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A NOTE ON THE DENSITY OF s-DIMENSIONAL SETS

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Abstract. Sets in Euclidean spaces which are measurable with respect to Hausdorff s-dimensional measure with $0 < s < 1$ are shown to have an at most countable set of points where the exact s-density exists and is finite and non-zero.

The purpose of this note is to give a slight improvement to the result of Marstrand [2] which states that for $0 < s < 1$, the set of points of an s-set at which the exact density exists and is finite is of s-measure 0. This result was given a simpler proof by Falconer [1, p. 55–56] and the theorem below uses his notation and definitions. Indeed, this stronger result is patterned on his proof.

Definitions. $B_r(y) = \{x\colon d(x,y) \le r\}$ is the *closed ball* about x of radius r.

$H^s_\delta(E) = \inf \sum_{i=1}^\infty |U_i|$ where the inf is over all countable covers of E with $\operatorname{diam}(U_i) \le \delta$.

$H^s(E) = \lim_{\delta\to 0} H^s_\delta(E)$ is the *Hausdorff outer measure* of E.

$D^s(E,x) = \lim_{r\to 0} H^s(E \cap B_r(x))/(2r)^s$ is the *density* of E at x provided the limit exists.

The *upper density* and *lower density* are defined using the lim sup and lim inf respectively.

Theorem. *Suppose E is an s-measurable set in Euclidean n-dimensional space with $0 < s < 1$. Then the set of points x where the density $D^s(E,x)$ exists and is finite and non-zero is an at most countable set of points.*

Proof. Given a natural number k and $0 < s < 1$, suppose there is a set E in a Euclidean space for which the set of points A where $D^s(E,x)$ is finite and non-zero is an uncountable set. Let A_j be the set of all $x \in A$ for which $r < 1/j$ implies $(2r)^s/j < H^s(E \cap B_r(x)) < (2r)^s \cdot j$. Since $A = \bigcup A_j$ there is a natural number N so that A_N is uncountable and thus there is a point $y \in A_N$ which is an accumulation point of A_N. Let $0 < \eta < 1$ and let $x \in A_N$ with $d(x,y) = r$ and $r(1+\eta) < 1/N$. Let $A_{r,\eta}(y) = B_{r(1+\eta)}(y) \backslash B_{r(1-\eta)}(y)$, the annulus centered at y. Note (as in [1]) that

$$\begin{aligned}(2r)^{-s}H^s(E \cap A_{r,\eta}(y)) &= (2r)^{-s}H^s(E \cap B_{r(1+\eta)}(y)) - (2r)^{-s}H(E \cap B_{r(1-\eta)}(y)) \\ &\to D^s(E,y)((1+\eta)^s - (1-\eta)^s).\end{aligned}$$

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Then, since $B_{r\eta/2}(x) \subset A_{r,\eta}(y)$, it follows that

$$r^s\eta^s/N < H^s(E \cap B_{r\eta/2}(x)) \leq H^s(E \cap A_{r,\eta}(y))$$

and on dividing the terms of this inequality by $(2r)^s$ and letting r approach 0, it follows that

$$\eta^s/(N \cdot 2^s) \leq D^s(E,y)((1+\eta)^s - (1-\eta)^s) = D^s(E,y)(2s\eta + O(\eta^2))$$

which is impossible for sufficiently small η. This contradiction shows that each A_j consists of isolated points and from this it also follows that A is an at most countable set. □

To complete this picture, suppose that $\{x_i\}$ is a sequence of points in a Euclidean space and $\{d_i\}$ is a sequence of extended real numbers with $0 < d_i \leq \infty$. The following example shows that it is possible for a set to have exact s-density d_i at each point x_i.

Example. Given points x_i in Euclidean k-dimensional space, numbers d_i with $0 < d_i \leq \infty$ and $s \in (0,1)$, there is a set E of finite s-measure so that $D^s(E, x_i) = d_i$.

Construction. The set will be constructed on a sequence of line segments parallel to one of the axes. First, let $0 < d < \infty$ and $s \in (0,1)$ be given. Place a closed set X_N of s-measure $2^s \cdot d(N^{-s} - (N+1)^{-s})$ in each interval $((N+1)^{-1}, N^{-1})$. Then $E_d = \bigcup X_i \cup \{0\}$ is a closed set. Since $2^s \cdot d \cdot \sum_{n=N}^{\infty}(1/n^s - 1/(n+1)^s)/(2r)^s = d \cdot N^{-s}/r^s$, if $1/(N+1) \leq r < 1/N$, it follows that

$$d \cdot (N+1)^{-s}/r^s \leq H^s(E_d \cap (-r,r))/(2r)^s \leq d \cdot N^{-s}/r^s.$$

Since the opposite sides of this inequality approach d as r approaches 0, the density of E_d is d at 0. If it is desired to have a set E_∞ of finite measure and s-density ∞ at the origin, let $d_n \uparrow \infty$ and construct closed sets $X_N \subset ((N+1)^{-1}, N^{-1})$ of s-measure $2^s \cdot d_N \cdot (N^{-s} - (N+1)^{-s})$. Then the same calculations show that the set $E_\infty = \bigcup X_i \cup \{0\}$ has density ∞ at 0. If the d_n are chosen so that

$$\sum d_n(n^{-s} - (n+1)^{-s}) < \infty,$$

then the s-measure of the set E_∞ will be finite. Note that E_d can be considered to be a subset of E^k of points on the first coordinate axis with s-density d at the origin. Suppose $\{x_i\}$ and $\{d_i\}$ are given. Let $E_1 = F_1 = E_{d_1} + x_1$. Let $r_1 = 1$ and let $r_n = \min\{2^{-n}, d(x_i, x_j) : i < j \leq n\}$. Suppose E_{n-1} and F_{n-1} have been constructed. Let $\varepsilon_n > 0$ with $\varepsilon_n < r_n/2^n$ so that $H^s(B_{\varepsilon_n}(x_n) \cap E_{n-1}) < 2^{-n} \cdot r_{n+1}^s$ and $H^s(B_{\varepsilon_n}(0) \cap E_{d_n}) < 2^{-n} \cdot r_{n+1}^s$. Let $F_n = (E_{d_n} \cap B_{\varepsilon_n}(0)) + x_n$ and let $E_n = (E_{n-1} \backslash B_{\varepsilon_n}(x_n)) \cup F_n$. Then $E = \limsup E_n$ is the required set. It clearly has finite s-measure and if $r > 0$ is given with $r_{n+1}(1-2^{-n}) \leq r < r_n(1-2^{-n})$, then $H^s(F_i \cap B_r(x_i)) - 2^{-n+1} \cdot r_{n+1}^s \leq H^s(E \cap B_r(x_i)) \leq H_s(F_i \cap B_r(x_i)) + 2^{-n+1} \cdot r_{n+1}^s$. Dividing the terms in this inequality by $(2r)^s$ and letting $r \to 0$, the two sides of the resulting inequality approach d_i. It follows that the s-density of E at x_i is d_i.

Some natural questions which arise are:

1. Is it possible that the set in the example above be constructed so that it has the exact preassigned s-density only at the preassigned points?

2. What kind of structure does the set of points where the s-density is infinite have?

3. Is there a generalization for $s > 1$; for example, is it possible that the set of points is always of σ-finite $n-1$ measure where $n-1 < s < n$?

References

1. K. J. Falconer, *The Geometry of Fractal Sets*, Cambridge Univ. Press, 1985. MR **88d:**28001
2. J. M. Marstrand, *Some fundamental properties of plane sets of fractional dimension*, Proc. Lond. Math. Soc. (3) **4** (1954), 257–302. MR **16:**121g

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