EXTENDIBILITY OF HOMOGENEOUS POLYNOMIALS ON BANACH SPACES

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Abstract. We study the $n$-homogeneous polynomials on a Banach space $X$ that can be extended to any space containing $X$. We show that there is an upper bound on the norm of the extension. We construct a predual for the $n$-homogeneous polynomials on $X$ and we characterize the extendible 2-homogeneous polynomials on $X$ when $X$ is a Hilbert space, an $L_1$-space or an $L_\infty$-space.

1. Introduction

There is no Hahn-Banach theorem for $n$-homogeneous polynomials on a Banach space when $n > 1$. For example, the 2-homogeneous polynomial $P(x) = \sum x_k^2$ on $l_2$ cannot be extended to any $C(K)$ space containing $l_2$, as the presence of the Dunford-Pettis property on $C(K)$ would force $P$ to be weakly sequentially continuous [1]. Even when a homogeneous polynomial can be extended to a larger space, it can happen that the norm cannot be preserved [2, 9]. The Aron-Berner extension process for homogeneous polynomials on a Banach space $X$ [2] avoids these obstacles by requiring that the extension be defined only for a very restricted class of spaces containing $X$—essentially, the bidual $X^{**}$ and spaces closely related to it (see also [1, 4, 7, 10, 13]). A similar observation applies to the ultrapower method of Dineen-Timoney and Lindström-Ryan [6, 8].

We propose to study those homogeneous polynomials on a Banach space $X$ that extend to every space containing $X$. We do not require that the norm be preserved. We show that, if a homogeneous polynomial, $P$, on $X$ can be extended to every space containing $X$, then it is possible to put an upper bound on the norms of the extensions. This enables us to put a natural norm on the space $\mathcal{P}_c(^nX)$ of “extendible” $n$-homogeneous polynomials. We show that $\mathcal{P}_c(^nX)$ is complete in this norm. We then construct a predual for the Banach space $\mathcal{P}_c(^nX)$. We conclude with some special classes of spaces. We characterize the extendible 2-homogeneous polynomials on $X$ when $X$ is a Hilbert space, an $L_1$-space or an $L_\infty$-space.

All the Banach spaces considered can be taken over the real or complex numbers. $\mathcal{P}(^nX; Y)$ denotes the Banach space of bounded, $n$-homogeneous polynomials from $X$ into $Y$. Thus if $P \in \mathcal{P}(^nX; Y)$, then there exists a unique bounded symmetric multilinear mapping $A : X^n \rightarrow Y$ such that $P(x) = A(x, \ldots, x)$ for every $x \in X$. 

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1023
When $Y$ is the scalar field we denote this space by $P^n(X)$. For $P \in P^n(X)$ we can define a bounded linear operator $T : X \rightarrow P^{n-1}(X)$ by $T(x)(y) = A(x, y, \ldots, y)$, where $A$ is the multilinear form on $X^n$ that generates $P$ as described above. We shall refer to $T$ as the linear operator associated with $P$.

2. Extendibility of linear operators and homogeneous polynomials

We begin with the description of a process that will enable us to “paste together” a family of spaces containing $X$ in a coherent way. Let $j_\alpha : X \rightarrow Z_\alpha$ be a family of embeddings, indexed by $\alpha \in A$. By an amalgamation of this family we shall mean a triple consisting of a Banach space $Z$, an embedding $j : X \rightarrow Z$ and a family of bounded linear operators $i_\alpha : Z_\alpha \rightarrow Z$ with $\|i_\alpha\| \leq 1$, such that $i_\alpha \circ j_\alpha = j$ for every $\alpha \in A$. Where there is no danger of confusion, we shall refer to the amalgamation $(Z, j, \{i_\alpha\})$ simply as “the amalgamation $Z$”.

An amalgamation may be constructed as follows: first, form the $l_1$-sum $(\sum_\alpha Z_\alpha)_1$ and let $k_\alpha$ denote the usual embedding of $Z_\alpha$ into this space. Let

$$N = \left\{(j_\alpha x_\alpha) \in \left(\sum_\alpha Z_\alpha\right)_1 : x_\alpha \in X, \sum_\alpha x_\alpha = 0 \right\}.$$ 

It is easy to see that $N$ is a closed subspace of $(\sum_\alpha Z_\alpha)_1$. Let $Z$ be the quotient space:

$$Z = \left(\sum_\alpha Z_\alpha\right)_1 / N,$$

and let $\pi$ be the quotient mapping of $(\sum_\alpha Z_\alpha)_1$ onto $Z$. For each $\alpha \in A$ let $i_\alpha = \pi \circ k_\alpha$. Thus, we have $\|i_\alpha\| \leq 1$ for every $\alpha$. To define an embedding of $X$ into $Z$ we note first that for all $\alpha, \beta \in A$, we have $k_\alpha j_\alpha(x) - k_\beta j_\beta(x) \in N$ for every $x \in X$. Therefore $\pi k_\alpha j_\alpha(x) = \pi k_\beta j_\beta(x)$ and we can define

$$j(x) = \pi k_\alpha j_\alpha(x) \text{ for any } \alpha \in A.$$ 

To see that $j$ is an embedding, let $x \in X$. It is clear from the definition of $j(x)$ that $\|j(x)\| \leq \|x\|$. On the other hand, if we choose any $\alpha \in A$, we have, by the definition of the quotient norm,

$$\|j(x)\| = \inf \{\|k_\alpha j_\alpha(x) + u\| : u \in N\}.$$ 

Now if $u \in N$, then $u = (j_\beta x_\beta)$ where $\sum_\beta x_\beta = 0$ and so

$$\|k_\alpha j_\alpha(x) + u\| = \|j_\alpha(x) + j_\alpha(x_\alpha)\| + \sum_{\beta \neq \alpha} \|j_\beta(x_\beta)\|$$

$$= \|x + x_\alpha\| + \sum_{\beta \neq \alpha} \|x_\beta\| \geq \|x + x_\alpha + \sum_{\beta \neq \alpha} x_\beta\|$$

$$= \|x + \sum_{\beta} x_\beta\| = \|x\|,$$

since $\sum x_\beta = 0$. Therefore $\|j(x)\| = \|x\|$ for every $x \in X$. Finally, it follows from the definition of $j$ that $i_\alpha \circ j_\alpha = j$ for every $\alpha$. Thus $(Z, j, (j_\alpha))$ meets all the requirements for an amalgamation of the family of embeddings $(j_\alpha)$. 
We shall say that a bounded \( n \)-homogeneous polynomial \( P : X \to Y \) is extendible if, for every embedding of \( X \) into a Banach space \( Z \), there exists an extension of \( P \) to a bounded \( n \)-homogeneous polynomial \( Q : Z \to Y \). With the help of amalgamations, we can show that we can control the norm of the extension independently of the space \( Z \):

**Proposition 1.** If \( P \in \mathcal{P}(\pi X; Y) \) is extendible, then there exists \( C > 0 \) such that for every embedding of \( X \) into a Banach space \( Z \), there is an extension \( Q \in \mathcal{P}(\pi Z; Y) \) of \( P \) with \( \|Q\| \leq C \).

**Proof.** If this were false, then for every \( n \in \mathbb{N} \) there would exist a Banach space \( Z_n \) and an embedding \( j_n : X \to Z_n \) such that every extension of \( P \) to \( Z_n \) has norm at least \( n \). Let \( (Z, j, (i_n)) \) be an amalgamation of this sequence of embeddings. Now \( P \) has a bounded extension, \( Q \), defined on \( Z \). But then \( Q \circ i_n \) is an extension of \( P \) to \( Z_n \), and hence

\[
\|Q\| \geq \|Q \circ i_n\| \geq n \quad \text{for every } n \in \mathbb{N},
\]

which is impossible. This concludes the proof.

Let \( \mathcal{P}_e(\pi X; Y) \) denote the subset of \( \mathcal{P}(\pi X; Y) \) consisting of all the extendible \( n \)-homogeneous polynomials. \( \mathcal{P}_e(\pi X; Y) \) is a vector space on which we can define a norm by letting \( e(P) \) be the smallest positive real number \( C \) with the property that for every space \( Z \) containing \( X \) there is an extension \( Q \in \mathcal{P}(\pi Z; Y) \) of \( P \) with \( \|Q\| \leq C \). We have \( \|P\| \leq e(P) \) for every \( P \in \mathcal{P}_e(\pi X; Y) \).

**Proposition 2.** \( (\mathcal{P}_e(\pi X; Y), e) \) is complete.

**Proof.** It suffices to show that every absolutely summable series is convergent. Accordingly, suppose that \( \sum_k e(P_k) < \infty \), where \( P_k \in \mathcal{P}_e(\pi X; Y) \) for every \( k \). Then \( \sum P_k \) is absolutely summable in \( \mathcal{P}(\pi X; Y) \) and hence it converges in this space to a bounded \( n \)-homogeneous polynomial \( P \). Let \( Z \) be a space that contains \( X \). For each \( k \) there exists an extension \( Q_k \) of \( P_k \) to \( Z \), with \( \|Q_k\| \leq e(P_k) \). It follows that the series \( \sum Q_k \) converges in \( \mathcal{P}(\pi Z; Y) \) to a polynomial \( Q \). It is clear that \( Q \) extends \( P \), and so \( P \in \mathcal{P}_e(\pi X; Y) \). Furthermore, we have \( \|Q\| \leq \sum \|Q_k\| \leq \sum e(P_k) \).

Finally, we have

\[
e\left(P - \sum_{k=1}^N P_k\right) = e\left(\sum_{k>N} P_k\right) \leq \sum_{k>N} e(P_k),
\]

and hence \( \sum P_k \) converges to \( P \) in \( (\mathcal{P}_e(\pi X; Y), e) \). This concludes the proof.

We shall see that the uniform norm is equivalent to the norm \( e \) on \( \mathcal{P}_e(\pi X) \) if and only if every polynomial in \( \mathcal{P}(\pi X) \) is extendible.

3. **The predual of the space of extendible polynomials**

We begin with the construction of a predual for the space \( \mathcal{L}_e(\pi^2 X \times Y) \) of extendible bilinear forms on \( X \times Y \). We can define a complete norm, \( e \), on this space exactly as in the previous section. We shall denote the projective norm on the tensor product \( X \otimes Y \) by \( \pi_{X,Y} \). We refer the reader to [3] for further details on tensor products. Now if \( i,j \) are embeddings of \( X,Y \) into \( W,Z \) respectively, then \( i \otimes j \) is an algebraic embedding of \( X \otimes Y \) into \( W \otimes Z \). Thus we may identify \( X \otimes Y \) with a vector subspace of \( W \otimes Z \). For \( u \in X \otimes Y \) we have \( \pi_{W,Z}(u) \leq \pi_{X,Y}(u) \).

The norms \( \pi_{W,Z} \) and \( \pi_{X,Y} \) are equivalent on \( X \otimes Y \) if and only if every bounded
bilinear form on $X \times Y$ has a bounded extension to $W \times Z$. For an element $u$ of $X \otimes Y$, we define
\[
\eta(u) = \inf \{ \pi_{W,Z}(u) : X \subset W, Y \subset Z \},
\]
the infimum being taken over all pairs of embeddings $X \hookrightarrow W$, $Y \hookrightarrow Z$. We claim that $\eta$ is a reasonable crossnorm on $X \otimes Y$. Obviously, we have $\eta(\lambda u) = |\lambda|\eta(u)$. To see that $\eta$ is subadditive, let $u_1, u_2 \in X \otimes Y$ and let $\varepsilon > 0$. For each $r = 1, 2$ there exists a pair of embeddings $i_r : X \hookrightarrow W_r$, $j_r : Y \hookrightarrow Z_r$, such that $\pi_{W_r,Z_r}(u_r) \leq \eta(u_r) + \varepsilon/2$. Let $(W,i,(k_r))$, $(Z,j,(l_r))$ respectively be amalgamations of these embeddings. Then we have $i \otimes j = (k_r \otimes l_r) \circ (i_r \otimes j_r)$ for each $r$ and since $\|k_r\|$, $\|l_r\| \leq 1$ it follows that $\pi_{W,Z} \leq \pi_{W_r,Z_r}$ for each $r$. Therefore
\[
\eta(u_1 + u_2) \leq \pi_{W,Z}(u_1 + u_2) \leq \pi_{W,Z}(u_1) + \pi_{W,Z}(u_2) \leq \eta(u_1) + \eta(u_2) + \varepsilon,
\]
and hence $\eta(u_1 + u_2) \leq \eta(u_1) + \eta(u_2)$. Next, suppose that $\eta(u) = 0$ for some $u \in X \otimes Y$. Then there is a sequence of pairs of embeddings $X \hookrightarrow W_r$, $Y \hookrightarrow Z_r$ such that $\pi_{W_r,Z_r}(u) < 1/r$. Arguing as above, let $W, Z$ be amalgamations of these sequences. Then we have $\pi_{W,Z}(u) \leq \pi_{W,Z}(u) < 1/r$ for every $k$ and so $\pi_{W,Z}(u) = 0$, whence $u = 0$. This shows that $\eta$ is a norm. To see that $\eta$ is a reasonable crossnorm, note first that if $x \in X$, $y \in Y$, then $\pi_{W,Z}(x \otimes y) = \|x\|\|y\|$ for all $W, Z$ and so $\eta(x \otimes y) = \|x\|\|y\|$. Secondly, let $\varphi \in X^*$, $\psi \in Y^*$. Let $u \in X \otimes Y$ with $\eta(u) < 1$. Choose embeddings $X \hookrightarrow W$, $Y \hookrightarrow Z$ such that $\pi_{W,Z}(u) < 1$ and let $\varphi_W$, $\psi_Z$ be Hahn-Banach extensions of $\varphi$, $\psi$ to $W$, $Z$ respectively. Then
\[
|\varphi \otimes \psi(u)| = |\varphi_W \otimes \psi_Z(u)| \leq \|\varphi_W\|\|\psi_Z\| = \|\varphi\|\|\psi\|.
\]
It follows that $\eta$ is a reasonable crossnorm.

**Proposition 3.** $(X \hat{\otimes}_Y Y)^* \text{ is isometrically isomorphic to } L_c(2X \times Y)$.

**Proof.** Every bounded linear functional on $X \hat{\otimes}_Y Y$ is the linearization, $\hat{T}$, of a bounded bilinear form $T$ on $X \times Y$. Since $\hat{T}$ is $\eta$-continuous, we have $|\hat{T}(u)| \leq \eta^*(\hat{T})\eta(u) \leq \eta^*(\hat{T})\pi_{W,Z}(u)$ for every $u \in X \otimes Y$ and every pair of embeddings $X \hookrightarrow W$, $Y \hookrightarrow Z$. Hence $\hat{T}$ is continuous on $X \otimes Y$ for the norm induced by $W \otimes_{\pi} Z$ and it follows from the Hahn-Banach theorem that $T$ extends to a bilinear form $S$ on $W \times Z$ with $\|S\| \leq \eta^*(\hat{T})$. Therefore $T$ is extendible and $e(T) \leq \eta^*(\hat{T})$. Conversely, if $T \in L_c(2X \times Y)$ and $X \hookrightarrow W$, $Y \hookrightarrow Z$, then $T$ extends to $S$ in $L(2W \times Z)$ with $\|S\| \leq e(T)$. It follows that $|\hat{T}(u)| = |\hat{S}(u)| \leq e(T)\pi_{W,Z}(u)$ for $u \in X \otimes Y$ and hence $\hat{T}$ is $\eta$-continuous, with $\eta^*(\hat{T}) \leq e(T)$. Therefore $\eta^*(\hat{T}) = e(T)$. This completes the proof.

We now construct a predual for the space of extendible $n$-homogeneous polynomials. The dual of the $n$-fold symmetric tensor product $\bigotimes^n_{\pi} X$ is the Banach space $L_s(nX)$ of symmetric, continuous $n$-linear forms on $X$. Since this latter space is isomorphic to $P(n,X)$, it follows that $\bigotimes^n_{s,\pi} X$ can be considered as an “isomorphic” predual of $P(n,X)$. We prefer to renorm $\bigotimes^n_{s,\pi} X$ so this becomes an isometry, as follows: for $u \in \bigotimes^n_{s,\pi} X$, we will define
\[
\|u\|_s = \sup \{|A(u)| : A \in L_s(nX) \text{ and } \|\hat{A}\| = 1\},
\]
where $\hat{A}$ denotes the $n$-homogeneous polynomial associated with $A$. This norm is equivalent to the projective norm on $\bigotimes^n_{s,\pi} X$. Let us write $x^n$ for the tensor
Then, if $u$ belongs to the uncompleted symmetric tensor product $\bigotimes_{x,\pi}^n X$, it follows from the polarization formula that $u$ can be expressed as a linear combination of tensors of the form $x^n$. We then have another formula for $\|u\|_\pi$:

$$\|u\|_\pi = \inf \left\{ \sum_{j=1}^m |\lambda_j| \|x_j\|^n : u = \sum_{j=1}^m \lambda_j x_j^n \right\}.$$ 

We shall denote by $\tilde{X}_n^{(n)}$ the space $\bigotimes_{x,\pi}^n X$ with this equivalent norm. Then $\mathcal{P}(nX)$ is the dual space of $\tilde{X}_n^{(n)}$. The mapping $x \mapsto x^n$ is a “universal” continuous $n$-homogeneous polynomial on $X$: for every $P \in \mathcal{P}(nX)$ there is a unique $\tilde{P} \in (\tilde{X}_n^{(n)})^*$ with the same norm, such that

$$P(x) = \tilde{P}(x^n) \quad \text{for every } x \in X.$$ 

We refer to [11, 12] for further details.

Now the space $\tilde{X}_n^{(n)}$ suffers the same defect as the projective tensor product $X \otimes Y$ in that its norm does not respect embeddings of $X$ into larger spaces. Proceeding as we did for $X \otimes Y$, we can define a norm, $\eta$, on $X^{(n)}$ by

$$\eta(u) = \inf \left\{ \sum_{j=1}^m |\lambda_j| \|w_j\|^n : u = \sum_{j=1}^m \lambda_j w_j^n, \ w_j \in W, X \subset W \right\},$$

the infimum being taken over all embeddings $X \hookrightarrow W$. We denote by $X_n^{(n)}$ the homogeneous product space $X^{(n)}$ with this norm, and we denote the completion of this space by $\tilde{X}_n^{(n)}$. We then have $(\tilde{X}_n^{(n)})^* = \mathcal{P}_\varepsilon(nX)$:

**Proposition 4.** $P \in \mathcal{P}(nX)$ is extendible if and only if the linear form $\tilde{P}$ on $X^{(n)}$ is continuous with respect to the norm $\eta$. The correspondence $P \leftrightarrow \tilde{P}$ is an isometric isomorphism of $(\mathcal{P}_\varepsilon(nX), \varepsilon)$ with $(\tilde{X}_n^{(n)})^*$.

The proof is similar to the proof of Proposition 3. D. Cardano and I. Zalduendo have shown that integral polynomials are extendible. This also follows from the above proposition—the $n$-homogeneous polynomial $P$ is integral precisely when the linear form $\tilde{P}$ is continuous with respect to the injective norm $\varepsilon$ and since $\eta$ is a reasonable crossnorm, we have $\varepsilon \leq \eta$.

**Corollary 5.** The norm $\varepsilon$ is equivalent to the uniform norm on $\mathcal{P}_\varepsilon(nX)$ if and only if every $n$-homogeneous polynomial on $X$ is extendible.

**Proof.** If $\mathcal{P}(nX) = \mathcal{P}_\varepsilon(nX)$, then, since $\|\cdot\| \leq \varepsilon$, it follows that these norms are equivalent. Conversely, suppose these norms are equivalent on $\mathcal{P}_\varepsilon(nX)$. Since the uniform norm is the dual of the projective norm and $\varepsilon$ is the dual of $\eta$, it follows that the projective norm is equivalent to $\eta$ on $X^{(n)}$. Therefore these norms yield the same dual, hence $\mathcal{P}(nX) = \mathcal{P}_\varepsilon(nX)$. This concludes the proof.

Thus, if every $n$-homogeneous polynomial on $X$ is extendible, then there exists a positive constant $C$, depending only on $X$ and $n$, such that for every embedding $X \hookrightarrow Z$, every $P \in \mathcal{P}(nX)$ has an extension to $Z$ whose norm is at most $C\|P\|$. On the other hand, if $X$ supports an $n$-homogeneous polynomial that is not extendible, then, using an amalgamation argument, we see that there is an embedding $X \hookrightarrow Z$ and a sequence of $n$-homogeneous polynomials $P_k$ on $X$ such that every extension of $P_k$ to $Z$ has norm at least equal to $k$. 

4. EXAMPLES

We begin with a simple observation:

**Lemma 6.** If \( P \in \mathcal{P}(2^X) \) is extendible, then the associated linear operator \( T : X \to X^* \) is 2-summing.

**Proof.** There exists an indexing set \( I \) and an embedding \( j : X \to I^1_\infty \). Let \( Q \in \mathcal{P}(2I^1_\infty) \) be an extension of \( P \) and let \( S \) be the associated linear operator. Since every bounded linear operator from an \( \mathcal{L}_\infty \)-space into an \( \mathcal{L}_1 \)-space is 2-summing, \( S \) is 2-summing. Hence \( T = j^* \circ S \circ j \) is 2-summing. This concludes the proof.

In general, this condition is not sufficient for \( P \) to be extendible. However, for \( \mathcal{L}_1 \)-spaces, it is enough:

**Proposition 7.** Suppose \( X \) is an \( \mathcal{L}_1 \)-space. Then \( P \in \mathcal{P}(2^X) \) is extendible if and only if the associated linear operator \( T : X \to X^* \) is 2-summing.

**Proof.** If \( T \) is 2-summing, then there exists a probability measure \( \mu \) and a factorization of \( T \) as \( u \circ J_2 \circ v \), where \( v \in \mathcal{L}(X, L_\infty(\mu)) \), \( u \in \mathcal{L}(L_2(\mu), X^*) \) and \( J_2 : L_\infty(\mu) \to L_2(\mu) \) is the canonical inclusion. Now let \( j : X \to Z \) be an embedding. Since \( L_\infty(\mu) \) is injective, \( v \) extends to an operator \( \tilde{v} : Z \to L_\infty(\mu) \) and then \( S = u \circ J_2 \circ \tilde{v} \) extends \( T \). Consider the restriction of \( S^* \) to \( X \). We have \( S^*|_X = v^* \circ J_2^* \circ u^*|_X \). Now, since \( X \) is an \( \mathcal{L}_1 \)-space, the operator \( u^*|_X : X \to L_2(\mu) \) is 1-summing, and hence 2-summing. Therefore this operator extends to an operator \( w : Z \to L_2(\mu) \). Let \( R : Z \to Z^* \) be the composition \( \tilde{v}^* \circ J_2^* \circ w \) and let \( Q \) be the 2-homogeneous polynomial on \( I^1_\infty \) associated with \( R \). Then \( Q \) extends \( P \). This concludes the proof.

Next, we consider linear operators and 2-homogeneous polynomials on a Hilbert space \( H \). The situation concerning extendibility of linear operators on Hilbert spaces is particularly simple. In general, 2-summing operators are extendible, since they factor through \( L_\infty(\mu) \) spaces [5]. Conversely, suppose that \( T : H \to H \) is extendible. Let \( j : H \to I^1_\infty \) be an embedding. \( T \) extends to an operator \( S : I^1_\infty \to H \). But every such operator is 2-summing, and hence \( T \) is 2-summing. Therefore, the spaces of extendible and 2-summing operators on \( H \) coincide.

The situation with 2-homogeneous polynomials is different. Let \( P \) be an extendible 2-homogeneous polynomial on \( H \) and let \( T : H \to H \) be the associated linear operator. Let \( j : H \to I^1_\infty \) be an embedding. \( P \) extends to a 2-homogeneous polynomial \( Q \) on \( I^1_\infty \) which has an associated linear operator \( S : I^1_\infty \to (I^1_\infty)^* \) with the property that \( j^* \circ S \circ j = T \). Since \((I^1_\infty)^* \) is an \( \mathcal{L}_1 \)-space, \( S : I^1_\infty \to (I^1_\infty)^* \) and \( j^* : (I^1_\infty)^* \to H \) are both 2-summing, and hence \( j^* \circ S \) is nuclear [5, p. 119]. Therefore \( T \) is nuclear and it follows that \( P \) is also nuclear. Thus every extendible 2-homogeneous polynomial on \( H \) is nuclear. On the other hand, it follows easily from the Hahn-Banach theorem that nuclear polynomials are extendible. In summary, we have:

**Proposition 8.** Let \( H \) be a Hilbert space.

(a) A bounded linear operator from \( H \) into \( H \) is extendible if and only if it is 2-summing.

(b) A bounded 2-homogeneous polynomial on \( H \) is extendible if and only if it is nuclear.
Finally, let $X$ be an $\mathcal{L}_\infty$-space. Then $X^{**}$ is injective and so, by the Aron-Berner extension theorem [2] we have:

**Proposition 9** (Aron and Berner). Suppose $X$ is an $\mathcal{L}_\infty$-space. Then every $P \in \mathcal{P}(^*X)$ is extendible.

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