REMARKS ON THE NON-COHEN-MACAULAY LOCUS OF NOETHERIAN SCHEMES

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ABSTRACT. In this paper we give a notion of polynomial type $p(X)$ of a Noetherian scheme X and define the function $dp : X \longrightarrow \mathbb{Z}$ by $dp(x) = \dim O_{X,x} - p(O_{X,x})$ for all $x \in X$. Then we show that if X admits a dualizing complex and X is equidimensional, dp is (lower) semicontinuous; moreover, in that case, the non-Cohen-Macaulay locus $\mathrm{nCM}(X) = \{x \in X \mid O_{X,x}$ is not Cohen-Macaulay$\}$ is biequidimensional iff dp is constant on $\mathrm{nCM}(X)$.

1. INTRODUCTION

Let X be a noetherian scheme with the structure sheaf O_X. The non-Cohen-Macaulay locus $\mathrm{nCM}(X)$ of X is defined by $\mathrm{nCM}(X) = \{x \in X \mid O_{X,x}$ is not a Cohen-Macaulay ring$\}$. It is well-known that if X admits a dualizing complex then $\mathrm{nCM}(X)$ is a closed subscheme of X ([H1]). Then, we say that $\mathrm{nCM}(X)$ is biequidimensional if it is equidimensional and equicodimensional, i.e. all irreducible components of $\mathrm{nCM}(X)$ have the same dimension and codimension. Therefore it is natural to ask following questions:

(i) How can one estimate the dimension of $\mathrm{nCM}(X)$?

(ii) When is $\mathrm{nCM}(X)$ biequidimensional?

The purpose of this paper is to deal with these questions. First, for the local scheme $X = \mathrm{Spec}\, A$, where A is a local ring, the question (i) has been studied in [C1] and [C2]. There we defined the polynomial type of a local ring A as follows. Let $\underline{x}$ be a system of parameters and $\underline{n} = (n_1, \ldots, n_d)$ a d-tuple of positive integers. Then the least degree of all polynomials in $\underline{n}$ bounding the function

$$I(\underline{n}; \underline{x}) = \ell(A/(x_1^{n_1}, \ldots, x_d^{n_d})A) - n_1 \ldots n_d e(\underline{x}; A)$$

from above (where $\ell(N)$ is the length of the A-module N and $e(\underline{x}; A)$ is the Serre multiplicity of A with respect to $\underline{x}$) is independent of the choice of the system of parameters $\underline{x}$. This invariant is called the polynomial type of A and denoted by $p(A)$ (see [C2] for details). We will extend this notion to schemes and show in Section 2 (Theorem 2.4) that the polynomial type of the scheme X is equal to dim $\mathrm{nCM}(X)$ provided X is equidimensional and admits a dualizing complex. Second, relative to the polynomial type we can define on X a function $dp : X \longrightarrow \mathbb{Z}$ by $dp(x) = \dim O_{X,x} - p(O_{X,x})$ for all $x \in X$, where $\mathbb{Z}$ is the set of all integers. Then

we will prove in Section 3, under the same assumptions for X as above, that this function is (lower) semicontinuous (Theorem 3.2). As a consequence of this result we get a criterion for the biequidimensionality of $\mathrm{nCM}(X)$ which says that $\mathrm{nCM}(X)$ is biequidimensional if and only if dp is constant on $\mathrm{nCM}(X)$ (Theorem 3.4).

The reader may consult [G], [H1] and [H2] for unexplained terminology and notations.

2. The polynomial type of a scheme

First, we recall the definition of polynomial type of a local ring A given in [C2]. Let $\underline{x} = (x_1, \ldots, x_d)$ be a system of parameters of A and $\underline{n} = (n_1, \ldots, n_d)$ a d-tuple of positive integers. We consider the difference between the length and the multiplicity

$$I(\underline{n}; \underline{x}) = \ell(A/(x_1^{n_1}, \ldots, x_d^{n_d})A) - n_1 \ldots n_d e(\underline{x}; A)$$

as a function in $\underline{n}$. Then the least degree of all polynomials bounding $I(\underline{n}; \underline{x})$ from above is independent of the choice of $\underline{x}$. This numerical invariant is called the polynomial type of A and denoted by $p(A)$. For further use in this paper, we need the following remarks and results from [C1] and [C2] about the polynomial type of a local ring A.

(2.1) If A is a Cohen-Macaulay ring, we set $p(A) = -\infty$. Then, if A is not Cohen-Macaulay, $0 \le p(A) \le \dim A - 1$.

(2.2) Denote by $\mathfrak{a}_i(A)$ the annihilator of the i-th local cohomology module $H_{\mathfrak{m}}^i(A)$ of A with respect to the maximal ideal $\mathfrak{m}$. Put $\mathfrak{a}(A) = \mathfrak{a}_0(A) \ldots \mathfrak{a}_{d-1}(A)$, where $d = \dim A$. Suppose that A admits a dualizing complex. Then by [H1], the non-Cohen-Macaulay locus $\mathrm{nCM}(A)$ of A is a closed set in $\mathrm{Spec}\, A$. Then the following two assertions are true (see [C1, Theorem 1.2] and [C2, Theorem 3.3]):
  (i) $\dim \mathrm{nCM}(A) \le p(A) = \dim A/\mathfrak{a}(A)$.
  (ii) If A is equidimensional then $\dim \mathrm{nCM}(A) = p(A)$.

(2.3) With the same method that we used in the proof of Theorem 4.1 in [C2], we can show the following result: Suppose that A admits a dualizing complex and k is a positive integer. Then the following conditions are equivalent:
  (i) $p(A) \le k$.
  (ii) For any $\mathfrak{p} \in \mathrm{Spec}\, A$ with $\dim A/\mathfrak{p} > k$, $A_{\mathfrak{p}}$ is Cohen-Macaulay and $\dim A_{\mathfrak{p}} + \dim A/\mathfrak{p} = \dim A$.

(2.4) (see [C2, Corollary 5.3]). Let $\mathfrak{p}$ be a prime ideal of A. Then $p(A) \ge p(A_{\mathfrak{p}}) + \dim A/\mathfrak{p}$.

Now we want to extend the notion of polynomial type to a noetherian scheme.

(2.5) Definition. Let X be a noetherian scheme. We set $p(x) = p(O_{X,x})$ for any $x \in X$ and define the polynomial type of X by

$$p(X) = \sup_{x \in X} p(x).$$

(2.6) Remark. As in the local case, we have $-\infty \le p(X) \le \dim X - 1$, and X is a Cohen-Macaulay scheme (i.e. $O_{X,x}$ is Cohen-Macaulay for all $x \in X$) iff $p(X) = -\infty$. Therefore $p(x) \ge 0$ if and only if $x \in \mathrm{nCM}(X)$.

Denote by F the set of all closed points of X. Then we have an easy lemma.

(2.7) Lemma. *Let X, F be as above. Then*

$$p(X) = \sup_{x \in F} p(x).$$

Proof. It is enough to prove this for the affine scheme $X = \operatorname{Spec} A$, where A is a noetherian ring. Let $x \in X$ be an arbitrary point corresponding to a prime ideal $\mathfrak{p} \subseteq A$. Then there exists a maximal ideal $\mathfrak{m}$ of A such that $\mathfrak{p} \subseteq \mathfrak{m}$. Hence, by (2.4), $p(A_{\mathfrak{p}}) \leq p(A_{\mathfrak{m}}) - \dim(A_{\mathfrak{m}}/\mathfrak{p}A_{\mathfrak{m}}) \leq p(A_{\mathfrak{m}})$. Therefore $p(X) \leq \sup_{x \in F} p(O_{X,x})$. $\qquad\square$

Similarly to the local case, we have the following theorem which gives a geometric meaning of the polynomial type of a scheme.

(2.8) Theorem. *Let X be a noetherian scheme admitting a dualizing complex. Then*
   (i) $p(X) \geq \dim \operatorname{nCM}(X)$.
   (ii) If X is locally equidimensional, then the equality holds in (i).

Proof. Note first that since X admits a dualizing complex, X is of finite dimension and $\operatorname{nCM}(X)$ is a closed subscheme of X by [H1]. We only need to prove our claim for the affine scheme $X = \operatorname{Spec} A$.

(i). Let $\mathfrak{p}$ be a prime ideal of A contained in $\operatorname{nCM}(A)$ such that $\dim A/\mathfrak{p} = \dim \operatorname{nCM}(A)$. Let $\mathfrak{m}$ be a maximal ideal of A such that $\mathfrak{p} \subseteq \mathfrak{m}$ and $\dim(A_{\mathfrak{m}}/\mathfrak{p}A_{\mathfrak{m}}) = \dim(A/\mathfrak{p})$. As $\mathfrak{p}A_{\mathfrak{m}} \in \operatorname{nCM}(A_{\mathfrak{m}})$, we get $\dim \operatorname{nCM}(A_{\mathfrak{m}}) \geq \dim \operatorname{nCM}(A)$. The converse inequality $\dim \operatorname{nCM}(A_{\mathfrak{n}}) \leq \dim \operatorname{nCM}(A)$ is easily shown to hold true for any maximal ideal $\mathfrak{n}$ of A. So we get $\dim \operatorname{nCM}(A_{\mathfrak{m}}) = \dim \operatorname{nCM}(A)$. Hence by (2.2), (i) we get $p(A) \geq p(A_{\mathfrak{m}}) \geq \dim \operatorname{nCM}(A_{\mathfrak{m}}) = \dim \operatorname{nCM}(A)$.

(ii). Now, by (2.7) we may assume that there exists a maximal ideal $\mathfrak{n}$ of A such that $p(A) = p(A_{\mathfrak{n}})$. Then by (2.2), (ii) we get $p(A) = p(A_{\mathfrak{n}}) = \dim \operatorname{nCM}(A_{\mathfrak{n}}) \leq \dim \operatorname{nCM}(A)$. Thus $p(A) = \dim \operatorname{nCM}(A)$, as required. $\qquad\square$

(2.9) Example. This example shows that the equality $p(X) = \dim \operatorname{nCM}(X)$ does not hold without the assumption that X is equidimensional. For $d \geq 2$, we set $A = k[X_1, \ldots, X_d, Y]/(X_1 Y, \ldots, X_d Y)$, where $X_1, \ldots, X_d$ and Y are indeterminates over an algebraically closed field k. Then $\dim A = d$. It is easy to check that $\operatorname{nCM}(A) = \{\mathfrak{m}\}$, where $\mathfrak{m} = (X_1, \ldots, X_d, Y)A$; thus $\dim \operatorname{nCM}(A) = 0$ and $p(A) = p(A_{\mathfrak{m}})$. Let $\underline{y} = (y_1, \ldots, y_d)$ be a system of parameters of $A_{\mathfrak{m}}$ defined by $y_i = X_i A_{\mathfrak{m}}$ for $i = 1, \ldots, d-1$ and $y_d = (X_d + Y)A_{\mathfrak{m}}$. Then we can show in $A_{\mathfrak{m}}$ that $I(\underline{n}; \underline{y}) = n_d + 1$ is a polynomial of degree 1. Therefore $p(A) = 1$.

3. Biequidimensionality of nCM(X)

We start with the following fact.

(3.1) Lemma. *Let X be a noetherian scheme of finite dimension. Then for all $x \in X$ we have*
   (i) $p(X) \geq p(x) + \dim \overline{\{x\}}$.
   (ii) If X is locally equidimensional and admits a dualizing complex, then $\dim O_{X,x} - p(x) \geq \dim O_{X,y} - p(y)$ for any specialization y of x.

Proof. For $x \notin \operatorname{nCM}(X)$, there is nothing to prove. Thus we may assume that $x \in \operatorname{nCM}(X)$.

(i). Let y be an arbitrary point in $\overline{\{x\}}$. By (2.4) we have $p(X) \geq p(y) \geq p(x) + \dim O_{\overline{\{x\}},y}$. Therefore $p(X) \geq p(x) + \dim \overline{\{x\}}$.

(ii). It follows from the proof of (i) that

$$\dim O_{X,x} - p(x) \geq \dim O_{X,x} + \dim O_{\overline{\{x\}},y} - p(y).$$

Since X admits a dualizing complex, X is catenary by [H1]. Hence, we deduce from the local equidimensionality of X that $\dim O_{\overline{\{x\}},y} = \dim O_{X,y} - \dim O_{X,x}$. Therefore $\dim O_{X,x} - p(x) \geq \dim O_{X,y} - p(y)$, as required. $\qquad\square$

Now let $\mathbb{Z}$ be the set of all integers. We consider the function $p \ : X \longrightarrow \mathbb{Z}$ defined by $p(x) = p(O_{X,x})$ for all $x \in X$. Then Lemma 3.1, (i) leads us to the following question: is the function p semicontinuous? Unfortunately, it is not true in general. But, with the same method as used in [C-M], we can prove that if X admits a dualizing complex then the function p is (upper) semicontinuous if and only if $p(X) \leq 1$. On the other hand, (ii) of Lemma 3.1 leads us to study the function $dp \ : X \longrightarrow \mathbb{Z}$ defined by $dp(x) = \dim O_{X,x} - p(x)$ for all $x \in X$. We have the following theorem.

(3.2) Theorem. *Let X be a noetherian scheme admitting a dualizing complex. Suppose that X is locally equidimensional. Then the function $dp \ : X \longrightarrow \mathbb{Z}$ is (lower) semicontinuous.*

To prove this theorem we need the following lemma.

(3.3) Lemma. *Let X be as in Theorem 3.2. Let $x \in \mathrm{nCM}(X)$. Set $V_x = \{z \mid z$ is a generic point of $\mathrm{nCM}(X)$ such that $x \in \overline{\{z\}}\}$. Then*

$$dp(x) = \min_{z \in V_x} dp(z).$$

Proof. It is enough to prove the lemma in case X is an affine scheme. Suppose that $X = \mathrm{Spec}\, A$. Let $\mathfrak{p}$ be a prime ideal of A corresponding to x. Then

$$V_x = \{\mathfrak{q} \in \mathrm{nCM}(A) \mid \mathfrak{q} \subseteq \mathfrak{p} \text{ and } \mathfrak{q} \text{ is minimal in } \mathrm{nCM}(A)\}.$$

Let $\mathfrak{q}_o \in V_x$ be such that $dp(\mathfrak{q}_o) = \min_{\mathfrak{q} \in V_x} dp(\mathfrak{q})$. Since $dp(\mathfrak{p}) \leq dp(\mathfrak{q}_o)$ by (3.1), (ii), it remains to show that $dp(\mathfrak{p}) \geq dp(\mathfrak{q}_o)$. Note that by the choice of $\mathfrak{q}_o$, if $\mathfrak{q}'$ is a prime ideal properly contained in $\mathfrak{q}_o$ then $\mathfrak{q}' \notin \mathrm{nCM}(A)$. Morerover, since A is equidimensional and catenary, it follows by (2.1) and (2.3) that $p(A_{\mathfrak{q}_o}) = 0$. Therefore, if we can prove that $\dim(A/\mathfrak{q}_o)_\mathfrak{p} \geq p(A_\mathfrak{p})$ then

$$\dim A_\mathfrak{p} - \dim A_{\mathfrak{q}_o} = \dim(A/\mathfrak{q}_o)_\mathfrak{p} \geq p(A_\mathfrak{p}),$$

and the above inequality follows. On the other hand, by virtue of (2.3), to prove that $\dim(A/\mathfrak{q}_o)_\mathfrak{p} \geq p(A_\mathfrak{p})$ we only have to show that if $\mathfrak{q}$ is a prime ideal contained in $\mathfrak{p}$ with $\dim(A/\mathfrak{q})_\mathfrak{p} > \dim(A/\mathfrak{q}_o)_\mathfrak{p}$ then $\mathfrak{q} \notin \mathrm{nCM}(A)$. In fact, suppose that there is a prime ideal $\mathfrak{q} \in \mathrm{nCM}(X)$ contained in $\mathfrak{p}$ with $\dim(A/\mathfrak{q})_\mathfrak{p} > \dim(A/\mathfrak{q}_o)_\mathfrak{p}$. Without loss of generality we may assume that $\mathfrak{q} \in V_x$. Therefore $\dim A_\mathfrak{q} - p(A_\mathfrak{q}) \geq \dim A_{\mathfrak{q}_o}$. This implies that

$$\dim A_\mathfrak{p} - \dim A_\mathfrak{q} + p(A_\mathfrak{q}) \leq \dim A_\mathfrak{p} - \dim A_{\mathfrak{q}_o} = \dim(A/\mathfrak{q}_o)_\mathfrak{p};$$

thus, since $\dim(A/\mathfrak{q}_o)_\mathfrak{p} < \dim(A/\mathfrak{q})_\mathfrak{p} = \dim A_\mathfrak{p} - \dim A_\mathfrak{q}$,

$$\dim(A/\mathfrak{q}_o)_\mathfrak{p} + p(A_\mathfrak{q}) < \dim(A/\mathfrak{q}_o)_\mathfrak{p}.$$

This contradicts the inequality $p(A_\mathfrak{q}) \geq 0$. The lemma is proved. $\qquad\square$

We are now ready to prove the theorem.

Proof of Theorem (3.2). As in the proof of Lemma 3.3, it is enough to show our claim for the affine case $X = \operatorname{Spec} A$. We have to prove that for each non-negative integer t the set $U_t = \{\mathfrak{p} \in A \mid dp(\mathfrak{p}) \geq t\}$ is an open set. By (3.1), (ii) U_t is stable under generalization. Therefore, by [M], (22.B) we only need to prove that if $\mathfrak{p} \in U_t$ then U_t contains a non-empty open set of the irreducible closed set $V(\mathfrak{p})$. Since $\operatorname{nCM}(A)$ is closed, there exists a finite set of prime ideals $W = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ such that $V(\mathfrak{p}_1), \ldots, V(\mathfrak{p}_n)$ are the different irreducible components of $\operatorname{nCM}(A)$. We set $V = \{\mathfrak{p}_i \in W \mid \mathfrak{p}_i \not\subseteq \mathfrak{p}\}$ and $P = \bigcap_{\mathfrak{p}_i \in V} \mathfrak{p}_i$. Then there always exists an element $f \in P \setminus \mathfrak{p}$. Let $Y_f = \operatorname{Spec}(A/\mathfrak{p})_f$. We will show that for all $\mathfrak{q} \in Y_f$, $dp(\mathfrak{q}) \geq t$. In fact, by Lemma (3.3) we get $dp(\mathfrak{q}) = \min\limits_{\substack{\mathfrak{p}_i \in W \\ \mathfrak{p}_i \subseteq \mathfrak{q}}} dp(\mathfrak{p}_i)$ and $dp(\mathfrak{p}) = \min\limits_{\mathfrak{p}_i \in W \setminus V} dp(\mathfrak{p}_i)$. But, it is easy to see by the choice of f that $W \setminus V = \{\mathfrak{p}_i \in W \mid \mathfrak{p}_i \subseteq \mathfrak{q}\}$. Thus $dp(\mathfrak{q}) = dp(\mathfrak{p}) \geq t$, and the proof of Theorem (3.2) is complete. $\square$

Theorem (3.2) leads to a criterion for the biequidimensionality of $\operatorname{nCM}(X)$ as follows. Recall that $\operatorname{nCM}(X)$ is said to be biequidimensional if it is equidimensional and equicodimensional, i.e. all irreducible components of $\operatorname{nCM}(X)$ have the same dimension and codimension.

(3.4) Theorem. *Let X be as in Theorem (3.2). Then the following conditions are equivalent:*
  (i) $\operatorname{nCM}(X)$ is a biequidimensional scheme.
  (ii) $dp\mid_{\operatorname{nCM}(X)} : \operatorname{nCM}(X) \longrightarrow \mathbb{Z}$ is a constant function.

Proof. (i) $\Rightarrow$ (ii). Let Z be an arbitrary irreducible component of $\operatorname{nCM}(X)$ with the unique generic point z. Since X is locally equidimensional and catenary, $p(z) = 0$ by (2.1) and (2.3). Therefore $dp(z) = \operatorname{codim} \operatorname{nCM}(X)$, since $\operatorname{nCM}(X)$ is equicodimensional. Thus, we may assume that $X = \operatorname{Spec} A$, where A is an equidimensional ring which admits a dualizing complex. We have to prove that $dp(\mathfrak{p}) = dp(\mathfrak{q})$ for two arbitrary prime ideals $\mathfrak{p} \subset \mathfrak{q}$ contained in $\operatorname{nCM}(X)$. Now, by localizing at $\mathfrak{q}$, we may assume without any loss of generality that A is a local ring with the maximal ideal $\mathfrak{q}$. Since $dp(\mathfrak{p}) \geq dp(\mathfrak{q})$ by (3.1), (ii), it remains to show that $dp(\mathfrak{q}) \geq dp(\mathfrak{p})$. This is equivalent to proving that $p(A_\mathfrak{p}) \geq p(A) - \dim A/\mathfrak{p}$. But, applying (2.3) to the ring $A_\mathfrak{p}$, we only have to verify that there exists a prime ideal $\mathfrak{p}_o$ contained in $\mathfrak{p}$ such that $\dim(A/\mathfrak{p}_o)_\mathfrak{p} \geq p(A) - \dim A/\mathfrak{p}$ and $A_{\mathfrak{p}_o}$ is non-Cohen-Macaulay. In fact, let $\mathfrak{a}(A) = \mathfrak{a}_o(A) \ldots \mathfrak{a}_{d-1}(A)$, where $\mathfrak{a}_i(A)$ is the annihilator of $H^i_\mathfrak{q}(A)$ and $d = \dim A$. Since A is equidimensional and catenary, we have $\operatorname{nCM}(A) = V(\mathfrak{a}(A))$ by [Sch], 2.4.6. Moreover, since $\operatorname{nCM}(A)$ is equidimensional, there exists a prime ideal $\mathfrak{p}_o$ containing $\mathfrak{a}(A)$ such that $\mathfrak{p}_o \subseteq \mathfrak{p}$ and $\dim A/\mathfrak{p}_o = \dim A/\mathfrak{a}(A)$. Therefore $A_{\mathfrak{p}_o}$ is non-Cohen-Macaulay, and $\dim A/\mathfrak{p}_o = p(A)$ by (2.2). Then $\dim(A/\mathfrak{p}_o)_\mathfrak{p} = \dim A/\mathfrak{p}_o - \dim A/\mathfrak{p} = p(A) - \dim A/\mathfrak{p}$, as required.

(ii) $\Rightarrow$ (i). Let Z be an abitrary irreducible component of $\operatorname{nCM}(X)$ with the unique generic point z. As in the proof of (i) $\Rightarrow$ (ii) we have $dp(z) = \dim O_{X,z}$. Therefore $\operatorname{nCM}(X)$ is equicodimensinal since dp is a constant function on $\operatorname{nCM}(X)$. Suppose now that $\operatorname{nCM}(X)$ is not equidimensional. Then, as X is locally equidimensional and catenary, there exists a point $x \in \operatorname{nCM}(X)$ such that $O_{X,x}/J_x$ is not an equidimensional local ring, where J is a sheaf of ideals defining $\operatorname{nCM}(X)$. Put $O_{X,x} = A, J_x = I \subseteq A$. Let $\mathfrak{m}$ the maximal ideal of A. Since A/I is not equidimensional, there exists a prime ideal $\mathfrak{p} \in \operatorname{Ass}(A/I)$ such that $\dim A/\mathfrak{p} < \dim A/I$ and $p(\mathfrak{p}) = 0$. Therefore, $dp(\mathfrak{p}) = \dim A_\mathfrak{p} - p(\mathfrak{p}) = \dim A_\mathfrak{p} = \dim A - \dim(A/\mathfrak{p}) >$

$\dim A - \dim(A/I) = \dim A - p(A) = dp(\mathfrak{m})$ by (2.2), (ii). Thus the function dp is not a constant on $\mathrm{nCM}(X)$. The theorem is proved. $\qquad\square$

(3.5) Remarks. The following example shows that Theorem (3.4) does not hold without the assumption of the equidimensionality of X. Let $X_1, \ldots, X_5$ be indeterminates over an algebraic closed field k. We put $R = k[X_1, \ldots, X_5]$, $\mathfrak{p}_1 = (X_1, X_2, X_3, X_4)R$, $\mathfrak{p}_2 = (X_1, X_2, X_4, X_5)R$ and $\mathfrak{p}_3 = (X_1, X_3, X_4, X_5)R$. We consider a ring A defined by $A = S^{-1}(R/\mathfrak{a})$, where $\mathfrak{a} = (X_1, X_2)R \cap (X_3, X_4)R \cap (X_1, X_4, X_5)R$ and $S = R \setminus (\mathfrak{p}_1 \cup \mathfrak{p}_2 \cup \mathfrak{p}_3)$. Then, $X = \mathrm{Spec}(A)$ is not equidimensional. It is not difficult to check that $\mathrm{nCM}(X) = \{\mathfrak{p}_1 A, \mathfrak{p}_2 A, \mathfrak{p}_3 A\}$; hence $\mathrm{nCM}(X)$ is biequidimensional. On the other hand, we can easily show that $p(A_{\mathfrak{p}_1}) = 0$ and $p(A_{\mathfrak{p}_2}) = p(A_{\mathfrak{p}_3}) = 1$. Therefore $dp(A_{\mathfrak{p}_2}) = dp(A_{\mathfrak{p}_3}) = 1 \neq 2 = dp(A_{\mathfrak{p}_1})$; so dp is not constant on $\mathrm{nCM}(X)$.

However, we do not have any example which shows the necessary of the equidimensionality of X in Theorem (3.2). Thus we have the following question, whose answer we still do not know:

Question. Let X be a noetherian scheme admitting a dualizing complex. Is the function $dp : X \longrightarrow \mathbb{Z}$ semicontinuous ?

Acknowledgment

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