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CHARACTERIZATIONS OF CONTRACTION C-SEMIGROUPS

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ABSTRACT. A C-semigroup $\{T(t)\}_{t\geq 0}$ is of contractions if $\|T(t)x\| \leq \|Cx\|$ for $t \geq 0$, $x \in X$. Using the Hille-Yosida space, we completely characterize the generators of contraction C-semigroups. We also give the Lumer-Phillips theorem for C-semigroups in several special cases.

1. INTRODUCTION

The notion of exponentially bounded C-semigroup was introduced by Davies and Pang [1]. Recently, the theory of C-semigroup has been extensively developed by many authors [2, 7, 9]. This theory allows us to study many ill-posed abstract Cauchy problems.

The starting point of this paper is to try to give an answer to the question asked by R. deLaubenfels in [3, Open question 6.10]: Does there exist an analogue of the Lumer-Phillips theorem for C-semigroups? Since the Lumer-Phillips theorem characterizes the generators of contraction C_0-semigroups, this gives us the motivation to make a suitable definition for the contractions of C-semigroups and then characterize the generators.

On the other hand, many works have generalized the Hille-Yosida theorem to C-semigroups. Earlier, Davies and Pang [1] gave a chracterization of an exponentially bounded C-semigroup under the assumption that $R(C)$ is dense in X. Later, Tanaka and Miyadera [7] generalized their results to the case of $R(C)$ not dense, and they gave a sufficient and necessary condition for a closed linear operator with dense domain to be the generator of an exponentially bounded C-semigroup. After defining the contraction C-semigroup, we are also interested in characterizing the generator by the Hille-Yosida type theorem. Here the main difficulty we meet with is that the generator may not be densely defined, we choose the Hille-Yosida space to give an additional condition on the generator.

This paper is organized as follows. §2 is devoted to some preliminaries on C-semigroups. In §3 we characterize the generators of contraction C-semigroups in general cases, and under the assumption that $C(D(A))$ is dense in $R(C)$, the characterization can be simplified. §4 deals with several special cases of $\rho(A) \neq \varnothing$ or

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$\overline{R(C)} = X$, we obtain both the Hille-Yosida theorem and the Lumer-Phillips theorem in such cases. This means that we partly give the answer to the question mentioned above in the affirmative.

2. Preliminaries

Throughout this paper, X will be a Banach space. The space of all bounded linear operators on X will be denoted by $B(X)$, and C will always be an injective operator in $B(X)$. For an operator A, we will write $D(A)$ for its domain, $R(A)$ for its range and $\rho(A)$ for its resolvent set, and we will write $\bar{E}$ for the closure of a subspace of X, E.

First, we recall the definition of C-semigroups.

Definition 2.1. A strongly continuous family $\{T(t)\}_{t\geq 0} \subset B(X)$ is called a *C-semigroup* if $T(t+s)C = T(t)T(s)$ for $t, s \geq 0$ and $T(0) = C$. $\{T(t)\}_{t\geq 0}$ is *exponentially bounded* if there exist $M < \infty$ and $\omega \in R$ such that $\|T(t)\| \leq Me^{\omega t}$.

The *generator of* $\{T(t)\}_{t\geq 0}$, A, is defined by

$$Ax = C^{-1}\left[\lim_{t\downarrow 0} \frac{1}{t}(T(t)x - Cx)\right]$$

with

$$D(A) = \left\{x \in X\colon \lim_{t\downarrow 0} \frac{1}{t}(T(t)x - Cx) \text{ exists and is in } R(C)\right\}.$$

The complex number λ is in $\rho_C(A)$, the *C-resolvent* of A, if $(\lambda - A)$ is injective and $R(C) \subseteq R(\lambda - A)$.

Lemma 2.2 ([4, 7])**.** *Suppose A generates a C-semigroup $\{T(t)\}_{t\geq 0}$ satisfying $\|T(t)\| \leq Me^{\omega t}$. Then*

(a) *A is a closed linear operator with $\overline{D(A)} \supseteq R(C)$;*

(b) *$\forall x \in X$, $T(t)x = Cx + A\int_0^t T(s)x ds$, which implies $T(\cdot)x$ is a mild solution for the abstract Cauchy problem*

$$\frac{d}{dt}u(t) = Au(t), \qquad u(0) = x; \tag{1}$$

(c) *$\forall x \in D(A)$ and $t \geq 0$, $T(t)x \in D(A)$ with $AT(t)x = T(t)Ax$;*

(d) *$A = C^{-1}AC$, where $D(C^{-1}AC) = \{x \in X\colon Cx \in D(A)$ and $ACx \in R(C)\}$;*

(c) *$(\omega, \infty) \subseteq \rho_C(A)$. For every $r > \omega$ and $n \in \mathbf{N}$, $D((r-A)^{-n}) \supseteq R(C)$ and*

$$(r-A)^{-n}C = \frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-rt}T(t)\,dt \tag{2}$$

which implies $\|(r-\omega)^n(r-A)^{-n}C\| \leq M$.

Next we need to introduce the Hille-Yosida space for an operator, for the details we refer to [4].

Definition 2.3. Suppose A has no eigenvalues in $(0, \infty)$. The *Hille-Yosida space for A*, Z_0, is the Banach space defined by

$$Z_0 = \{x \in X\colon \text{ The Cauchy problem (1) has a bounded uniformly continuous mild solution } u(\cdot, x)\}$$

with

$$\|x\|_{Z_0} = \sup\{\|u(t,x)\|;\ t \geq 0\} \quad \text{for } x \in Z_0.$$

And the *weak Hille-Yosida space* for A, Y, is the Banach space defined by

$$Y = \{x \in X\colon x \in R((s-A)^n)\ \forall\ s>0, n \in \mathbf{N} \text{ with}$$
$$\|x\|_Y = \sup\{s^n\|(s-A)^{-n}x\|; s>0, n+1 \in \mathbf{N}\} < \infty\}.$$

The relation between Z_0 and Y is as follows.

Lemma 2.4. *Suppose A has no eigenvalues in $(0,\infty)$, and Z_0 and Y are defined as above. Then*

(a) *$Z_0 \subset Y$ and $\|x\|_{Z_0} = \|x\|_Y$ for all $x \in X$;*
(b) *Z_0 is the closure, in Y, of $D(A|_Y)$, where $D(A|_Y) = \{x \in Y \cap D(A) : Ax \in Y\}$;*
(c) *$A|_{Z_0}$ generates a contraction C_0-semigroup on Z_0.*

3. Characterizations of contraction C-semigroups

A C-semigroup $\{T(t)\}_{t\geq 0}$ is *of contractions* if $\|T(t)x\| \leq \|Cx\|$ for $t \geq 0$ and $x \in X$. In this section, we give the characterizations of the generators of contraction C-semigroups. We start with the following

Proposition 3.1. *Suppose A generates a contraction C-semigroup, then*

(a) *$(0,\infty) \subseteq \rho_C(A)$, and for $\lambda > 0$, $n \in \mathbf{N}$ and $x \in X$, $R(C) \subseteq R((\lambda - A)^n)$ with*
$$\lambda^n\|(\lambda-A)^{-n}Cx\| \leq \|Cx\|;$$
(b) *for every $x \in D(A)$, there exists an $x^* \in F(Cx)$, that is, $x^* \in X^*$, $\|x^*\| = \|Cx\|$ and $x^*(Cx) = \|Cx\|^2$, such that*
$$Re\langle CAx, x^*\rangle \leq 0,$$
where $\langle x, x^*\rangle$ denotes the value of x^* at x.

Proof. (a) follows directly from Lemma 2.2*(e)*.

Let $x \in D(A)$ and $x^* \in F(Cx)$. Then

$$Re\left\langle \frac{T(t)x - Cx}{t}, x^*\right\rangle \leq Re\left\langle \frac{T(t)x}{t}, x^*\right\rangle - \frac{\|Cx\|^2}{t} \leq 0 \quad \text{for } t > 0,$$

hence

$$Re\langle CAx, x^*\rangle = \lim_{t\downarrow 0} Re\left\langle \frac{T(t)x - Cx}{t}, x^*\right\rangle \leq 0.$$

This is (b). □

Remark 3.2. If an operator A with $CA \subseteq AC$ satisfies (b), we call A *C-dissipative*. Similar to the proof of [5, Chapter 1, Theorem 4.2], we can prove that A is C-dissipative if and only if $\|(\lambda - A)Cx\| \geq \lambda\|Cx\|\ \forall x \in D(A)$ and $\lambda > 0$. Note that if $\lambda\|(\lambda-A)^{-1}Cx\| \leq \|Cx\|$ for all $x \in X$, then for $x \in D(A)$,

$$\|(\lambda-A)Cx\| = \|C(\lambda-A)x\| \geq \|\lambda(\lambda-A)^{-1}C(\lambda-A)x\| = \lambda\|Cx\|.$$

Using the Hille-Yosida space, we can completely characterize the generators.

Theorem 3.3. *Let A be an operator on X. Then A generates a contraction C-semigroup if and only if A satisfies*

(a) $A = C^{-1}AC$;
(b) $(0, \infty) \subseteq \rho_C(A)$, $R(C) \subseteq R((\lambda - A)^n)$ *and* $\lambda^n \|(\lambda - A)^{-n}Cx\| \leq \|Cx\|$ *for* $\lambda > 0$, $n \in \mathbf{N}$ *and* $x \in X$;
(c) *for some* $\lambda \geq 0$, *the Hille-Yosida space for* $A - \lambda I$, *denoted by* Z_λ, *contains* $R(C)$.

Proof. For the necessity, Lemma 2.2(d) and Proposition 3.1 imply (a) and (b). It remains to show (c). Let $\lambda > 0$ and define $S(t) = e^{-\lambda t}T(t)$ for $t \geq 0$. Thus $\{S(t)\}_{t \geq 0}$ is a bounded uniformly strongly continuous C-semigroup, generated by $A - \lambda I$. By Lemma 2.2(b) and Definition 2.3, $R(C)$ is contained in the Hille-Yosida space for $A - \lambda I$, i.e., Z_λ.

Conversely, let $A_\lambda = A|_Z$. By Lemma 2.4, $A_\lambda - \lambda I$ generates a C_0-semigroup of contractions, $e^{t(A_\lambda - \lambda I)}$, on $(Z_\lambda, \|\cdot\|_{Z_\lambda})$, which implies e^{tA_λ} is also a C_0-semigroup on $(Z_\lambda \|\cdot\|_{Z_\lambda})$.

For $t \geq 0$, define $W(t) : X \to X$ by $W(t) = e^{tA_\lambda}C$; we show that $\{W(t)\}_{t \geq 0}$ is a C-semigroup generated by A.

In fact, by (a), $CA \subseteq AC$, so that C commutes with e^{tA_λ} for $t \geq 0$. Thus

$$W(t+s)Cx = e^{(t+s)A_\lambda}C^2x = e^{tA_\lambda}Ce^{sA_\lambda}Cx = W(t)W(s)x,$$

that is, $W(t+s)C = W(t)W(s)$.

Moreover, if $x \in D(A)$, then $Cx \in Z_\lambda \cap D(A)$ with $ACx = CAx \in Z_\lambda$, so that $Cx \in D(A_\lambda)$, which implies that $e^{tA_\lambda}Cx$ is differentiable and

$$Ae^{tA_\lambda}Cx = A_\lambda e^{tA_\lambda}Cx = e^{tA_\lambda}A_\lambda Cx = e^{tA_\lambda}CAx,$$

hence $W(t)x \in D(A)$ with $AW(t)x = W(t)Ax$. So $W(t)$ is generated by an extension of A. To show A is the generator, we only need to prove that A is closed. It is exactly as in the proof of [9, Lemma 2.2].

Finally, by (b) and the exponential formulas for C-semigroups, we have

$$\|W(t)x\| = \lim_{n \to \infty} \left\| \left(1 - \frac{t}{n}A\right)^{-n} Cx \right\| \leq \|Cx\|,$$

so that $\{W(t)\}$ is of contractions. $\square$

Condition (c) in Theorem 3.3 seems to be difficult to check, but in the case of $C(D(A))$ dense in $R(C)$, it can be omitted.

Theorem 3.4. *Suppose* $C(D(A))$ *is dense in* $R(C)$. *Then* A *generates a contraction* C*-semigroup if and only if* A *satisfies* (a) *and* (b) *in Theorem 3.3.*

Proof. We only need to show the sufficiency.

If $x \in D(A)$, then $Cx \in D(A)$ with $ACx = CAx$ by (a). By (b), $R(C) \subseteq Y$, the weak Hille-Yosida space for A, since Z_0 is the closure of $D(A|_Y)$ in Y by Lemma 2.4, we have $Cx \in Z_0$.

For all $x \in X$, there exists a sequence $\{x_n\} \subset D(A)$, such that $Cx_n \to Cx$, in X. Moreover, for $n, m \in \mathbf{N}$, by Lemma 2.4,

$$\|Cx_n - Cx_m\|_{Z_0} = \|Cx_n - Cx_m\|_Y \leq \|Cx_n - Cx_m\|,$$

so that $\{Cx_n\}$ is a Cauchy sequence in Z_0, which implies $R(C) \subseteq Z_0$. So Theorem 3.4 follows from Theorem 3.3. $\square$

From the proof above we know that in this case we can choose $\lambda = 0$ in Theorem 3.3(c).

Note that if $D(A)$ is dense in X, then $C(D(A))$ is dense in $R(C)$. However, [2, Example 6.2] gave an example of a C-semigroup whose generator A is not densely defined while $C(D(A))$ is dense in $R(C)$.

4. Special cases

In this section, we make some applications of the results from the preceding section. First we give a sufficient condition that $C(D(A))$ is dense in $R(C)$.

Lemma 4.1. *Suppose that A generates an exponentially bounded C-semigroup and there exists a sequence $\{\lambda_n\} \subset \rho(A)$, such that $\lambda_n \to +\infty$, then $C(D(A))$ is dense in $R(C)$.*

Proof. Let $\lambda \in \rho(A)$, then for $\forall x \in X$, $(\lambda - A)^{-1}Cx = C(\lambda - A)^{-1}x \in C(D(A))$. An estimation using Eq. (2) yields that $\lambda(\lambda - A)^{-1}Cx \to Cx$ as $\lambda \to +\infty$. So that $\lambda_n(\lambda_n - A)^{-1}Cx \to Cx (n \to \infty)$, and since $\lambda_n(\lambda_n - A)^{-1}Cx \in C(D(A))$, our result holds. $\square$

The next lemma will be needed in the sequel.

Lemma 4.2. *Let A be a closed linear operator with $CA \subseteq AC$. Suppose $0 \neq \lambda \in \rho_C(A)$ and $\lambda\|(\lambda - A)^{-1}Cx\| \leq \|Cx\|$, $\forall x \in X$. Then $R(\lambda - A) \supseteq \overline{R(C)}$, and $\lambda\|(\lambda - A)^{-1}x\| \leq \|x\|$ for all $x \in \overline{R(C)}$.*

Proof. Let $x \in \overline{R(C)}$. There exists a sequence $\{x_n\} \subset X$ such that $Cx_n \to x$ as $n \to \infty$. Define $x'_n = (\lambda - A)^{-1}Cx_n$, thus

$$\|x'_n - x'_m\| = \|(\lambda - A)^{-1}C(x_n - x_m)\| \leq \frac{1}{\lambda}\|C(x_n - x_m)\|$$

for $n, m \in \mathbf{N}$, so $\{x'_n\}$ is a Cauchy sequence. Suppose $x'_n \to x_0 \in X$ as $n \to \infty$. Since $(\lambda - A)x'_n = Cx_n$ and A is closed, it follows that $x_0 \in D(A)$ and $(\lambda - A)x_0 = x$. Moreover,

$$\begin{aligned}\lambda\|(\lambda - A)^{-1}x\| = \lambda\|x_0\| = \lim_{n\to\infty} \|x'_n\| &= \lim_{n\to\infty} \lambda\|(\lambda - A)^{-1}Cx_n\| \\ &\leq \lim_{n\to\infty} \|Cx_n\| = \|x\|,\end{aligned}$$

as desired. $\square$

Now we can apply Theorem 3.4 to the case of $\rho(A) \neq \varnothing$. It is remarked that since $\rho(A) \neq \varnothing$, $CA \subseteq AC$ implies $A = C^{-1}AC$.

Theorem 4.3. *Let A be an operator on X. Suppose that $(0, \infty) \subseteq \rho(A)$. Then A generates a contraction C-semigroup if and only if A satisfies*

(a) $CA \subseteq AC$;
(b) $\lambda\|(\lambda - A)^{-1}Cx\| \leq \|Cx\|$ *for* $\lambda > 0$ *and* $x \in X$;
(c) $C(D(A))$ *is dense in* $R(C)$.

Proof. Theorem 3.3 and Lemma 4.2 imply the necessity.

Conversely, we define an operator B on X by

$$D(B) = \{Cx : x \in D(A)\},\ Bx = Ax \quad \text{for } x \in D(B).$$

So that $D(B) = C(D(A))$ and $R(B) \subseteq R(C)$. For $\lambda > 0$, since $(\lambda - A)^{-1}Cx = C(\lambda - A)^{-1}x$, so $R(\lambda - B) \supseteq R(C)$ and $\lambda\|(\lambda - B)^{-1}Cx\| = \lambda\|(\lambda - A)^{-1}Cx\| \leq$

$\|Cx\|$. From Remark 3.2, we know that B is dissipative on $(\overline{R(C)}, \|\cdot\|)$. Since $\overline{D(B)} = \overline{C(D(A))} = \overline{R(C)}$, by [5, Chapter 1, Theorem 4.3], B is closable in $\overline{R(C)}$ (hence in X), and the closure of B in $(\overline{R(C)}, \|\cdot\|)$ (or X), $\bar{B}$, is dissipative on $(\overline{R(C)}, \|\cdot\|)$. By Lemma 4.2, $R(\lambda - \bar{B}) = \overline{R(C)}$ for $\lambda > 0$. Therefore, the Lumer-Phillips theorem for C_0-semigroups implies that $\bar{B}$ generates a contraction C_0-semigroup, $\{S(t)\}_{t\geq 0}$, on $(\overline{R(C)}, \|\cdot\|)$. Define $T(t) : X \to X$ by $T(t) = S(t)C$. Thus $\{T(t)\}_{t\geq 0}$ is a C-semigroup of contractions on X. For $x \in D(A)$,

$$\frac{T(t)x - Cx}{t} = \frac{S(t)Cx - Cx}{t} \to BCx = CAx,$$

so that an extension of A is the generator, and since $\rho(A) \neq \varnothing$, it is exactly A. □

Remark 4.4. (a) The conditions (a)–(c) in Theorem 4.3 are equivalent to (a), (c) and (b)′ A is C-dissipative. In fact, by Remark 3.2, (b)′ implies that $\|\lambda(\lambda - A)Cx\| \geq \lambda\|Cx\|$ $(\lambda > 0, x \in D(A))$. Since $(0, \infty) \subseteq \rho(A)$, for $\lambda > 0$,

$$\|Cx\| = \|(\lambda - A)C(\lambda - A)^{-1}x\| \geq \lambda\|C(\lambda - A)^{-1}x\| = \lambda\|(\lambda - A)^{-1}Cx\|,$$

which is (b).

(b) In [2, Theorem 3.3], it is claimed that if $\rho(A) \neq \varnothing$ and A generates a C-semigroup of $O(e^{\omega t})$, then $(\omega, \infty) \subset \rho(A)$. However, there appears to be a gap in the argument, because it fails to prove that, if C^{-1} and $(r - A)$ both have resolvents that commute, then $C^{-1}(r - A) = (r - A)C^{-1}$. Here is a counterexample, suggested by deLaubenfels himself. Take $X = BC([0, \infty))$, the space of all bounded continuous functions on $[0, \infty)$ with supremum norm. Define $(Af)(s) = -sf(s)$ with $D(A) = \{f \in X, Af \in X\}$ and $(Cf)(s) = \frac{s}{1+s}f(s)$ for $s \geq 0$. Then $\sigma(A)$(the spectrum of A) $= (-\infty, 0]$, and $C^{-1}(\lambda - A) = (\lambda - A)C^{-1}$ for all $\lambda \in \rho(A)$. It is obvious that the function $f(s) = \frac{1}{1+s}$ is in $D(C^{-1}A)$ but is not in $D(AC^{-1})$. Thus $C^{-1}A \neq AC^{-1}$. We do not know whether the claimed result remains true.

Let us consider the case when $\rho(A) \neq \varnothing$. Let $C = (r - A)^{-n}$, where $r \in \rho(A)$ and $n \in \mathbf{N} \cup \{0\}$. From [9, Lemma 6.1], we know $\rho_C(A) = \rho(A)$. Since $R(C) = D(A^n)$ and $C(D(A)) = D(A^{n+1})$, as a direct consequence of Theorem 4.3, we have

Corollary 4.5. *Suppose $r \in \rho(A) \neq \varnothing$, let $C = (r - A)^{-n}$, $n \in \mathbf{N} \cup \{0\}$. Then the following statements are equivalent.*

(a) *A generates a contraction C-semigroup*;
(b) *A satisfies*
  (i) $(0, \infty) \subseteq \rho_C(A)$,
  (ii) *$\forall x \in D(A^n)$ and $\lambda > 0$, $\|\lambda(\lambda - A)^{-1}x\| \leq \|x\|$,*
  (iii) *$D(A^{n+1})$ is dense in $D(A^n)$*;
(c) *A satisfies* (i), (iii) *and*
  (ii)′ *A is C-dissipative.*

In the case of $\overline{R(C)} = X$, the generator of a contraction C-semigroup is in fact the generator of a contraction C_0-semigroup.

Theorem 4.6. *Suppose $\overline{R(C)} = X$. Then the following assertions are equivalent*:

(a) *A generates a contraction C-semigroup $\{T(t)\}_{t\geq 0}$*;
(b) *A generates a contraction C_0-semigroup $\{S(t)\}_{t\geq 0}$ and $CA \subseteq AC$*;
(c) *A satisfies*
  (i) *A is closed and $CA \subseteq AC$,*

(ii) $(0,\infty) \subseteq \rho_C(A)$ *and* $\lambda\|(\lambda - A)^{-1}Cx\| \le \|Cx\|$ *for* $\lambda > 0, x \in X$,

(iii) $D(A)$ *is dense*;

(d) A *satisfies* (i), (iii) *and*

(ii)′ $(0,\infty) \subseteq \rho_C(A)$ *and* A *is* C*-dissipative.*

Proof. (a)⇒(c) and (c)⇒(d) are obvious.

(c)⇒(b). By Lemma 4.2, (ii) implies $R(\lambda - A) = X$ and $\lambda\|(\lambda - A)^{-1}x\| \le \|x\|$ for $\lambda > 0$ and $x \in X$, applying the Hille-Yosida theorem for C_0-semigroups to A gives (b).

(b)⇒(a). By defining $T(t) = S(t)C$, it is not hard to show that $\{T(t)\}_{t\ge 0}$ is a contraction C-semigroup generated by A.

(d)⇒(c). Since A is C-dissipative, $\forall x \in D(A)$, we have $\|(\lambda - A)Cx\| \ge \lambda\|Cx\|$, so that for $x \in R(\lambda - A)$,

$$\|Cx\| = \|(\lambda - A)C(\lambda - A)^{-1}x\| \ge \lambda\|C(\lambda - A)^{-1}x\| = \lambda\|(\lambda - A)^{-1}Cx\|.$$

Since $R(\lambda - A) \supseteq R(C)$, which is dense in X, a similar proof as that of Lemma 4.2 will do. □

Consider when B generates a contraction C_0-semigroup $\{S(t)\}_{t\ge 0}$ on $\overline{R(C)}$ and $CB \subseteq BC$. Define $T(t) = S(t)C$ $(t \ge 0)$, we get a contraction C-semigroup $\{T(t)\}_{t\ge 0}$ on X. Suppose A is the generator. It is not hard to verify that $B = A|_{\overline{R(C)}}$. For the converse, in the case of Theorem 4.3, we know it is true.

Open Question. Suppose A is the generator of a contraction C-semigroup on a Banach space X. Does there exist a restriction of A, A', which is a generator of a contraction C_0-semigroup on $\overline{R(C)}$?

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