ON THE EXTENDED HILBERT’S INEQUALITY

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Abstract. In this paper, it is shown that the extended Hilbert’s inequality for double series can be refined by the aid of the Euler-Maclaurin summation formula. The extreme cases \( p \to 1^+ \) and \( q \to +\infty \) are discussed.

1. Introduction

Let \( \{a_n\} \) and \( \{b_n\} \) be two sequences of nonnegative real numbers, \( p > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). If \( \sum_{n=1}^{\infty} a_n^p < +\infty \) and \( \sum_{n=1}^{\infty} b_n^q < +\infty \), then an extended Hilbert’s inequality may be written in the form

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} \leq \left( \frac{\pi}{\sin \frac{\pi}{p}} \right) \left( \sum_{n=1}^{\infty} a_n^p \right)^{\frac{1}{p}} \left( \sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}}. \tag{1}
\]

As is well known, the constant factor \( \pi/\sin \frac{\pi}{p} \) contained in (1) is best possible. In other words, \( \pi/\sin \frac{\pi}{p} \) cannot be replaced by any positive number smaller than it (cf. [1], [2]). But we may move the factor \( \pi/\sin \frac{\pi}{p} \) of the right-hand side of (1) to the inside of the summation and write it in the following form:

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} \leq \left\{ \sum_{n=1}^{\infty} \left( \pi/\sin \frac{\pi}{p} - \alpha_n(q) \right) a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \left( \pi/\sin \frac{\pi}{p} - \alpha_n(p) \right) b_n^q \right\}^{\frac{1}{q}} \tag{2}
\]

where \( \alpha_n(r) \downarrow 0 \) (\( r = p, q \)). Clearly, it will offer a refined form of (1). In this paper it will be shown that we can take \( \alpha_n(r) = \lambda/n^{1-\frac{1}{r}} \), where \( \lambda \) is a positive real number that is independent of \( r \). Furthermore, we prove also that \( \lambda = 1 - \gamma \), where \( \gamma \) is the Euler constant.

Before proving our results we need to define some functions. Throughout this paper we assume that \( x \in [1, +\infty) \) and \( r \in (1, +\infty) \).

Let us define the following functions:

\[
u(x) = x^{1-\frac{1}{r}} I(x),
\]

where \( I(x) \) is defined by

\[
I(x) = \int_0^{\frac{1}{x}} \frac{1}{1+t} \left( \frac{1}{t} \right)^{\frac{1}{r}} dt.
\]

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and

\[ v(x) = \int_{1}^{\infty} \rho(t) F(x, t) \, dt, \tag{5} \]

where \( \rho(t) \) and \( F(x, t) \) are defined respectively by

\[ \rho(t) = t - [t] - \frac{1}{2} \quad \text{and} \quad F(x, t) = \frac{(r + 1)xt + x^2}{r(x + t)^{2r + 1}}. \]

For convenience we define

\[ \lambda_r(x) = u(x) + v(x) - \frac{x}{2(x + 1)} \tag{6} \]

where \( u(x) \) and \( v(x) \) are defined respectively by (3) and (5). Particularly, in the case \( x = 1 \), \( \lambda_r(1) \) is denoted by \( \lambda(r) \). We know from (6) that

\[ \lambda(r) = u(1) + v(1) - \frac{1}{4}. \tag{7} \]

We will show that \( \lambda(r) \) can be written in the form

\[ \lambda(r) = J(r) + R(r), \tag{8} \]

where \( J \) and \( R \) are defined respectively by

\[ J(r) = \int_{0}^{1} \frac{1}{1 + t} \left( \frac{1}{r} \right)^{\frac{1}{2}} \, dt - \frac{13r + 2}{48r} \]

and

\[ R(r) = \frac{\theta}{5760} \left( 3 + \frac{20}{r} + \frac{18}{r^2} + \frac{4}{r^3} \right) \quad (0 < \theta < 1). \]

2. LEMMAS

The aim of the section is to prove the following inequalities are valid:

\[ \lambda_r(x) \geq \lambda(r) > \lambda \]

where \( \lambda \) is an infimum of \( \lambda(r) \).

**Lemma 1.** Let \( I(x) \) be the function defined by (4). Then

\[ I(x) \geq \frac{r(2r - 1)x^{\frac{1}{2}}}{(r - 1)((2r - 1)x + r - 1)}. \tag{9} \]

**Proof.** Using integration by parts we obtain

\[ I(x) = \frac{rx^{\frac{1}{2}}}{(r - 1)(x + 1)} + \frac{r}{r - 1} K(x), \tag{10} \]

where \( K(x) \) is defined by

\[ K(x) = \int_{0}^{\frac{1}{x}} \frac{t^{1 - \frac{1}{r}}}{(1 + t)^2} \, dt. \]

Define the functions \( f \) and \( g \) respectively by

\[ f(t) = \frac{1}{(1 + t)^2} \left( \frac{1}{x} \right)^{1 - \frac{1}{r}} \quad \text{and} \quad g(t) = (xt)^{1 - \frac{1}{r}}, \quad t \in \left[ 0, \frac{1}{x} \right]. \]
Evidently $f(t)$ is nonnegative and monotone decreasing in $[0, \frac{1}{r}]$ and $g(t)$ satisfies the constraint $0 \leq g(t) \leq 1$. According to Steffensen's inequality we have

\begin{equation}
\int_{\frac{1}{r}-c}^{\frac{1}{r}} f(t) \, dt \leq \int_{0}^{\frac{1}{r}} f(t) g(t) \, dt = K(x) \leq \int_{0}^{c} f(t) \, dt,
\end{equation}

where $c = \int_{0}^{\frac{1}{r}} g(t) \, dt = \int_{0}^{\frac{1}{r}} (xt)^{1-\frac{1}{r}} dt = \frac{r}{(2r-1)^{\frac{1}{r}}}$.

Hence we obtain from (5) that

\begin{equation}
\text{Lemma 1.}
\end{equation}

By Lemma 1 we obtain easily the following inequality:

\begin{equation}
\text{Lemma 2.}
\end{equation}

Substituting it in the second term of the right-hand side of (10) we obtain after simplification that (9) is valid.

\begin{proof}
At first, consider the function $u(x)$ defined by (3). Taking derivatives and after simplification we have

\begin{equation}
u'(x) = u'(x) = r - \frac{1}{r} x^{-\frac{1}{r}} I(x) - \frac{x^{1-\frac{1}{r}}}{1+x}.
\end{equation}

By Lemma 1 we obtain easily the following inequality:

\begin{equation}
u'(x) \geq r/(x+1)((2r-1)x + (r-1)).
\end{equation}

Define the functions $F_1$ and $F_2$ by

\begin{equation}
F_1(t) = \frac{r+1}{r(x+t)^{2t^{1/r}}}
\end{equation}

and

\begin{equation}
F_2(t) = \frac{2x}{(x+t)^{3t^{1/r}}}, \quad t \in [1, +\infty).
\end{equation}

Obviously $F_i(t) \downarrow 0$ ($t \to +\infty$) and after calculations $F_i''(t) > 0$. In the paper [4] it has been proved that

\begin{equation}
-\frac{1}{8} F_1(1) < \int_{1}^{\infty} \rho(t) F_i(t) \, dt < -\frac{1}{12} F_i \left( \frac{3}{2} \right),
\end{equation}

where $\lambda_i(n)$ is defined by (7).

By (11) we obtain easily the following inequality:

\begin{equation}
\text{Lemma 1.}
\end{equation}

Hence we obtain from (5) that

\begin{equation}
\nu'(x) = \int_{1}^{\infty} \rho(t) \left( \frac{x + rt + t - xr}{r(x+t)^{3t^{1/r}}} \right) \, dt
\end{equation}

\begin{equation}
= \int_{1}^{\infty} \rho(t) F_1(t) \, dt - \int_{1}^{\infty} \rho(t) F_2(t) \, dt
\end{equation}

\begin{equation}
> -\frac{1}{8} F_1(1) + \frac{1}{12} F_2 \left( \frac{3}{2} \right)
\end{equation}

\begin{equation}
= -\frac{r+1}{8r(x+1)^2} + \frac{4x}{3(2x+3)^3} \left( \frac{2}{3} \right)^{\frac{1}{r}}.
\end{equation}

We can obtain from (13) and (14) that

\begin{equation}
\lambda_i(x) = u'(x) + v'(x) - \frac{1}{2(x+1)^2}
\end{equation}

\begin{equation}
> \frac{(-2r^2 + 3r + 1)x + (3r^2 + 4r + 1)}{8r(x+1)^2((2r-1)x + (r-1))} + \frac{4x}{3(2x+3)^3} \left( \frac{2}{3} \right)^{\frac{1}{r}}.
\end{equation}
By direct computations we have the following conclusions (see Notes at the end of this paper):

When \( r \geq 4, (\frac{2}{3})^\frac{1}{r} > \frac{2}{3}, \lambda_r'(x) > 0 \) is true. And when \( 1 < r < 4, (\frac{2}{3})^\frac{1}{r} < \frac{2}{3}, \lambda_r'(x) > 0 \) is also true. This implies that \( \lambda_r(x) \) is monotone increasing. Whence (12) is valid.

**Lemma 3.** Let \( \lambda(r) \) be the function defined by (8). Then if \( \lambda = \inf \{ \lambda(r) \} \) we have

\[
\lim_{r \to \infty} \lambda(r) = \lambda.
\]

**Proof.** Evidently the function \( J(r) \) is continuously differentiable in \((1, +\infty)\). Hence

\[
J'(r) = \frac{1}{r^2} \int_0^1 \ln t \left( \frac{1}{t} \right)^{\frac{1}{r}} dt + \frac{1}{24r^2}.
\]

Substituting \( t = e^{-y} \) in (15) we obtain easily that

\[
J'(r) = -\frac{1}{r^2} \int_0^{+\infty} ye^{-\alpha y} \left( \frac{1}{1 + e^{-y}} \right) dy + \frac{1}{24r^2} < 0,
\]

where \( \alpha = 1 - \frac{1}{r} \). Hence the function \( J(r) \) is monotone decreasing. Clearly \( R(r) \) is also monotone decreasing. Thus

\[
\lambda = \inf \{ \lambda(r) \} = \lim_{r \to \infty} \lambda(r). \quad \square
\]

By Lemma 3, we obtain at once the following results:

\[
\lambda(r) > \lambda = \ln 2 - \frac{13}{48} + \frac{\theta}{1920} \quad (0 < \theta < 1).
\]

### 3. Main Results

**Theorem 1.** Let \( q \geq p > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). If \( 0 < \sum_{n=1}^{\infty} a_n^p < +\infty \) and \( 0 < \sum_{n=1}^{\infty} b_n^q < +\infty \), then

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m b_n \left( \frac{m+n}{m+n} \right) < \left\{ \sum_{n=1}^{\infty} \left( \frac{\pi}{\sin \frac{\pi}{p} - \frac{\lambda}{n^r}} \right) a_n^p \right\} ^\frac{1}{p} \left\{ \sum_{n=1}^{\infty} \left( \frac{\pi}{\sin \frac{\pi}{q} - \frac{\lambda}{n^r}} \right) b_n^q \right\} ^\frac{1}{q},
\]

where \( \lambda = 1 - \gamma \) and \( \gamma \) is the Euler constant. \( \lambda \) is the largest constant that keeps (17) valid and is independent of \( r \) (\( r = p, q \)).
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Proof. We may apply Hölder’s inequality to estimate the left-hand side of (17) as follows:

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_mb_n}{m+n} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m}{(m+n)^\frac{1}{p}} \left( \frac{m}{n} \right)^{\frac{1}{q}} \frac{b_n}{(m+n)^\frac{1}{q}} \leq \left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m^p}{m+n} \left( \frac{m}{n} \right)^{\frac{1}{q}} \right\} \left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{b_n^q}{m+n} \left( \frac{m}{n} \right)^{\frac{1}{q}} \right\}^{\frac{1}{q}} \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_r(n) a_n^p \right)^{\frac{1}{p}} \left( \sum_{n=1}^{\infty} \omega_p(n) b_n^q \right)^{\frac{1}{q}},
\]

where \( \omega_r(n) \) \( (r = p, q) \) is defined by

\[
\omega_r(n) = \sum_{m=1}^{\infty} \frac{1}{m+n} \left( \frac{n}{m} \right)^{\frac{1}{q}}.
\]

Applying the Euler-Maclaurin summation formula to \( \omega_r(n) \) and using the relation \( \sin \frac{\pi}{p} = \sin \frac{\pi}{q} \), we obtain

\[
\omega_r(n) = \int_1^{\infty} g(t) \, dt + \int_1^{\infty} \rho(t)g'(t) \, dt = \int_0^{\infty} g(t) \, dt - \int_0^1 g(t) \, dt + \int_1^{\frac{1}{t}} g(t) \, dt + \int_1^{\infty} \rho(t)g'(t) \, dt,
\]

where the function \( g(t) \) is defined by

\[
g(t) = \frac{1}{n+t} \left( \frac{n}{t} \right)^{\frac{1}{q}}, \quad t \in (0, +\infty).
\]

Note that

\[
\int_0^{\infty} g(t) \, dt = \pi/\sin \frac{\pi}{p}, \quad \int_0^1 g(t) \, dt = \int_0^{\frac{1}{t}} \frac{1}{1+t} \left( \frac{1}{t} \right)^{\frac{1}{q}} \, dt = u(n)/n^{1-\frac{1}{q}}
\]

and

\[
\int_1^{\infty} \rho(t)g'(t) \, dt = v(n)/n^{1-\frac{1}{q}},
\]

where \( u(x) \) and \( v(x) \) are the functions defined by (3) and (5) respectively. Hence

\[
\omega_r(n) = \pi/\sin \frac{\pi}{p} - \left( u(n) + v(n) - \frac{n}{2(n+1)} \right) / n^{1-\frac{1}{q}}
\]

(18)

\[
= \pi/\sin \frac{\pi}{p} - \lambda_r(n)/n^{1-\frac{1}{q}},
\]

where \( \lambda_r(n) \) is the function defined by (6).

In view of (12) we have

\[
\omega_r(n) \leq \pi/\sin \frac{\pi}{p} - \lambda(r)/n^{1-\frac{1}{q}}.
\]
When $n = 1$ it follows from (18) that
\[
\lambda(r) = \lambda_r(1) = \pi / \sin \frac{\pi}{p} - \omega_r(1).
\]
Applying the Euler-Maclaurin summation formula to $\omega_r(1)$ we have
\[
\omega_r(1) = \sum_{m=1}^{\infty} \frac{1}{1 + m} \left( \frac{1}{m} \right)^{\frac{1}{2}} = \int_{1}^{\infty} f(t) \, dt + \frac{1}{2} f(1) + \sum_{k=1}^{s-1} -\rho_s,
\]
where $f(t) = \frac{1}{1 + t} \left( \frac{1}{t} \right)^{\frac{1}{2}}$, $\sum_{k=1}^{s-1} = \sum_{k=1}^{s-1} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(1)$ and the $B_j$'s are the Bernoulli numbers, viz. $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$ etc., and $\rho_s$ is the remainder of the form
\[
\rho_s = \frac{B_{2s}}{(2s)!} f^{(2s-1)}(1) \quad (0 < \theta < 1).
\]
For $s = 2$ we obtain (8) from (20). By virtue of Lemma 3 and (19) we get that
\[
\omega_r(n) < \pi / \sin \frac{\pi}{p} - \lambda / n^{1 - \frac{1}{2}}.
\]
It remains to show that $\lambda = 1 - \gamma$, where $\gamma$ is the Euler constant. For $n = 1$, using the Euler-Maclaurin summation formula we obtain from (18) that
\[
\lambda(r) = \pi / \sin \frac{\pi}{p} - \left\{ \sum_{m=1}^{k-1} \frac{1}{1 + m} \left( \frac{1}{m} \right)^{\frac{1}{2}} + \sum_{m=k}^{\infty} \frac{1}{1 + m} \left( \frac{1}{m} \right)^{\frac{1}{2}} \right\}
\]
\[
= \pi / \sin \frac{\pi}{p} - \left\{ \sum_{m=1}^{k-1} \frac{1}{1 + m} \left( \frac{1}{m} \right)^{\frac{1}{2}} + \int_{k}^{\infty} f(t) \, dt + \frac{1}{2} f(k) - \frac{\theta}{12} f'(k) \right\}
\]
\[
= \int_{0}^{k} f(t) \, dt - \sum_{m=1}^{k-1} \frac{1}{1 + m} \left( \frac{1}{m} \right)^{\frac{1}{2}} - \frac{1}{2} f(k) + \frac{\theta}{12} f'(k) \quad (0 < \theta < 1),
\]
where $f(t) = \frac{1}{1 + t} \left( \frac{1}{t} \right)^{\frac{1}{2}}$. In accordance with the definition of the Euler constant $\gamma$, i.e.
\[
\sum_{m=0}^{k-1} \frac{1}{1 + m} = \gamma + \ln(k - 1) + \varepsilon_{k-1} \quad (\varepsilon_{k-1} \to 0, \text{ if } k \to +\infty)
\]
and by Lemma 3 we obtain from (22)
\[
\lambda = \lim_{r \to \infty} \lambda(r) = \int_{0}^{k} \frac{1}{1 + t} \, dt - \sum_{m=1}^{k-1} \frac{1}{1 + m} - \frac{1}{2(k + 1)} - \frac{\theta}{12(1 + k)^2}
\]
\[
= 1 - \gamma + \Delta R,
\]
where \( \Delta R \) is the error of the form
\[
\Delta R = \ln \frac{k+1}{k-1} - \varepsilon_{k-1} - \frac{1}{2(1+k)} - \frac{\theta}{12(1+k)^2} \quad (0 < \theta < 1).
\]
This implies that the bigger we take the value of \( k \), the smaller the value of \( |\Delta R| \).

Let \( k \to +\infty \). Then we obtain that \( \lambda = 1 - \gamma \).

Based on Lemma 3 and (21), it follows that \( \lambda \) is the largest constant that keeps (17) valid and is independent of \( r \) \((r = p, q)\).

Thus we have completed the proof of the theorem.

The value of \( \lambda \) is given numerically as follows:
\[
\lambda = 0.422784335098467 \ldots
\]

In particular, in the case \( r = 2 \), it follows from (8) that
\[
\lambda(2) = J(2) + R(2) = \frac{\pi}{2} - \frac{7}{24} + \frac{\theta}{320}.
\]

In view of (19) we have
\[
\omega_2(n) \leq \pi - \lambda(2)/\sqrt{n}.
\]

Therefore we obtain a sharp result of Hilbert’s inequality.

**Theorem 2.** If \( 0 < \sum_{n=1}^{\infty} a_n^2 < +\infty \) and \( 0 < \sum_{n=1}^{\infty} b_n^2 < +\infty \), then
\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \left\{ \sum_{n=1}^{\infty} (\pi - \alpha/\sqrt{n}) a_n^2 \right\}^{\frac{1}{2}} \left\{ \sum_{n=1}^{\infty} (\pi - \alpha/\sqrt{n}) b_n^2 \right\}^{\frac{1}{2}},
\]
where \( \alpha = \frac{\pi}{2} - \frac{7}{24} + \frac{\theta}{320} \) \((0 < \theta < 1)\).

Finally, the extreme cases \( p \to 1^+ \) and \( q \to +\infty \) are discussed. Note that \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( q \geq p > 1 \). In the paper [3] it has been proved that \( \lambda(p) > \frac{1}{p-1} \), where \( \lambda(p) \) is defined by (20). Now we may prove that \( \lambda(p) \sim \frac{1}{p-1} \) when \( p \to 1^+ \).

In fact, for \( r = p \) and \( x = 1 \) we consider the function defined by (4) and denote it by \( h(p) \). We have
\[
h(p) = \int_0^1 \frac{1}{1+t} \left( \frac{1}{t} \right)^{\frac{1}{p}} \, dt.
\]

From (10) we obtain
\[
h(p) = \frac{p}{2(p-1)} + \frac{p}{p-1} k(1), \quad \text{where } k(1) = \int_0^1 \frac{t^{1-\frac{1}{p}}}{(1+t)^2} \, dt.
\]

Use (11) to estimate \( k(1) \). When \( x = 1 \) we have
\[
c = \int_0^1 g(t) \, dt = \int_0^1 t^{1-\frac{1}{p}} \, dt = \frac{p}{2(p-1)},
\]
\[
\int_{1-c}^1 f(t) \, dt = \int_{1-c}^1 \frac{dt}{(1+t)^2} = \frac{p}{2(3p-2)}
\]
and \( \int_c^1 f(t) \, dt = \frac{p}{3p-1} \). Hence
\[
\frac{p}{2(3p-2)} \leq k(1) \leq \frac{p}{3p-1}.
\]
Since \( \lim_{p \to 1^+} \frac{p}{p^2 - 2} = \lim_{p \to 1^+} \frac{p}{p - 1} = \frac{1}{2} \), it follows that \( \lim_{p \to 1^+} k(1) = \frac{1}{2} \).

Whence \( \lim_{p \to 1^+} (p - 1) h(p) = 1 \). It follows from (8) that \( \lim_{p \to 1^+} \lambda(p)/\frac{1}{p - 1} = 1 \).

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**Notes**

\( \lambda'(x) > g(x) \), where

\[
g(x) = \frac{(-2r^2 + 3r + 1)x + (3r^2 + 4r + 1)}{8r(x + 1)^2((2r - 1)x + (r - 1))} + \frac{4x}{3(2x + 3)^3} \left( \frac{2}{3} \right)^2, \quad r > 1, x \geq 1.
\]

When \( r \geq 4 \), \( \left( \frac{2}{3} \right)^2 > \frac{9}{16} \). Hence

\[
g(x) > \frac{(-2r^2 + 3r + 1)x + (3r^2 + 4r + 1)}{8r(x + 1)^2((2r - 1)x + (r - 1))} + \frac{6x}{5(2x + 3)^3}
\]

\[
= \frac{A_1x^4 + A_2x^3 + A_3x^2 + A_4x + A_5}{40r(x + 1)^2(2x + 3)^3((2r - 1)x + (r - 1))} > 0,
\]

where

\[
A_1 = 16r^2 + 72r + 40, \quad A_2 = 556r + 220,
A_3 = 192r^2 + 1386r + 450, \quad A_4 = 588r^2 + 1437r + 405,
A_5 = 13r(3r^2 + 4r + 1)
\]

When \( 1 < r < 4 \), \( \left( \frac{2}{3} \right)^\frac{1}{r} > \frac{2}{3} \). Hence

\[
g(x) > \frac{-2r^2 + 3r + 1)x + (3r^2 + 4r + 1)}{8r(x + 1)^2((2r - 1)x + (r - 1))} + \frac{8x}{9(2x + 3)^3}
\]

\[
= \frac{B_1x^4 + B_2x^3 + B_3x^2 + B_4x + B_5}{72r(x + 1)^2(2x + 3)^3((2r - 1)x + (r - 1))} > 0,
\]

where

\[
B_1 = -16r^2 + 152r + 72, \quad B_2 = -112r^2 + 1068r + 396,
B_3 = 256r^2 + 2562r + 810, \quad B_4 = 1036r^2 + 2609r + 729,
B_5 = 243(3r^2 + 4r + 1).
\]

Consequently, we have \( \lambda'(x) > g(x) > 0 \).

**References**


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