Hardy’s Theorem
For the n-Dimensional Euclidean Motion Group

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Abstract. An uncertainty principle, due to Hardy, for Fourier transform pairs on \( \mathbb{R} \) says that if the function \( f \) is “very rapidly decreasing”, then the Fourier transform cannot also be “very rapidly decreasing” unless \( f \) is identically zero. In this paper we state and prove an analogue of Hardy’s theorem for the \( n \)-dimensional Euclidean motion group.

1. Introduction

It is a well-known simple fact that if a function \( f \) on \( \mathbb{R} \) is compactly supported then its Fourier transform \( \hat{f} \) cannot also be compactly supported, unless \( f = 0 \). More generally, we have the following principle in classical Fourier analysis: If the function \( f \) is “very rapidly decreasing” then the Fourier transform cannot also be “very rapidly decreasing”, unless \( f \) is identically zero. The following result of Hardy makes the rather vague statement above precise:

**Theorem 1.1** (Hardy). Suppose \( f \) is a measurable function on \( \mathbb{R} \) such that

\[
|f(x)| \leq Ce^{-\alpha x^2}, \quad |\hat{f}(\xi)| \leq C e^{-\beta \xi^2}, \quad x, \xi \in \mathbb{R},
\]

where \( \alpha, \beta \) and \( C \) are positive constants. If \( \alpha \beta > \frac{1}{4} \) then \( f = 0 \) a.e. If \( \alpha \beta < \frac{1}{4} \) there are infinitely many linearly independent functions satisfying (1.1), and if \( \alpha \beta = \frac{1}{4} \) then \( f(x) = C e^{-\alpha x^2} \).

For a proof of the above theorem see [2], Theorem 3.2. Hardy’s theorem is also valid in \( \mathbb{R}^n \) (see [8] for a proof). A generalization of Hardy’s theorem, due to Cowling and Price, asserts that if \( a, b \) are nonnegative constants such that \( ab \geq \frac{1}{4} \), then the only \( f \in S' \) satisfying \( \|e^{ax^2} f\|_p + \|e^{by^2} \hat{f}\|_q < \infty \) for \( 1 \leq p, q \leq \infty \) with at least one of them finite, is \( f = 0 \). On the other hand, if \( ab < \frac{1}{4} \), there are infinitely many \( f \in S \) satisfying \( \|e^{ax^2} f\|_p + \|e^{by^2} \hat{f}\|_q < \infty \) (see [1]). Another theorem of this kind is due to A. Beurling [5], which says that if \( f \in L^1(\mathbb{R}) \) is such that

\[
\int \int_{\mathbb{R}^2} |f(x)\hat{f}(y)| e^{\frac{1}{4}|x+y|} \, dx \, dy < \infty,
\]

then \( f = 0 \) a.e. One can see that Hardy’s theorem can be deduced from this more general theorem of Beurling. This

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class of results can also be viewed as some sort of “uncertainty principle”. For an elaboration of this point of view, see [6], [8] and the bibliographies in those papers.

Suppose \( G \) is a “sufficiently nice” connected Lie group with Haar measure \( m \), and \( \hat{G} \) its unitary dual. Then by the abstract Plancherel theorem we know that there exist a measure structure and a unique measure \( \mu \) on \( \hat{G} \) such that for all \( f \in L^1(G) \cap L^2(G) \),

\[
\int_G |f(x)|^2 \, dm(x) = \int_{\hat{G}} \text{tr}(\pi(f)\pi(f)^\ast) \, d\mu(\pi),
\]

where for \( f \) in \( L^1(G) \) we define the group Fourier transform \( \hat{f} \) of \( f \) by

\[
\hat{f}(\pi) = \pi(f) = \int_G f(x)\pi(x) \, dm(x), \quad \pi \in \hat{G}
\]

(the integral being interpreted suitably). Therefore we can ask the following question in this more general set up : Suppose \( f \) is an \( L^1 \)-function on \( G \) such that both \( f \) and \( \hat{f} \) decay “very rapidly” at infinity. Then is \( f = 0 \) a.e.?

Analogues of Hardy’s theorem for the Heisenberg group \( \mathcal{H}_n \) and the Euclidean motion group of the plane \( M(2) \), have been proved in [8]. In the next two sections we shall state and prove an analogue of Hardy’s theorem for the \( n \)-dimensional Euclidean motion group, \( M(n) \), \( n \geq 2 \). While the proof in [8] for \( M(2) \) proceeds by reducing the theorem to the Euclidean case, the proof here is more direct and involves some simple estimates of the \( K \)-finite matrix coefficients of irreducible representations.

Finally, we remark that, in [7], an analogue of Hardy’s theorem is proved for a subclass of connected noncompact semi-simple Lie groups and all symmetric spaces of the noncompact type.

2. Description of the unitary dual of \( M(n) \)

The group \( G = M(n) \) is the semi-direct product of \( \mathbb{R}^n \) with the special orthogonal group \( K = SO(n) \). A typical element of \( G \) is denoted by \((a, k)\) where \( a \in \mathbb{R}^n \) and \( k \in K \). If \( da \) denotes Lebesgue measure on \( \mathbb{R}^n \) and \( dk \) normalized Haar measure on \( K \), then Haar measure on \( G \) is given by \( da \, dk \). The natural action of \( K \) on \( \mathbb{R}^n \) is denoted by \( k \cdot \nu \), where \( k \in K \) and \( \nu \in \mathbb{R}^n \). (Since the ‘natural’ action is left multiplication by the matrix \( k \), \( \mathbb{R}^n \) should really be thought of as the space of column vectors.) For any unexplained terminology and notation in this section the reader may refer to [4].

We shall now describe \( \hat{G} \), the unitary dual of \( G \).

Let \( \nu \in \mathbb{R}^n \) and \( \nu \neq 0 \). Let \( U_\nu \) denote the stabilizer of \( \nu \) in \( K \) under the natural action of \( K \) on \( \mathbb{R}^n \). Then \( U_\nu \) is conjugate to the subgroup \( \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \ : \ A \in SO(n-1) \right\} \). We identify this subgroup with \( SO(n-1) \). Fix an irreducible unitary representation \( \lambda \) of \( U_\nu \) acting on \( \mathbb{C}^{d_\lambda} \). Let \( H(K, \lambda) \) be the vector space of all measurable functions \( \psi : K \rightarrow \mathbb{C}^{d_\lambda} \) such that \( \psi(uk) = \lambda(u)(\psi(k)) \) for \( u \in U_\nu \), \( k \in K \) and \( \int_K \|\psi(k)\|^2 \, dk < \infty \). Here \( \| \cdot \| \) denotes the norm on \( \mathbb{C}^{d_\lambda} \). It is easy to see that \( H(K, \lambda) \) is a Hilbert space with respect to the inner product defined by

\[
(\psi_1, \psi_2) = d_\lambda \int_K \langle \psi_1(k), \psi_2(k) \rangle \, dk
\]
where \( \langle ., . \rangle \) denotes the usual inner product on \( \mathbb{C}^{d_\lambda} \) and \( \psi_1, \psi_2 \in H(K, \lambda) \). Define \( T_{\nu, \lambda} \) on \( H(K, \lambda) \) by

\[
(2.1) \quad (T_{\nu, \lambda}(a,k)\psi)(k_o) = e^{i(k_o^{-1} \cdot a)} \psi(k_o k), \ \psi \in H(K, \lambda)
\]

for \( a \in \mathbb{R}^n, k, k_o \in K \). We also use \( \langle ., . \rangle \) to denote the inner product on \( \mathbb{R}^n \). One can easily verify that \( T_{\nu, \lambda} \) is a unitary representation of \( G \) on \( H(K, \lambda) \). Further, it can be shown that (see [3], [4]):

(a) For \( \nu \neq 0 \) and any \( \lambda \in \widehat{U}_\nu \), the representation \( T_{\nu, \lambda} \) is irreducible.

(b) Every infinite dimensional irreducible unitary representation of \( G \) is equivalent to some \( T_{\nu, \lambda} \) with \( \nu \) and \( \lambda \) as above.

(c) Given two non-zero vectors \( \nu, \nu_1 \in \mathbb{R}^n \) and representations \( \lambda \in \widehat{U}_\nu \) and \( \lambda_1 \in \widehat{U}_{\nu_1} \), the representations \( T_{\nu, \lambda} \) and \( T_{\nu_1, \lambda_1} \) are equivalent if and only if \( \nu \) and \( \nu_1 \) belong to the same \( K \)-orbit (i.e. \( \nu, \nu_1 \) have the same Euclidean norm) and the representations \( \lambda \) and \( \lambda_1 \) are equivalent under the obvious identification of \( U_\nu \) with \( U_{\nu_1} \).

If \( ||\nu|| = ||\nu_1|| = r, r \in \mathbb{R}^+ \), then by abuse of notation we denote the \( n \)-tuple \( (0, 0, \ldots, 0, r)^t \) also by \( r \). Here \( || \cdot || \) denotes the Euclidean norm on \( \mathbb{R}^n \) and \( t \) denotes the transpose. In this case we write \( U_r \) for \( U_\nu \) and note that \( U_r \) consists precisely of the matrices \( \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \) with \( A \in SO(n-1) \). Hence, we adopt the notation \( U_r = SO(n-1) \). We then choose the representative of the equivalence class of \( T_{\nu, \lambda} \) as \( T_{r, \lambda} \). Apart from these infinite dimensional representations \( T_{r, \lambda} \), the finite dimensional unitary representations of \( K \) also yield finite dimensional unitary representations of \( G \), but these do not enter into the Plancherel formula (see [4] for details).

The Plancherel measure \( \mu \) is supported on the subset of \( \widehat{G} \) given by \( \{ T_{r, \lambda} : \lambda \in \widehat{SO(n-1)} \} \) and \( r \in \mathbb{R}^+ \) and on each “piece” \( \{ T_{r, \lambda} : r \in \mathbb{R}^+ \} \) with \( \lambda \in \widehat{SO(n-1)} \) fixed, it is given by \( C_n r^{n-1} dr \), where \( C_n \) is a constant depending only on \( n \).

Before we end this section we state the following lemma, from complex analysis, that plays a crucial role in the proof of our main theorem:

**Lemma 2.1.** Suppose \( h \) is an entire function on \( \mathbb{C} \) such that \( h(z) = O(e^{a|z|^2}) \) for \( z \in \mathbb{C} \) and \( h(t) = O(e^{-at^2}) \) for \( t \in \mathbb{R} \) where \( a \) is a positive constant. Then \( h(z) = Const.e^{-a|z|^2}, z \in \mathbb{C} \).

Applying the following result (see [10], pp.175) to the even and odd parts of the function separately, the lemma follows: Let \( h \) be an entire function on \( \mathbb{C} \) such that \( h(z) = O(e^{a|z|}) \) for \( z \in \mathbb{C} \) and \( h(t) = O(e^{-at}) \) for \( t \in \mathbb{R}^+ \), where ‘\( a \)’ is a positive constant. Then \( h(z) = Const.e^{-az}, z \in \mathbb{C} \).

3. Analogue of Hardy’s Theorem for \( M(n) \)

We will now define the group Fourier transform on \( G = M(n) \). Given a function \( f \) in \( L^1(G) \) and \( \pi \in \widehat{G} \) the group Fourier transform \( \hat{f} \) of \( f \) at \( \pi \) is the operator

\[
\hat{f}(\pi) = \pi(f) = \int_{\mathbb{R}^n} \int_K f(a,k) \pi(a,k) dk da
\]
(the integral being interpreted suitably, see [9]). Then by the Plancherel theorem we know that for \( f \in L^1 \cap L^2(G) \), \( \hat{f} \) is a Hilbert-Schmidt operator for almost all \( \pi \) (with respect to the Plancherel measure) and we denote its Hilbert-Schmidt norm by \( \|\hat{f}(\pi)\|_{HS} \). We now state and prove an analogue of Hardy’s theorem for \( G \).

**Theorem 3.1.** Suppose \( f \) is a measurable function on \( G \) satisfying the following estimates:

\[
|f(a, k)| \leq Ce^{-\alpha \|a\|^2}, \quad (a, k) \in G, \tag{3.2}
\]

\[
\|\hat{f}(T_r, \lambda)\|_{HS} \leq C e^{-\beta r^2}, \quad r \in \mathbb{R}^+, \tag{3.3}
\]

for some positive constants \( C, \alpha, \beta \) and \( \mathbb{C} \) where \( C_\lambda \) depends only on \( \lambda \). If \( \alpha \beta > \frac{1}{4} \) then \( f = 0 \) a.e.

*(Remark 3.2. Since functions on \( \mathbb{R}^n \) can be thought of as functions on \( G \) invariant under right action by \( K \), Hardy’s theorem for \( \mathbb{R}^n \) shows that \( \frac{1}{4} \) is the best possible constant.)*

**Proof.** Observe that by identifying \(-r\) with the \( n\)-tuple \((0, \cdots, 0, -r)^t\) for \( r \in \mathbb{R}^+ \) we can define \( T_{-r, \lambda} \). Now, \( T_{-r, \lambda} \) and \( T_{r, \lambda} \) are equivalent as representations of \( G \). Hence \( \|\hat{f}(T_{-r, \lambda})\|_{HS} = \|\hat{f}(T_{r, \lambda})\|_{HS} \) and we thus have

\[
\|\hat{f}(T_{r, \lambda})\|_{HS} \leq C e^{-\beta r^2}, \quad r \in \mathbb{R}. \tag{3.4}
\]

For \( r \in \mathbb{R} \) and \( \lambda \in SO(n-1) \), let \( S = \{ e_i^\lambda : i \in \mathbb{N} \} \) be a basis of \( H(K, \lambda) \) consisting of \( K \)-finite vectors. (For fixed \( \lambda \), notice that the representation \( T_{r, \lambda} \) restricted to \( K \) is just the right regular action of \( K \) on \( H(K, \lambda) \).) Note that if \( \phi \) is a \( K \)-finite vector, then \( \phi \in C^\infty(K, \mathbb{C}^{d_k}) \). It suffices to show that for any fixed \( i \) and \( j \), the condition \( \alpha \beta > \frac{1}{4} \) implies \( \langle \hat{f}(T_{r, \lambda})e_i^\lambda, e_j^\lambda \rangle = 0 \) as a function of \( r \) and \( \lambda \). Fix \( i_o, j_o \in \mathbb{N} \) and consider for \( r \in \mathbb{R} \),

\[
\langle \hat{f}(T_{r, \lambda})e_i^\lambda, e_j^\lambda \rangle = \int_K \int_{\mathbb{R}^n} f(a, k)(T_{r, \lambda}(a, k)e_i^\lambda, e_j^\lambda) da dk. \tag{3.5}
\]

Let \( \Phi_{\alpha, \beta}^{i_o, j_o}(a, k) = \langle T_{r, \lambda}(a, k)e_i^\lambda, e_j^\lambda \rangle \) for \( r \in \mathbb{R}, \lambda \in SO(n-1), i_o, j_o \in \mathbb{N}, \) and \( (a, k) \in G \). Then by definition of \( T_{r, \lambda} \), we have

\[
\Phi_{\alpha, \beta}^{i_o, j_o}(a, k) = d_\lambda \int_K \langle (T_{r, \lambda}(a, k)e_i^\lambda)(k_o), e_j^\lambda(k_o) \rangle dk_o.
\]

\[
= d_\lambda \int_K e^{i(k_o^{-1}r, a)} \langle e_i^\lambda(k_o), e_j^\lambda(k_o) \rangle dk_o.
\]

\[
= d_\lambda \int_K e^{i(r, k_o-a)} \langle e_i^\lambda(k_o), e_j^\lambda(k_o) \rangle dk_o.
\]

Here the real number \( r \) is identified with \((0, \cdots, 0, r)^t\) and \( \langle \cdot, \cdot \rangle \) denotes both inner product on \( \mathbb{R}^n \) as well as \( \mathbb{C}^{d_k} \). Notice that the integral on the right-hand side makes sense even when \( r \in \mathbb{C} \) where we identify \( r \in \mathbb{C} \) with \((0, \cdots, 0, r)^t \) in \( \mathbb{C}^n \) and \( \langle \cdot, \cdot \rangle \) now denotes inner product on \( \mathbb{C}^n \) also. Hence, with \( (a, k) \) fixed, the function \( \Phi_{\alpha, \beta}^{i_o, j_o}(a, k) \) of the variable \( r \) extends to the whole complex plane. One can easily see that for fixed \( (a, k) \), \( z \mapsto \Phi_{\alpha, \beta}^{i_o, j_o}(a, k) \) is an entire function on \( \mathbb{C} \). Moreover, for
\[ z \in \mathbb{C}, \]
\[ |\Phi_{z,\lambda}^{\rho}(a, k)| \leq d_{\lambda} \int_{K} |e^{i(z, k, a)} \| e_{\lambda}^\rho(k_{o}k) \| e_{\lambda}^\rho(k_{o})| \, dk_{o} \]
\[ \leq A \int_{K} e^{-||(I_{m} z)\varepsilon_{n}, k, a\rangle} \, dk_{o} \]
where \( e_{n} = (0, \cdots, 0, 1)^{t} \) in \( \mathbb{R}^{n} \), \((a, k) \in G \), and \( A \) is a constant which depends on \( \lambda, i_{o}, j_{o} \). (Notice that \( e_{\lambda}^\rho \) and \( e_{\lambda}^{\rho} \) are continuous functions on \( K \) and hence bounded.) Since \( f \) satisfies (3.4) and \( \beta > \frac{1}{4a} \), we have
\[ |\langle \hat{f}(T_{r,\lambda})e_{\lambda}^\rho, e_{\lambda}^{\rho} \rangle| \leq C e^{-\beta r^{2}} \leq C e^{-\frac{r^{2}}{4a}}, \quad r \in \mathbb{R}. \]
By definition of \( \Phi_{\rho}^{\rho}(a, k) \) we have from (3.5),
\[ \langle \hat{f}(T_{r,\lambda})e_{\lambda}^\rho, e_{\lambda}^{\rho} \rangle = \int_{K} \int_{\mathbb{R}^{n}} f(a, k) \Phi_{\rho}^{\rho}(a, k) \, da \, dk. \]
Since \( f \) satisfies (3.2) and from (3.7), \( |\Phi_{\rho}^{\rho}(a, k)| \leq A e^{||a||} \), we conclude that the function \( r \mapsto \langle \hat{f}(T_{r,\lambda})e_{\lambda}^\rho, e_{\lambda}^{\rho} \rangle \) can be extended to the whole of \( \mathbb{C} \) and indeed it can be proved that \( z \mapsto \langle \hat{f}(T_{z,\lambda})e_{\lambda}^\rho, e_{\lambda}^{\rho} \rangle \) is an entire function. Further, a simple calculation using (3.2) and (3.7) shows that
\[ |\langle \hat{f}(T_{z,\lambda})e_{\lambda}^\rho, e_{\lambda}^{\rho} \rangle| \leq \int_{K} \int_{\mathbb{R}^{n}} |f(a, k)| |\Phi_{\rho}^{\rho}(a, k)| \, da \, dk \]
\[ \leq A \int_{K} \int_{\mathbb{R}^{n}} e^{-\alpha|a|^{2}} \left( \int_{K} e^{-||(I_{m} z)\varepsilon_{n}, k, a\rangle} \, dk_{o} \right) \, da \, dk \]
\[ = A \int_{K} \int_{\mathbb{R}^{n}} e^{-\alpha|a|^{2}} e^{-||(I_{m} z)\varepsilon_{n}, a\rangle} \, da \, dk \]
\[ = A \int_{\mathbb{R}^{n}} e^{-\alpha|a|^{2}} e^{-||(I_{m} z)\varepsilon_{n}, a\rangle} \, da \]
\[ = A e^{\frac{|(I_{m})_{\varepsilon_{n}}|^{2}}{4\alpha}} \int_{\mathbb{R}^{n}} e^{-\left(\sqrt{\alpha a^{2}} + \frac{(I_{m})_{\varepsilon_{n}}^{2}}{2\sqrt{\alpha}} \right) a} \, da \]
\[ \leq A' e^{\frac{|(I_{m})^{4}}{4\alpha}} \leq A' e^{\frac{|z|^{2}}{4\alpha}} \]
for \( z \in \mathbb{C} \) and some constants \( A, A' \).

It is clear from (3.8) and (3.10) that the function \( z \mapsto \langle \hat{f}(T_{z,\lambda})e_{\lambda}^\rho, e_{\lambda}^{\rho} \rangle \) satisfies the hypothesis of Lemma 2.1. Hence, it follows that \( \langle \hat{f}(T_{r,\lambda})e_{\lambda}^\rho, e_{\lambda}^{\rho} \rangle = \text{Const}. e^{-\frac{r^{2}}{4\alpha}} \).

Hence from (3.4) \( |\langle \hat{f}(T_{r,\lambda})e_{\lambda}^\rho, e_{\lambda}^{\rho} \rangle| = | \text{Const}. e^{-\frac{r^{2}}{4\alpha}} | \leq C_{\lambda} e^{-\beta r^{2}} \); and since \( \beta - \frac{1}{4a} > 0 \), we see that \( \langle \hat{f}(T_{r,\lambda})e_{\lambda}^\rho, e_{\lambda}^{\rho} \rangle \equiv 0 \) as a function of \( r \). Since \( i_{o}, j_{o} \) and \( \lambda \) were arbitrary, \( \hat{f}(T_{r,\lambda}) \equiv 0 \) for all \( r \in \mathbb{R}^{+} \) and \( \lambda \in SO(n - 1) \). Hence by the one-to-one property of the group Fourier transform we get that \( f = 0 \) a.e. This completes the proof of the theorem.

(Actually an examination of the proof shows that we have proved the following stronger result: Let \( \delta_{1}, \delta_{2} \in \overline{K} \) and \( \chi_{\delta_{1}} \) and \( \chi_{\delta_{2}} \) the corresponding characters. Then \( T_{\delta_{1}}(\chi_{\delta_{1}})T_{\delta_{2}}(f)T_{\delta_{2}}(\chi_{\delta_{2}}) \) is a finite rank operator (with rank bounded by a constant depending only on \( \delta_{1}, \delta_{2}, \lambda \)). This operator is zero on the orthogonal complement of...
a subspace whose dimension is again bounded by a constant depending only on $\delta_1$, $\delta_2$, $\lambda$. Suppose in this context that $\alpha$ and $\beta$ are positive constants such that $\alpha \beta > \frac{1}{4}$ and that $|f(a,k)| \leq C e^{-\alpha \|a\|^2}$ and $\|T_{r,\lambda}(\chi_{\delta_1})T_{r,\lambda}(f)T_{r,\lambda}(\chi_{\delta_2})\|_{HS} \leq C_{\lambda,\delta_1,\delta_2} e^{-\beta r^2}$ where $C$ is a positive constant and $C_{\lambda,\delta_1,\delta_2}$ is a positive constant depending only on $\delta_1$, $\delta_2$, $\lambda$. Then $f \equiv 0.$

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