6] (Theorem 1) that \( \mathcal{U} \) is properly contained in \( B \), the algebra of bounded real valued functions defined on \( \mathbb{Z} \). A \( \mathbb{Z} \)-flow \( X \) is minimal if there are no closed invariant non-void proper subsets. This condition is equivalent to: For every \( x \in X \) and open set \( U \) containing \( x \), the set \( \{ n \in \mathbb{Z} : a^n(x) \in U \} \) is syndetic. From this it is easy to see that for a minimal flow \( f \) every \( f_x \) is a minimal function. It follows that every minimal flow \( X \) has \( A \neq B \). Moreover, if we take Cartesian products, subflows, or factors of flows satisfying \( A \subseteq \mathcal{U} \) then the resulting flow will also have \( A \subseteq \mathcal{U} \) and so \( E \neq \beta \). We know, both from specific construction (see for instance [7]) and from the Jewitt-Krieger theorem, that there are many minimal subshifts of positive entropy.

Before giving the next example it is convenient to introduce some terminology. Let \((X, d)\) be a compact metric space and let \( \sigma \) be a homeomorphism of \( X \). Say that a point \( x \in X \) has an arithmetic subset \( S = a + L \mathbb{Z} \subseteq \mathbb{Z} \) such that if \( n, m \in S \) then \( d(\sigma^n x, \sigma^m x) < \epsilon \). We say \((X, \sigma)\) clusters arithmetically, or is of class \( A \), if every point of \( X \) has an arithmetically clustered orbit. The property of arithmetically clustered is unchanged if \( d \) is replaced by some other topologically equivalent (hence uniformly equivalent, since \( X \) is compact) metric.

**Proposition 4.**

1. If \((X, \sigma)\) and \((Y, \tau)\) are in \( \mathcal{A} \) then \((X \times Y, \sigma \times \tau)\) is in \( \mathcal{A} \).
2. If \((X, \sigma)\) is in \( S \) and if \((Y, \tau)\) is a factor of \((X, \sigma)\) then \((Y, \tau)\) is in \( S \).
3. The 2-shift does not cluster arithmetically.

**Proof.**

1. Let \((x, y) \in X \times Y\) and let \( \epsilon > 0 \). We use the same symbol \( d \) to denote the metric on each space. It suffices to find an arithmetic set \( S \) such that \( n, m \in S \) implies \( d(\sigma^n x, \sigma^m x) < \epsilon \) and \( d(\tau^n y, \tau^m y) < \epsilon \). Let \( S_x = a + L \mathbb{Z} \) be the appropriate arithmetic set for \( x \) and \( \epsilon \). Choose \( \delta > 0 \) so that if \( y_1, y_2 \) in \( Y \) and \( d(y_1, y_2) < \delta \) then \( d(\tau^k y_1, \tau^k y_2) < \epsilon \) for \( 0 \leq k \leq L - 1 \). Now choose \( S_Y = b + M \mathbb{Z} \) so that \( m, n \in S_Y \) implies \( d(\tau^m y, \tau^n y) < \delta \). Choose \( u \in \mathbb{Z} \) and \( 0 \leq k \leq L - 1 \) so that \( a + uL = b + k \). Let \( S = a + uL + LM \mathbb{Z} \). Observe that \( S \subseteq S_x \) so \( n, m \in S \) implies \( d(\sigma^n x, \sigma^m x) < \epsilon \). Write \( S \) as \( b + k + LM \mathbb{Z} \). If \( n, m \in S \) then \( n - k, m - k \in S_Y \) so \( d(\tau^{n-k} y, \tau^{m-k} y) < \delta \) yielding \( d(\tau^n y, \tau^m y) < \epsilon \).

2. This is a direct consequence of uniform continuity.

3. The point \( 0, 1, 1, 0, 0, 0, 1, 1, 1, \ldots \) does not have an arithmetically clustered orbit.

**Remark.**

Theorem 1 implies that no zero dimensional \( \mathcal{A} \) flow can have its enveloping semigroup equal to \( \beta \). The dimensional assumption is superfluous. Suppose \((X, \sigma)\) clusters arithmetically and suppose the function \( f \) is in the corresponding algebra \( A(X) \). It is easy to show that for every \( \epsilon > 0 \) there is an arithmetic set \( S \) such that if \( n, m \in S \) then \( |f(n) - f(m)| < \epsilon \). It follows that \( A \neq B \) and so \( E \neq \beta \).

The next example, a mild variant of one provided by B. Weiss at the 1995 CBMS conference in Bakersfield, is a non-minimal subshift (of the 2-shift) of positive entropy. It belongs to the class \( A \) and so \( E \neq \beta \).

Let \( m = m_1, m_2, \ldots \in \{0, 1, 2, 3, \ldots, 9\}^\mathbb{N} = M \). With each \( m \in M \) we associate \( I_m \subseteq \mathbb{Z} \) defined by \( i \in I_m \) if and only if there exist \( n \in \mathbb{N} \) and \( k \in \mathbb{Z} \) such that
$|i - k 10^n - \sum_{j=1}^n m_j 10^{i-j-1}| < n$. For example, if $m = 3, 2, 5 \ldots$ then, for any $k$, $3 + 10k \in I_m$, $\{22, 23, 24\} + 100k \subset I_m$, $\{52, 525\} + 1000k \subset I_m$. We let $x \in X \subset \{0, 1\}^\mathbb{Z}$ if there is some $m \in M$ such that $x_i = 0$ for each $i \in I_m$. We will say such an $x$ and $m$ are paired. In general neither $x$ nor $m$ determines the other. We let $\sigma$ be the shift: $\sigma(x)_i = x_{i+1}$.

**Proposition 5.** $X$ is a closed shift invariant set and the flow $(X, \sigma)$ has positive topological entropy.

**Proof.** If $x \in X$ and $m \in M$ are paired in the above manner then $m - 1$ and $m + 1$ are paired with $\sigma(x)$ and $\sigma^{-1}(x)$ respectively, where $m + 1$ is obtained by changing the leading string of nines (if there is one) to zeroes and then increasing the first non-nine digit (if there is one) by 1, and $m - 1$ is defined analogously so that $(m + 1) - 1 = m$.

Now suppose $x^{(\nu)}$ is a sequence of points in $X$ converging to a point $x$ in $\{0, 1\}^\mathbb{Z}$. We will show $x \in X$. Let $m^{(\nu)} \in M$ be paired with $x^{(\nu)}$. Since $M$ is compact, we can assume (passing to a subsequence if necessary) that $m^{(\nu)}$ converges to some $m \in M$. It is easily seen that $x$ and $m$ are paired. This shows $X$ is closed.

Next let $X_1$ be the set of $x$ paired with $1 = 1, 1, 1, \ldots$. If we look at the entries in the interval $[1, 10^k]$ we see that ones are specified at $I_1 \cap [1, 10^k]$ and any pattern of zeroes and ones may appear at the remaining entries. The set $I$ consists of $\{1, 11, 21, \ldots\} \cup \{10, 12, 101, 112, \ldots\} \cup \{109, 113, 1109, 1113, \ldots\} \cup \ldots$. Thus

$$\lceil \text{card}(I \cap [1, 10^k]) \rceil < 10^k \left(\frac{1}{10} + \frac{2}{10^2} + \frac{2}{10^3} + \ldots\right)$$

$$= 10^k \left(\frac{1}{10} + \frac{2}{10^2} \frac{1}{1 - \frac{1}{10}}\right) = \frac{11}{90} \times 10^k.$$

A finite block $x_1, \ldots, x_n$ in $\{0, 1\}^n$ is called an admissible $n$-block if there is some two-side continuation $\ldots, x_1, \ldots, x_n, \ldots$ to an element of $X$. We have just shown that for $n = 2^k$ the number of admissible $n$-blocks is at least $2^\pi n$. This, as is well-known, implies that $\sigma$ has positive entropy.

It is obvious from the construction that $(X, \sigma)$ belongs to $\mathcal{A}$.

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