PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 4, April 1998, Pages 1199–1204
S 0002-9939(98)04144-6

HARDY'S THEOREM FOR THE n-DIMENSIONAL EUCLIDEAN MOTION GROUP

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(Communicated by J. Marshall Ash)

Abstract. An uncertainty principle, due to Hardy, for Fourier transform pairs on $\mathbb{R}$ says that if the function f is "very rapidly decreasing", then the Fourier transform cannot also be "very rapidly decreasing" unless f is identically zero. In this paper we state and prove an analogue of Hardy's theorem for the n-dimensional Euclidean motion group.

1. Introduction

It is a well-known simple fact that if a function f on $\mathbb{R}$ is compactly supported then its Fourier transform $\tilde{f}$ cannot also be compactly supported, unless $f = 0$. More generally, we have the following principle in classical Fourier analysis : If the function f is "very rapidly decreasing" then the Fourier transform cannot also be "very rapidly decreasing", unless f is identically zero. The following result of Hardy makes the rather vague statement above precise:

Theorem 1.1 (Hardy). *Suppose f is a measurable function on $\mathbb{R}$ such that*

$$| f(x) | \leq Ce^{-\alpha x^2}, \ | \tilde{f}(\xi) | \leq Ce^{-\beta \xi^2}, \ x, \xi \in \mathbb{R}, \tag{1.1}$$

where α, β and C are positive constants. If $\alpha\beta > \frac{1}{4}$ then $f = 0$ a.e. If $\alpha\beta < \frac{1}{4}$ there are infinitely many linearly independent functions satisfying (1.1), and if $\alpha\beta = \frac{1}{4}$ then $f(x) = Ce^{-\alpha x^2}$.

For a proof of the above theorem see [2], Theorem 3.2. Hardy's theorem is also valid in $\mathbb{R}^n$ (see [8] for a proof). A generalization of Hardy's theorem, due to Cowling and Price, asserts that if a, b are nonnegative constants such that $ab \geq \frac{1}{4}$, then the only $f \in \mathcal{S}'$ satisfying $\|e^{ax^2} f\|_p + \|e^{by^2} \hat{f}\|_q < \infty$ for $1 \leq p, q \leq \infty$ with at least one of them finite, is $f = 0$. On the other hand, if $ab < \frac{1}{4}$, there are infinitely many $f \in \mathcal{S}$ satisfying $\|e^{ax^2} f\|_p + \|e^{by^2} \hat{f}\|_q < \infty$ (see [1]). Another theorem of this kind is due to A. Beurling [5], which says that if $f \in L^1(\mathbb{R})$ is such that $\int\int_{\mathbb{R}^2} | f(x)\hat{f}(y) | e^{|x\ y|} dx\ dy < \infty$, then $f = 0$ a.e. One can see that Hardy's theorem can be deduced from this more general theorem of Beurling. This

Received by the editors April 4, 1995 and, in revised form, September 3, 1996.
1991 *Mathematics Subject Classification.* Primary 22Exx; Secondary 22E30, 43A80.
Key words and phrases. Uncertainty principle, Fourier transform pairs, very rapidly decreasing, Euclidean motion group.

class of results can also be viewed as some sort of "uncertainty principle". For an elaboration of this point of view, see [6], [8] and the bibliographies in those papers.

Suppose G is a "sufficiently nice" connected Lie group with Haar measure m, and $\widehat{G}$ its unitary dual. Then by the abstract Plancherel theorem we know that there exist a measure structure and a unique measure μ on $\widehat{G}$ such that for all $f \in L^1(G) \cap L^2(G)$,

$$\int_G \mid f(x) \mid^2 dm(x) \;=\; \int_{\widehat{G}} tr(\pi(f)\pi(f)^*) d\mu(\pi),$$

where for f in $L^1(G)$ we define the group Fourier transform $\hat{f}$ of f by

$$\hat{f}(\pi) = \pi(f) \;=\; \int_G f(x)\pi(x) dm(x), \;\; \pi \in \widehat{G}$$

(the integral being interpreted suitably). Therefore we can ask the following question in this more general set up : Suppose f is an L^1-function on G such that both f and $\hat{f}$ decay "very rapidly" at infinity. Then is $f \;=\; 0$ a.e.?

Analogues of Hardy's theorem for the Heisenberg group $\mathcal{H}_n$ and the Euclidean motion group of the plane $M(2)$, have been proved in [8]. In the next two sections we shall state and prove an analogue of Hardy's theorem for the n-dimensional Euclidean motion group, $M(n)$, $n \geq 2$. While the proof in [8] for $M(2)$ proceeds by reducing the theorem to the Euclidean case, the proof here is more direct and involves some simple estimates of the K-finite matrix coefficients of irreducible representations.

Finally, we remark that, in [7], an analogue of Hardy's theorem is proved for a subclass of connected noncompact semi-simple Lie groups and all symmetric spaces of the noncompact type.

2. Description of the unitary dual of $M(n)$

The group $G \;=\; M(n)$ is the semi-direct product of $\mathbb{R}^n$ with the special orthogonal group $K \;=\; SO(n)$. A typical element of G is denoted by (a, k) where $a \in \mathbb{R}^n$ and $k \in K$. If da denotes Lebesgue measure on $\mathbb{R}^n$ and dk normalized Haar measure on K, then Haar measure on G is given by $da\; dk$. The natural action of K on $\mathbb{R}^n$ is denoted by $k \cdot \nu$, where $k \in K$ and $\nu \in \mathbb{R}^n$. (Since the 'natural' action is left multiplication by the matrix k, $\mathbb{R}^n$ should really be thought of as the space of column vectors.) For any unexplained terminology and notation in this section the reader may refer to [4].

We shall now describe $\widehat{G}$, the unitary dual of G.

Let $\nu \in \mathbb{R}^n$ and $\nu \neq 0$. Let U_ν denote the stabilizer of ν in K under the natural action of K on $\mathbb{R}^n$. Then U_ν is conjugate to the subgroup $\{\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \;:\; A \in SO(n-1)\}$. We identify this subgroup with $SO(n-1)$. Fix an irreducible unitary representation λ of U_ν acting on $\mathbb{C}^{d_\lambda}$. Let $H(K, \lambda)$ be the vector space of all measurable functions $\psi : K \to \mathbb{C}^{d_\lambda}$ such that $\psi(uk) \;=\; \lambda(u)(\psi(k))$ for $u \in U_\nu$, $k \in K$ and $\int_K \|\psi(k)\|^2 dk < \infty$. Here $\| \cdot \|$ denotes the norm on $\mathbb{C}^{d_\lambda}$. It is easy to see that $H(K, \lambda)$ is a Hilbert space with respect to the inner product defined by

$$(\psi_1, \psi_2) \;=\; d_\lambda \int_K \langle \psi_1(k), \psi_2(k) \rangle dk$$

where $\langle .,. \rangle$ denotes the usual inner product on $\mathbb{C}^{d_\lambda}$ and ψ_1, $\psi_2 \in H(K,\lambda)$. Define $T_{\nu,\lambda}$ on $H(K,\lambda)$ by

$$(T_{\nu,\lambda}(a,k)\psi)(k_o) \;=\; e^{i\langle k_o^{-1}\cdot\nu,a\rangle}\;\psi(k_o k),\;\; \psi \in H(K,\lambda) \tag{2.1}$$

for $a \in \mathbb{R}^n$, $k, k_o \in K$. We also use $\langle .,. \rangle$ to denote the inner product on $\mathbb{R}^n$. One can easily verify that $T_{\nu,\lambda}$ is a unitary representation of G on $H(K,\lambda)$. Further, it can be shown that (see [3], [4]):

(a) For $\nu \neq 0$ and any $\lambda \in \widehat{U_\nu}$, the representation $T_{\nu,\lambda}$ is irreducible.

(b) Every infinite dimensional irreducible unitary representation of G is equivalent to some $T_{\nu,\lambda}$ with ν and λ as above.

(c) Given two non-zero vectors ν, $\nu_1 \in \mathbb{R}^n$ and representations $\lambda \in \widehat{U_\nu}$ and $\lambda_1 \in \widehat{U_{\nu_1}}$, the representations $T_{\nu,\lambda}$ and T_{ν_1,λ_1} are equivalent if and only if ν and ν_1 belong to the same K-orbit (i.e. ν, ν_1 have the same Euclidean norm) and the representations λ and λ_1 are equivalent under the obvious identification of U_ν with U_{ν_1}.

If $\|\nu\| = \|\nu_1\| = r$, $r \in \mathbb{R}^+$, then by abuse of notation we denote the n-tuple $(0,0,\cdots,0,r)^t$ also by r. Here $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$ and t denotes the transpose. In this case we write U_r for U_ν and note that U_r consists precisely of the matrices $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ with $A \in SO(n-1)$. Hence, we adopt the notation $U_r \;=\; SO(n-1)$. We then choose the representative of the equivalence class of $T_{\nu,\lambda}$ as $T_{r,\lambda}$. Apart from these infinite dimensional representations $T_{r,\lambda}$, the finite dimensional unitary representations of K also yield finite dimensional unitary representations of G, but these do not enter into the Plancherel formula (see [4] for details).

The Plancherel measure μ is supported on the subset of $\widehat{G}$ given by $\{T_{r,\lambda} \;:\; \lambda \in SO(n-1)\widehat{\ }$ and $r \in \mathbb{R}^+\}$ and on each "piece" $\{T_{r,\lambda} \;:\; r \in \mathbb{R}^+\}$ with $\lambda \in SO(n-1)\widehat{\ }$ fixed, it is given by $C_n r^{n-1} dr$, where C_n is a constant depending only on n.

Before we end this section we state the following lemma, from complex analysis, that plays a crucial role in the proof of our main theorem:

Lemma 2.1. *Suppose h is an entire function on $\mathbb{C}$ such that $h(z) \;=\; O(e^{a|z|^2})$ for $z \in \mathbb{C}$ and $h(t) \;=\; O(e^{-at^2})$ for $t \in \mathbb{R}$ where a is a positive constant. Then $h(z) \;=\; Const.e^{-az^2}$, $z \in \mathbb{C}$.*

Applying the following result (see [10], pp.175) to the even and odd parts of the function separately, the lemma follows: Let h be an entire function on $\mathbb{C}$ such that $h(z) \;=\; O(e^{a|z|})$ for $z \in \mathbb{C}$ and $h(t) \;=\; O(e^{-at})$ for $t \in \mathbb{R}^+$, where 'a' is a positive constant. Then $h(z) \;=\; Const.e^{-az}$, $z \in \mathbb{C}$.

3. Analogue of Hardy's theorem for $M(n)$

We will now define the group Fourier transform on $G \;=\; M(n)$. Given a function f in $L^1(G)$ and $\pi \in \widehat{G}$ the group Fourier transform $\hat{f}$ of f at π is the operator

$$\hat{f}(\pi) \;=\; \pi(f) \;=\; \int_{\mathbb{R}^n}\int_K f(a,k)\pi(a,k)dkda \tag{3.1}$$

(the integral being interpreted suitably, see [9]). Then by the Plancherel theorem we know that for $f \in L^1 \cap L^2(G)$, $\hat{f}$ is a Hilbert-Schimdt operator for almost all π (with respect to the Plancherel measure) and we denote its Hilbert-Schimdt norm by $\|\hat{f}(\pi)\|_{HS}$. We now state and prove an analogue of Hardy's theorem for G.

Theorem 3.1. *Suppose f is a measurable function on G satisfying the following estimates:*

$$| f(a,k) | \leq Ce^{-\alpha\|a\|^2}, \ (a,k) \in G, \tag{3.2}$$

$$\|\hat{f}(T_{r,\lambda})\|_{HS} \leq C_\lambda e^{-\beta r^2}, \ r \in \mathbb{R}^+, \tag{3.3}$$

for some positive constants C_λ, α, β and C where C_λ depends only on λ. If $\alpha\beta > \frac{1}{4}$ then $f = 0$ a.e.

(*Remark* 3.2. Since functions on $\mathbb{R}^n$ can be thought of as functions on G invariant under right action by K, Hardy's theorem for $\mathbb{R}^n$ shows that $\frac{1}{4}$ is the best possible constant.)

Proof. Observe that by identifying $-r$ with the n-tuple $(0, \cdots, 0, -r)^t$ for $r \in \mathbb{R}^+$ we can define $T_{-r,\lambda}$. Now, $T_{-r,\lambda}$ and $T_{r,\lambda}$ are equivalent as representations of G. Hence $\|\hat{f}(T_{-r,\lambda})\|_{HS} = \|\hat{f}(T_{r,\lambda})\|_{HS}$ and we thus have

$$\|\hat{f}(T_{r,\lambda})\|_{HS} \leq C_\lambda e^{-\beta r^2}, \ r \in \mathbb{R}. \tag{3.4}$$

For $r \in \mathbb{R}$ and $\lambda \in SO(n-1)\widehat{\ }$, let $S = \{ e_i^\lambda : i \in \mathbb{N} \}$ be a basis of $H(K,\lambda)$ consisting of K-finite vectors. (For fixed λ, notice that the representation $T_{r,\lambda}$ restricted to K is just the right regular action of K on $H(K,\lambda)$.) Note that if ϕ is a K-finite vector, then $\phi \in C^\infty(K, \mathbb{C}^{d_\lambda})$. It suffices to show that for any fixed i and j, the condition $\alpha\beta > \frac{1}{4}$ implies $(\hat{f}(T_{r,\lambda})e_i^\lambda, e_j^\lambda) \equiv 0$ as a function of r and λ. Fix $i_o, j_o \in \mathbb{N}$ and consider for $r \in \mathbb{R}$,

$$(\hat{f}(T_{r,\lambda})e_{i_o}^\lambda, e_{j_o}^\lambda) = \int_K \int_{\mathbb{R}^n} f(a,k)(T_{r,\lambda}(a,k)e_{i_o}^\lambda, e_{j_o}^\lambda)dadk. \tag{3.5}$$

Let $\Phi_{r,\lambda}^{i_o,j_o}(a,k) = (T_{r,\lambda}(a,k)e_{i_o}^\lambda, e_{j_o}^\lambda)$ for $r \in \mathbb{R}$, $\lambda \in SO(n-1)\widehat{\ }$, $i_o, j_o \in \mathbb{N}$, and $(a,k) \in G$. Then by definition of $T_{r,\lambda}$, we have

$$\begin{aligned} \Phi_{r,\lambda}^{i_o,j_o}(a,k) &= d_\lambda \int_K \langle (T_{r,\lambda}(a,k)e_{i_o}^\lambda)(k_o), e_{j_o}^\lambda(k_o)\rangle dk_o \\ &= d_\lambda \int_K e^{i\langle k_o^{-1}\cdot r, a\rangle} \langle e_{i_o}^\lambda(k_o k), e_{j_o}^\lambda(k_o)\rangle dk_o \\ &= d_\lambda \int_K e^{i\langle r, k_o\cdot a\rangle} \langle e_{i_o}^\lambda(k_o k), e_{j_o}^\lambda(k_o)\rangle dk_o. \end{aligned} \tag{3.6}$$

Here the real number r is identified with $(0, \cdots, 0, r)^t$ and $\langle .,. \rangle$ denotes both inner product on $\mathbb{R}^n$ as well as $\mathbb{C}^{d_\lambda}$. Notice that the integral on the right-hand side makes sense even when $r \in \mathbb{C}$ where we identify $r \in \mathbb{C}$ with $(0, \cdots, 0, r)^t$ in $\mathbb{C}^n$ and $\langle .,. \rangle$ now denotes inner product on $\mathbb{C}^n$ also. Hence, with (a,k) fixed, the function $\Phi_{r,\lambda}^{i_o,j_o}(a,k)$ of the variable r extends to the whole complex plane. One can easily see that for fixed (a,k), $z \mapsto \Phi_{z,\lambda}^{i_o,j_o}(a,k)$ is an entire function on $\mathbb{C}$. Moreover, for

$z \in \mathbb{C}$,

$$\begin{aligned} | \Phi^{i_o,j_o}_{z,\lambda}(a,k) | &\leq d_\lambda \int_K | e^{i\langle z, k_o\cdot a\rangle} || e^\lambda_{i_o}(k_o k) || e^\lambda_{j_o}(k_o) | dk_o \\ &\leq A \int_K e^{-\langle (Im\ z)e_n, k_o\cdot a\rangle} dk_o \end{aligned} \tag{3.7}$$

where $e_n = (0,\cdots,0,1)^t$ in $\mathbb{R}^n$, $(a,k) \in G$, and A is a constant which depends on λ, i_o, j_o. (Notice that $e^\lambda_{i_o}$ and $e^\lambda_{j_o}$ are continuous functions on K and hence bounded.) Since f satisfies (3.4) and $\beta > \frac{1}{4\alpha}$, we have

$$| (\hat{f}(T_{r,\lambda})e^\lambda_{i_o}, e^\lambda_{j_o}) | \leq Ce^{-\beta r^2} \leq Ce^{-\frac{r^2}{4\alpha}},\ r \in \mathbb{R}. \tag{3.8}$$

By definition of $\Phi^{i_o,j_o}_{r,\lambda}(a,k)$ we have from (3.5),

$$(\hat{f}(T_{r,\lambda})e^\lambda_{i_o}, e^\lambda_{j_o}) = \int_K \int_{\mathbb{R}^n} f(a,k)\Phi^{i_o,j_o}_{r,\lambda}(a,k)dadk. \tag{3.9}$$

Since f satisfies (3.2) and from (3.7), $| \Phi^{i_o,j_o}_{z,\lambda}(a,k) | \leq Ae^{|z||a||}$, we conclude that the function $r \mapsto (\hat{f}(T_{r,\lambda})e^\lambda_{i_o}, e^\lambda_{j_o})$ can be extended to the whole of $\mathbb{C}$ and indeed it can be proved that $z \mapsto (\hat{f}(T_{z,\lambda})e^\lambda_{i_o}, e^\lambda_{j_o})$ is an entire function. Further, a simple calculation using (3.2) and (3.7) shows that

(3.10)

$$\begin{aligned} | (\hat{f}(T_{z,\lambda})e^\lambda_{i_o}, e^\lambda_{j_o}) | &\leq \int_K \int_{\mathbb{R}^n} | f(a,k) || \Phi^{i_o,j_o}_{z,\lambda}(a,k) | dadk \\ &\leq A \int_K \int_{\mathbb{R}^n} e^{-\alpha\|a\|^2} (\int_K e^{-\langle (Im\ z)e_n, k_o\cdot a\rangle} dk_o) dadk \\ &= A \int_K \int_{\mathbb{R}^n} e^{-\alpha\|a\|^2} e^{-\langle (Im\ z)e_n, k_o\cdot a\rangle} dadk_o \\ &= A \int_{\mathbb{R}^n} e^{-\alpha\|a\|^2} e^{-\langle (Im\ z)e_n, a\rangle} da \\ &= Ae^{\frac{\|(Im\ z)e_n\|^2}{4\alpha}} \int_{\mathbb{R}^n} e^{-\langle \sqrt{\alpha}a + \frac{(Im\ z)e_n}{2\sqrt{\alpha}}, \sqrt{\alpha}a + \frac{(Im\ z)e_n}{2\sqrt{\alpha}}\rangle} da \\ &\leq A' e^{\frac{|Im\ z|^2}{4\alpha}} \\ &\leq A' e^{\frac{|z|^2}{4\alpha}} \end{aligned}$$

for $z \in \mathbb{C}$ and some constants A, A'.

It is clear from (3.8) and (3.10) that the function $z \mapsto (\hat{f}(T_{z,\lambda})e^\lambda_{i_o}, e^\lambda_{j_o})$ satisfies the hypothesis of Lemma 2.1. Hence, it follows that $(\hat{f}(T_{r,\lambda})e^\lambda_{i_o}, e^\lambda_{j_o}) = Const.e^{-\frac{r^2}{4\alpha}}$. Hence from (3.4) $| (\hat{f}(T_{r,\lambda})e^\lambda_{i_o}, e^\lambda_{j_o}) | = | Const.e^{-\frac{r^2}{4\alpha}} | \leq C_\lambda e^{-\beta r^2}$; and since $\beta - \frac{1}{4\alpha} > 0$, we see that $(\hat{f}(T_{r,\lambda})e^\lambda_{i_o}, e^\lambda_{j_o}) \equiv 0$ as a function of r. Since i_o, j_o and λ were arbitrary, $\hat{f}(T_{r,\lambda}) \equiv 0$ for all $r \in \mathbb{R}^+$ and $\lambda \in SO(n-1)\widehat{\ }$. Hence by the one-to-one property of the group Fourier transform we get that $f = 0$ a.e. This completes the proof of the theorem.

(Actually an examination of the proof shows that we have proved the following stronger result : Let $\delta_1, \delta_2 \in \widehat{K}$ and χ_{δ_1} and χ_{δ_2} the corresponding characters. Then $T_{r,\lambda}(\chi_{\delta_1})T_{r,\lambda}(f)T_{r,\lambda}(\chi_{\delta_2})$ is a finite rank operator(with rank bounded by a constant depending only on δ_1, δ_2, λ). This operator is zero on the orthogonal complement of

a subspace whose dimension is again bounded by a constant depending only on δ_1, δ_2, λ. Suppose in this context that α and β are positive constants such that $\alpha\beta > \frac{1}{4}$ and that $\mid f(a,k) \mid \leq Ce^{-\alpha\|a\|^2}$ and $\|T_{r,\lambda}(\chi_{\delta_1})T_{r,\lambda}(f)T_{r,\lambda}(\chi_{\delta_2})\|_{HS} \leq C_{\lambda,\delta_1,\delta_2}e^{-\beta r^2}$ where C is a positive constant and $C_{\lambda,\delta_1,\delta_2}$ is a positive constant depending only on δ_1, δ_2, λ. Then $f \equiv 0$.)

Acknowledgements

I thank Prof. A. Sitaram for suggesting this problem to me and for several useful conversations with him on the material covered in this paper. I also thank the referee for several comments which helped in improving the exposition.

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