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GROUPS ACTING ON CUBES AND KAZHDAN'S PROPERTY (T)

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Abstract. We show that a group G contains a subgroup K with $e(G,K) > 1$ if and only if it admits an action on a connected cube that is transitive on the hyperplanes and has no fixed point. As a corollary we deduce that a countable group G with such a subgroup does not satisfy Kazhdan's property (T).

Introduction

The celebrated theorem of Stallings (see [3] for a comprehensive treatment) says that a finitely generated group G has more than one end if and only if there exist a finite subgroup $K < G$ and a tree T on which G acts, such that

(i) G acts transitively on the edges of T,
(ii) K is the stabiliser of an edge, and
(iii) the action is essential, i.e., there is no global fixed point.

According to the theory of Bass and Serre the conditions (i)–(iii) are equivalent to the condition that G is isomorphic to either an amalgamated product $A *_K B$, where K is not equal to A or B, or an HNN extension $A*_K$. We describe this situation by saying that G splits over K.

There have been attempts by Scott [12] and Kropholler and Roller [6, 7] to define an end invariant $e(G,K)$ for a pair of groups $K < G$ that would indicate when G splits over K. Indeed, in both cases the relative end invariant of a pair (G,K) must be at least 2, when G splits over K, but the converse is known to be false.

To define the end invariant used by Scott one considers the set of K-invariant subsets of G, $\mathcal{P}(K\backslash G) := \{A \subset G \mid A = KA\}$, as an $\mathbb{F}_2G$-module, where $\mathbb{F}_2$ is the field of two elements, G acts by multiplication on the right and addition is the symmetric difference. A set $A \subset G$ is called K-finite, if $A \subset KF$ for some finite subset $F \subset G$; those sets form a submodule $\mathcal{F}(K\backslash G)$. Now the number of ends of the pair (G,K) is defined as the dimension of the G-fixed submodule of the quotient.

$$e(G,K) := \dim_{\mathbb{F}_2} \left(\frac{\mathcal{P}(K\backslash G)}{\mathcal{F}(K\backslash G)}\right)^G.$$

If G is a finitely generated group and Γ is a Cayley graph of G, then $e(G,K)$ measures the number of ends of the quotient space $K\backslash\Gamma$ (see [4]). In particular,

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$e(G, \{1\})$ is the classical number of ends of G, which is mentioned in the theorem of Stallings.

A subset $A \subset G$ for which the symmetric difference $A + Ag$ is K-finite for all $g \in G$ is called K-almost invariant; those subsets represent G-invariant elements in the quotient $\mathcal{P}(K\backslash G)/\mathcal{F}(K\backslash G)$. Observe that $e(G, K) \geq 1$ if and only if K has infinite index in G. To say that $e(G, K) > 1$ (or (G, K) is a multi-ended pair or K is a codimension-1 subgroup, in Sageev's terminology) means that there is a subset $A \subset G$, such that

(a) $A = KA$,
(b) A is K-almost invariant, and
(c) A is K-proper, i.e., neither A nor $G - A$ is K-finite.

Sageev [10] introduced a new geometric interpretation of this end invariant. He studied group actions on cubical complexes with a locally Euclidean metric that satisfy the global CAT(0) condition in the sense of Gromov and Bridson [1]. For cube complexes CAT(0) is a purely combinatorial condition which is analogous to the non-positive curvature condition for manifolds. Among other things it implies that the complex is contractible. Thus Sageev's theory includes group actions on trees as the one-dimensional case. He defined a notion of codimension one hyperplanes and showed that there is an induced action on the set of hyperplanes. His main theorem says that $e(G, K) > 1$ if and only if G admits an action on a CAT(0) cube complex, K is the stabiliser of an oriented hyperplane and the action satisfies a certain essentiality condition that guarantees condition (c) above. He further showed that when G is finitely generated and the complex is finite dimensional, then the group action is essential if and only if the orbits are unbounded.

In this paper we give a simplified version of Sageev's theory, where we consider only actions on connected cubes. In this version there is no restriction on the groups involved, and the essentiality condition can be expressed as succinctly as in the classical case by saying that the group action must not have a global fixed point.

Let M be an arbitrary set and let $\mathbf{2}$ denote the set $\{0, 1\}$. We define the full cube on M as the graph whose vertices are the functions $M \to \mathbf{2}$, and whose edges connect those pairs of functions which disagree on precisely one element of M. By a connected cube we mean any graph isomorphic to a connected component of a cube. For example the subgraph spanned by all vertices $M \to \mathbf{2}$ with finite support is a connected cube. For each element $m \in M$ the vertices $v\colon M \to \mathbf{2}$ fall into two half spaces, depending on whether $v(m)$ is 0 or 1; thus M may be interpreted as the set of hyperplanes. A group action by graph automorphisms on a connected cube induces an action on the set of hyperplanes in a natural way — this is false for group actions on full cubes.

Now we can state our main result.

Theorem. *A group G has a subgroup K with $e(G, K) > 1$ if and only if G admits an action on a connected cube that is transitive on the set of hyperplanes and has no fixed points.*

We will also show that K is contained in the stabilizer of a hyperplane as a subgroup of finite index.

In the course of the proof of this theorem we will embed the connected cube on M in the Hilbert space $\ell^2(M)$ of all square summable functions $M \to \mathbb{R}$. The condition $e(G, K) > 1$ then gives rise to an isometric action of G on this Hilbert

space, that has unbounded orbits. Serre defined the property (FH) for a group G to mean that any action of G by isometries on a Hilbert space has a global fixed point; clearly this implies that every orbit is bounded. For countable groups property (FH) is equivalent to Kazhdan's property (T) [2, Theorem 4.7]. Thus we have the following consequence of our theorem.

Corollary. *If G is a countable group and has a subgroup K with $e(G,K) > 1$, then G does not satisfy Kazhdan's property* (T).

This corollary was proved independently by Elek and Ramachandran [5] using different methods. A special case was also proved by Niblo and Reeves [8].

Sageev's original complex is isometrically embedded in the connected cube with the graph metric. As a consequence of our theorem, a group action on this complex is essential in the sense of Sageev if and only if the orbits of vertices are unbounded. The dimension of the connected cube only depends on the index of K in G and is always infinite. By contrast, Sageev's complex depends on the choice of a K-almost invariant set $A \subset G$, and the computation of its dimension is an interesting question.

For example, the most difficult part in the classical theory of Stallings and Dunwoody is to choose the appropriate almost invariant set, whose corresponding Sageev complex is one-dimensional. Kropholler and Roller [6] consider pairs (G,K) of Poincaré duality groups, where $e(G,K) \leq 2$, and define an obstruction $\mathrm{sing}_G(K)$ which vanishes precisely if G splits over a subgroup commensurable with K. Sageev [11, §3] has shown that for a word hyperbolic group G and a quasiconvex subgroup K with $e(G,K) > 1$ one can always find a finite dimensional cube complex on which the action of G is essential and cocompact. He informed us, however, that a geometric construction of Rubinstein and Wang [9] yields a pair (G,K) of finitely presented groups with $e(G,K) > 1$, where every Sageev complex is infinite dimensional.

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Groups and cubes

Given a group G and a G-set M, one can always define a linear action of G on the Banach space $\ell^p(M)$ of p-summable functions $f\colon M \to \mathbb{R}$ by defining $gf(m) := f(g^{-1}m)$. This action preserves the unit cube in $\ell^p(M)$, the set of functions with values in $[0,1]$. However, it also has a global fixed point, the origin, and is therefore of little interest to us. This suggests that we should use a slightly different construction of cubes, that takes into account the symmetries that are reflections in the hyperplanes.

A $\mathbf{2}$-set is a set H with a fixed point free involution, which we write as $h \mapsto h^*$. We write $\overline{h} := \{h, h^*\}$ and $\overline{H} := \{\overline{h} \mid h \in H\}$. On $\mathbf{2}$ we have a canonical fixed point free involution, the exchange of 0 and 1. Let

$$C(H) := \{v\colon H \to \mathbf{2} \mid v(h^*) = v(h)^* \text{ for all } h \in H\}.$$

We think of H as the set of half spaces, $\overline{H}$ the corresponding hyperplanes, and $C(H)$ the set of vertices of a full cube. For two vertices $v, w \in C(H)$ the set $\Delta(v,w) := \{\overline{h} \in \overline{H} \mid v(h) \neq w(h)\}$ comprises those hyperplanes that separate v from w.

Lemma 1. *For vertices $u, v, w \in C(H)$ we have $\Delta(u,w) = \Delta(u,v) + \Delta(v,w)$.*

Proof. For any $h \in H$ we have

$$u(h) \neq w(h) \;\Leftrightarrow\; u(h) \neq v(h) \;\text{ xor }\; v(h) \neq w(h).$$

Here we use xor to denote the exclusive or. □

Thus the relation

$$v \sim w \;:\Leftrightarrow\; \Delta(v,w) \text{ is finite}$$

is an equivalence relation. The classes of this relation are the connected components of $C(H)$. On these components, the function $d(v,w) := |\Delta(v,w)|$ is an integer valued metric (in fact the ℓ^1 metric.).

We say that a group G acts on a $\mathbf{2}$-set H if G acts on the set H and $g(h^*) = g(h)^*$ for all $h \in H$; in particular G acts on the quotient $\overline{H}$. We say that H is a transitive G-$\mathbf{2}$-set if the induced G-action on $\overline{H}$ is transitive. There is also a natural G-action on $C(H)$ given by $gv(h) := v(g^{-1}h)$.

Fix a group G. We consider two kinds of data:

(i) a subset $A \subset G$,
(ii) a triple (H, h, v) consisting of a transitive G-$\mathbf{2}$-set H, an element $h \in H$ and a vertex $v \in C(H)$,

and two conversion processes between them.

Given a subset $A \subset G$, let $H_A := \{gA, gA^* \mid g \in G\}$; this is a $\mathbf{2}$-set, where the involution * means taking complements in G, and G acts by left multiplication. For any $g \in G$ we define the characteristic vertex

$$\chi_g \colon H_A \to \mathbf{2}, \quad \chi_g(h) = 1 \text{ iff } g \in h.$$

Observe that $g_1\chi_{g_2} = \chi_{g_1 g_2}$; thus the characteristic vertices form a G-orbit. We obtain the triple (H_A, A, χ_1).

Conversely, we start with a triple (H, h, v) as in (ii) and define $A_H := \{g \in G \mid v(g^{-1}h) = 1\}$. Now $G_h \subset G_{A_H}$, because if $gh = h$, then for all $x \in G$

$$\begin{aligned} xv(h) = xv(gh) \;&\Leftrightarrow\; v(x^{-1}h) = v(x^{-1}gh) \\ &\Leftrightarrow\; x \in A_H \text{ iff } g^{-1}x \in A_H \;\Leftrightarrow\; gA_H = A_H. \end{aligned}$$

Furthermore, if $gh = h^*$ then $gA_H = A_H^*$, as

$$\begin{aligned} x \in A_H^* \;&\Leftrightarrow\; v(x^{-1}h) = 0 \;\Leftrightarrow\; v(x^{-1}h)^* = 1 \;\Leftrightarrow\; v(x^{-1}h^*) = 1 \\ &\Leftrightarrow\; v(x^{-1}gh) = 1 \;\Leftrightarrow\; g^{-1}x \in A_H \;\Leftrightarrow\; x \in gA_H. \end{aligned}$$

Thus the two processes are inverse to each other. We can recover H from A_H as $H_{A_H} := \{gA_H, gA_H^* \mid g \in G\}$. More precisely, we have a G-map $H \to H_{A_H}$ of $\mathbf{2}$-sets that sends h to A_H. Since both sets have the same number of orbits, this is actually a G-isomorphism of $\mathbf{2}$-sets. The vertex $\chi_1 \colon H_{A_H} \to \mathbf{2}$ corresponds to the chosen vertex $v \in C(H)$. Conversely, given a subset $A \subset G$, then by choosing $v := \chi_1 \in C(H_A)$ and $h := A$ we have $A = A_{H_A}$.

Next we study the characteristic orbit in the cube on H_A. The hyperplanes that separate two vertices in the G-orbit of χ_1 are given by the following simple formula.

Lemma 2. *For $x, y \in G$ we have $\Delta(\chi_x, \chi_y) = \{\overline{gA} \in \overline{H}_A \mid g^{-1} \in Ax^{-1} + Ay^{-1}\}$.*

Proof.

$$\begin{aligned}\overline{gA} \in \Delta(\chi_x, \chi_y) &\Leftrightarrow \chi_x(gA) \neq \chi_y(gA)\\ &\Leftrightarrow x \in gA \ \text{ xor } \ y \in gA\\ &\Leftrightarrow g^{-1} \in Ax^{-1} \ \text{ xor } \ g^{-1} \in Ay^{-1}\\ &\Leftrightarrow g^{-1} \in Ax^{-1} + Ay^{-1}.\end{aligned}$$

□

Let $G_A := \{g \in G \mid gA = A\}$ be the stabiliser of A and $G_{\overline{A}} := \{g \in G \mid gA = A \text{ or } gA = A^*\}$. If $Ax^{-1} + Ay^{-1}$ is contained in a finite union of right $G_{\overline{A}}$-cosets, then the set $\{g \in G \mid g^{-1} \in Ax^{-1} + Ay^{-1}\}$ is contained in a finite union of left $G_{\overline{A}}$-cosets, so only finitely many hyperplanes $\overline{gA}$ separate χ_x from χ_y. The converse is also true; thus we have

Corollary 3. *The orbit $G\chi_1$ is contained in a connected component of $C(H_A)$ if and only if A is $G_{\overline{A}}$-almost invariant.*

If A is itself $G_{\overline{A}}$-finite, say $A \subset G_{\overline{A}}F$, then $\Delta(\chi_1, \chi_g)$ contains at most $2|F|$ hyperplanes for any $g \in G$.

Corollary 4. *If A is $G_{\overline{A}}$-finite, then $G\chi_1$ is bounded.*

Proof of the Theorem

Assume that G is a group containing a subgroup K with $e(G, K) > 1$. Then there exists a subset A of G, satisfying conditions (a), (b) and (c); G acts on the **2**-set H_A and the orbit of the vertex χ_1 is contained in a connected component of $C(H_A)$ by Corollary 3. By condition (a), K is a subgroup of G_A, but we can show that the index is finite. Since A is proper we can find a $g \in G$ with $A \neq Ag$, and G_A stabilises $A + Ag$, which is K-finite by (b). Hence the next lemma says that if G has a fixed point in the component of χ_1, then condition (c) is violated.

Lemma 5. *If there exists a vertex $w \in C(H_A)$ which has finite distance from χ_1 and is fixed by the action of G, then A is not $G_{\overline{A}}$-proper.*

Proof. Assume that $w(A) = 0$. We show that A is $G_{\overline{A}}$-finite.

$$\begin{aligned}g \in A &\Leftrightarrow \chi_1(g^{-1}A) = 1 \Leftrightarrow \chi_1(g^{-1}A) \neq w(A)\\ &\Leftrightarrow \chi_1(g^{-1}A) \neq w(g^{-1}A), \quad \text{since } w = gw \text{ for all } g \in G,\\ &\Leftrightarrow \overline{g^{-1}A} \in \Delta(\chi_1, w).\end{aligned}$$

By hypothesis $\Delta(\chi_1, w)$ is finite; thus g^{-1} must lie in a finite union of left $G_{\overline{A}}$-cosets. Similarly, if $w(A) = 1$, then A^* is $G_{\overline{A}}$-finite. □

For the converse, we assume that G acts on a **2**-set H, transitively on $\overline{H}$, and there exists a vertex $v \in C(H)$, such that the orbit Gv is contained in a connected component C_0. For any element $h \in H$ the subset $A_H := \{g \in G \mid v(g^{-1}h) = 1\}$ is $G_{\overline{A}}$-almost invariant by Corollary 3. If A_H is not $G_{\overline{A}}$-proper, then, by Corollary 4, G has a bounded orbit.

It remains to show that the existence of a bounded orbit implies the existence of a fixed point. This would be immediate if our cube were a subset of some Hilbert space, as we will shortly see. Unfortunately, the ℓ^1-metric on an infinite connected cube is not complete, so the argument used for the Hilbert space cannot

be immediately generalised to our situation. Nonetheless, we can show the existence of a fixed point by the following argument. We will first embed C_0 in a Hilbert space so that the action of G on C_0 extends to it, then we will show that G has a fixed point in the Hilbert space as promised, and finally we will show that the fixed point actually lies in the image of C_0.

Let $\ell^2(\overline{H})$ denote the Hilbert space of square summable functions $s\colon \overline{H} \to \mathbb{R}$. We define a map $\rho\colon C_0 \to \ell^2(\overline{H})$ by

$$\rho(w)(\bar{h}) := \begin{cases} 1, & \text{if } w(h) \neq v(h), \\ 0, & \text{if } w(h) = v(h). \end{cases}$$

Remark. For any two points $w_1, w_2 \in C_0$, we have $\|\rho(w_1) - \rho(w_2)\|^2 = d(w_1, w_2)$. This follows immediately from Lemma 1.

For $s \in \ell^2(\overline{H})$, $h \in H$ and $g \in G$ define

$$gs(\bar{h}) := \begin{cases} s(g^{-1}\bar{h}), & \text{if } v(g^{-1}h) = v(h), \\ 1 - s(g^{-1}\bar{h}), & \text{if } v(g^{-1}h) \neq v(h). \end{cases}$$

It is easy to check that this defines an action of G on $\ell^2(\overline{H})$ by isometries, which is compatible with the action of G on C_0.

We now need the notion of a center of a bounded set. A bounded set B is by definition contained in some ball of finite radius r, and we define the radius $r(B)$ of the set to be the infimum of all such r. A point x is said to be a center for B if the ball of radius $r(B)$ around x contains B. It is a fact that any nonempty bounded set in a Hilbert space $\mathcal{H}$ has a unique centre. This lemma is due to Serre (see e.g. Lemma 3.8 in [2]), and we include a proof for the convenience of the reader.

Take any sequence (r_n) in $\mathbb{R}$, converging to $r(B)$, and (x_n) in $\mathcal{H}$, such that B lies in the ball of radius r_n around x_n. For any $\epsilon > 0$ we can choose m and n large enough so that $r_m^2, r_n^2 < r(B)^2 + \epsilon$. Put $y = \frac{1}{2}(x_m + x_n)$, and choose an element $b \in B$ such that $\|b - y\|^2 > r(B)^2 - \epsilon$. The parallelogram equality says that

$$4\,\|b - y\|^2 + \|x_m - x_n\|^2 = 2\,\|b - x_m\|^2 + 2\,\|b - x_n\|^2;$$

therefore

$$\|x_m - x_n\|^2 \leq 2(r_m^2 + r_n^2) - 4\,\|b - y\|^2 < 8\epsilon,$$

which means that (x_n) is a Cauchy sequence. As $\mathcal{H}$ is complete, the center exists, and since all these sequences converge, it must be unique.

Lemma 6. *If Gv is bounded, then G has a fixed point in C_0.*

Proof. If Gv is bounded in C_0, then $G\rho(v)$ is a bounded G-orbit in $\ell^2(\overline{H})$. Thus it has a G-invariant center $c\colon \overline{H} \to \mathbb{R}$. As G acts transitively on $\overline{H}$, we have for any $h_1, h_2 \in H$ that $c(\bar{h}_1)$ equals either $c(\bar{h}_2)$ or $1 - c(\bar{h}_2)$. So c takes at most two values in $\mathbb{R}$. By definition of $\ell^2(\overline{H})$, all but finitely many values of c must be 0, and the others must be 1; hence c lies in the image of ρ. $\square$

This completes the proof of our theorem.

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