SIMPLE QUOTIENTS OF HYPERBOLIC 3-MANIFOLD GROUPS

D. D. LONG AND A. W. REID

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Abstract. We show that hyperbolic 3-manifolds have residually simple fundamental group.

1. Introduction

Let $G$ be a finitely generated group and $X$ a property of groups, e.g. finite, simple, $p$-group. $G$ is said to be residually $X$, if for any element $g \neq 1$, there is a group $H$ with property $X$ and a surjective homomorphism $\phi : G \to H$ such that $\phi(g) \neq 1$.

Of interest to us are residual properties of groups $\pi_1(M)$ where $M$ is a compact orientable 3-manifold with infinite fundamental group. Now it is well-known that if $M$ is a hyperbolic 3-manifold, that is the quotient of hyperbolic 3-space by a torsion-free Kleinian group, then $\pi_1(M)$ is residually finite. In this note we prove a much stronger result which seems to have been unnoticed previously. First we make a definition.

Definition 1.1. Let $M$ be a compact orientable 3-manifold with infinite fundamental group and $\rho : \pi_1(M) \to SL(2, \mathbb{C})$ a faithful representation whose image lies in $SL(2, \mathbb{Q})$, where $\mathbb{Q}$ is the algebraic closure of $\mathbb{Q}$. In this case we define $\rho$ to be an algebraic representation.

With this we can state:

Theorem 1.2. Let $M$ be a compact orientable 3-manifold such that $\pi_1(M)$ admits an algebraic representation. Then $\pi_1(M)$ is residually simple.

A particular case of this is:

Corollary 1.3. Let $M$ be a finite volume hyperbolic 3-manifold. Then $\pi_1(M)$ is residually simple.

In fact we shall show more; the simple groups will all be of the type $PSL(2, F)$ for finite fields $F$ of prime cardinality. A corollary of Theorem 1.2 is a new proof of a result originally observed by Magnus, which follows by taking $M$ to be a handlebody:

Corollary 1.4. Nonabelian free groups are residually simple.

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2. General remarks

We start with some general remarks about algebraic representations of 3-manifold groups, and groups $PSL(2, \mathbb{F})$ which do not necessarily surject. This is just a reformulation of classical notions about linear groups; cf. [6]. For details on number fields and their completions, see [2] for instance.

Let $M$ be a compact orientable 3-manifold, and $\rho$ an algebraic representation of $\pi_1(M)$ into $SL(2, \mathbb{C})$. Denote the image of the group $\pi_1(M)$ under $\rho$ by $\Gamma$. Let $k = \mathbb{Q}(tr\gamma : \gamma \in \Gamma)$ denote the trace-field of $\Gamma$. Since $\rho$ is an algebraic representation, $k$ is a finite extension of $\mathbb{Q}$. Let $A\Gamma$ be the algebra

$$\{ \sum a_i\gamma_i : a_i \in k, \gamma_i \in \Gamma \}. $$

This is a quaternion algebra over $k$ as follows from [1]. We remark that for finite volume hyperbolic 3-manifolds $k$ is always a finite extension of $\mathbb{Q}$; cf. [4], Proposition 6.7.4.

By the classification theorem for quaternion algebras $A\Gamma$ is unramified at all but a finite number of places of $k$, [5]. In particular, for all but a finite number of prime ideals $\wp$ of $k$, $A \otimes_k k_{\wp} \cong M(2, k_{\wp})$. Thus on specifying an isomorphism between $A \otimes_k k_{\wp}$ and $M(2, k_{\wp})$, we induce a representation of $\Gamma$ into $SL(2, k_{\wp})$.

Since $M$ is compact, $\Gamma$ is finitely generated and finitely presented, therefore for all but finitely many prime ideals, we actually induce a representation of $\Gamma$ into $SL(2, O_{\wp})$ where $O_{\wp}$ are the $\wp$-adic integers in $k_{\wp}$, since only finitely many $k$-primes can divide denominators of elements of $\Gamma$.

Denote by $\pi_{\wp}$ a local uniformizing parameter for $O_{\wp}$. The unique maximal ideal of $O_{\wp}$ is $\pi O_{\wp}$, and $O_{\wp}/\pi O_{\wp}$ is a finite field with $p^n$ elements where $\wp$ divides $p$ for a rational prime $p$ and $n$ is the inertial degree of $\wp$. Therefore reduction induces a homomorphism (so far, not necessarily surjective) of $\Gamma$ into $SL(2, \mathbb{F}_{p^n})$, where $\mathbb{F}_{p^n}$ is the finite field with $p^n$ elements. By composing the map $\pi_1(M) \to \Gamma$, with the above, and then projectivising, we get a homomorphism $\phi_{\wp}$ of $\pi_1(M)$ into $PSL(2, \mathbb{F})$, for infinitely many finite fields $\mathbb{F}$.

**Lemma 2.1.** There are infinitely many $k$-primes $\wp$ such that the homomorphisms $\phi_{\wp}$ constructed above are nontrivial and map $\pi_1(M)$ into $PSL(2, \mathbb{F})$ where $\mathbb{F}$ has prime cardinality.

**Proof.** It is a well-known consequence of how prime ideals behave in finite extensions of $\mathbb{Q}$ that there are infinitely many rational primes that split completely in the finite extension $k/\mathbb{Q}$; see [2] Theorem 4.12 for example. Now a rational prime $p$ splits completely if and only if the inertial degree of the $k$-prime divisors of $p$ are all equal to 1. In particular we deduce that there are infinitely many rational primes $p$ with $\wp|p$ such that $\phi_{\wp}$ maps $\pi_1(M)$ into $PSL(2, \mathbb{F}_p)$ where $\mathbb{F}_p$ has $p$ elements.

It is also easy to see that infinitely many of these homomorphisms are non-trivial. For if $\gamma \in \Gamma$ is given, reduction of, for example, the $(1, 2)$-entry of $\gamma$ will be non-zero for all but a finite number of $k$-primes—since entries of $\gamma$ will be $\wp$-adic units for all but a finite number of $\wp$. Hence, the number of $k$-primes such that $\phi_{\wp}(\gamma)$ is trivial is finite.

In particular Lemma 2.1 implies that we may construct infinitely many non-trivial representations of $\pi_1(M)$ into groups $PSL(2, \mathbb{F})$ where the cardinalities of $\mathbb{F}$ are distinct primes.
3. Proof of Theorem 1.2

To prove Theorem 1.2 we shall make use of the description of subgroups of the groups $\text{PSL}(2, F)$ where $|F|$ is of odd prime cardinality. The following is deduced from [3], Theorem 6.25, together with the observation that all abelian subgroups of such $\text{PSL}(2, F)$ are cyclic.

**Theorem 3.1.** Let $p$ be an odd rational prime; then a complete list of subgroups of $\text{PSL}(2, F)$ where $|F| = p$ is

1. Cyclic groups of order $p$ and order $n$ where $n$ divides $\frac{(p+1)}{2}$.
2. Dihedral groups of order $n$ where $n$ is as in 1.
3. Semi-direct products of cyclic groups of order $p$ with cyclic groups of order $(p - 1)/2$.
4. $A_4$, $S_4$ or $A_5$.

We now show that the homomorphisms $\phi_\wp$ constructed above actually surject infinitely many groups $\text{PSL}(2, F)$ as in the statement of Lemma 2.1.

The group $\Gamma$ is never soluble of any finite degree. This follows for example by the fact that they contain free non-abelian groups, as they are non-elementary subgroups of $\text{SL}(2, \mathbb{C})$. Thus for the remainder of the proof, we fix some nontrivial element $\alpha$ which lies deep in the solubility series of $\pi_1(M)$.

Now suppose then that we are given some element $\gamma \in \Gamma$. As observed above, we can find (infinitely many) $k$-primes $\wp$ so that $\phi_\wp(\gamma) \neq 1$ and $\phi_\wp(\alpha) \neq 1$.

Since the homomorphic image of a term in the solubility series for a group lies in the same term of the solubility series of the image, the fact that the element $\alpha$ maps nontrivially means that the image of the group $\pi_1(M)$ cannot be of type 1, 2, 3 nor $A_4$ or $S_4$ in the list provided by Theorem 3.1 since these are all soluble of small fixed degree.

Thus the map $\phi_\wp$ will be shown to have been a surjection if we show that $\phi_\wp(\Gamma) \neq A_5$ for infinitely many $\wp$. Now if infinitely many of the homomorphisms constructed surject $A_5$, then since there are only finitely many normal subgroups in $\Gamma$ of index 60, it follows that for infinitely many of these homomorphisms the kernels coincide. However this is impossible. These homomorphisms were constructed by reducing $\Gamma$ modulo $\pi_\wp O_\wp$, hence if infinitely many homomorphisms had the same kernel this would mean that the elements in this matrix group were congruent to the identity modulo infinitely many $\pi_\wp O_\wp$, which is clearly false.

Thus we have $\phi_\wp$ that surjects $\pi_1(M)$ onto some $\text{PSL}(2, F)$ and which maps $\gamma$ non-trivially.

In fact we may conclude that under the homomorphisms $\phi_\wp$ constructed above, $\pi_1(M)$ surjects infinitely many of the simple groups $\text{PSL}(2, F)$, with $|F|$ of odd prime cardinality.

4. Application

A motivation for this result arises from trying to show that covers of hyperbolic 3-manifolds have positive first Betti number. With this in mind, an application of this result stems from the following question raised by D. Cooper. Here $inj(M)$ denotes the injectivity radius of $M$, which is simply half the length of the shortest closed geodesic in $M$.

**Question.** Is there a number $K > 0$ so that if $M$ is a closed hyperbolic 3-manifold and $inj(M) > K$, then $\text{rank}(H_1(M; \mathbb{Q})) > 0$?
An affirmative answer to this question, taken with the fact that \( \pi_1(M) \) is residually finite, implies in particular that every closed hyperbolic 3-manifold has a finite sheeted covering with positive first Betti number. However our main theorem shows that actually more would be true.

**Corollary 4.1.** If the above question has an affirmative answer, then every rational hyperbolic homology 3-sphere has infinite virtual Betti number.

**Proof.** We recall that \( M \) is said to have infinite virtual Betti number if given any integer \( N \), one can find a finite sheeted covering of \( M \) whose first Betti number is larger than \( N \). Equivalently, \( M \) has infinite virtual Betti number if the rank of \( H_2(\tilde{M}; \mathbb{Z}) \) is unbounded as \( \tilde{M} \) ranges over all finite covers of \( M \).

Since, given any constant \( C \), \( M \) has only finitely many geodesics of length at most \( C \), our main theorem implies that one can find regular covers of \( M \) having arbitrarily large injectivity radius for which the group of covering transformations has the form \( PSL(2, \mathbb{F}) \). An affirmative answer to the question implies that we may assume that these manifolds all have \( H_2(M_F; \mathbb{Z}) \) having rank at least one. The action of the covering group gives a series of representations \( \alpha_F : PSL(2, \mathbb{F}) \rightarrow GL(H_2(M_F; \mathbb{Z})) \).

Suppose to the contrary that the ranks of the groups \( H_2(M_F; \mathbb{Z}) \) were bounded, by \( P \) say. Then since there is a bound on the size of the finite subgroups in \( GL(P, \mathbb{Z}) \), and the sizes of the groups \( PSL(2, \mathbb{F}) \) are going to infinity, we would eventually see that some \( \alpha_F \) is nonfaithful, hence trivial. It follows that the fixed homology \( H_1(M_F; \mathbb{Q})^{PSL(2, \mathbb{F})} \cong H_1(M_F; \mathbb{Q}) \). However using the transfer map we see that the left hand side of this isomorphism is \( H_1(M; \mathbb{Q}) \), a contradiction, since we assumed that \( M \) was a rational homology sphere.

**References**


**Department of Mathematics, University of California, Santa Barbara, California 93106**

*E-mail address:* long@math.ucsb.edu

**Department of Mathematics, University of Texas, Austin, Texas 78712**

*E-mail address:* areid@math.utexas.edu