CHARACTERIZATIONS OF CONTRACTION $C$-SEMIGROUPS

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Abstract. A $C$-semigroup $\{T(t)\}_{t \geq 0}$ is of contractions if $\|T(t)x\| \leq \|Cx\|$ for $t \geq 0$, $x \in X$. Using the Hille-Yosida space, we completely characterize the generators of contraction $C$-semigroups. We also give the Lumer-Phillips theorem for $C$-semigroups in several special cases.

1. Introduction

The notion of exponentially bounded $C$-semigroup was introduced by Davies and Pang [1]. Recently, the theory of $C$-semigroup has been extensively developed by many authors [2, 7, 9]. This theory allows us to study many ill-posed abstract Cauchy problems.

The starting point of this paper is to try to give an answer to the question asked by R. deLaubenfels in [3, Open question 6.10]: Does there exist an analogue of the Lumer-Phillips theorem for $C$-semigroups? Since the Lumer-Phillips theorem characterizes the generators of contraction $C_0$-semigroups, this gives us the motivation to make a suitable definition for the contractions of $C$-semigroups and then characterize the generators.

On the other hand, many works have generalized the Hille-Yosida theorem to $C$-semigroups. Earlier, Davies and Pang [1] gave a characterization of an exponentially bounded $C$-semigroup under the assumption that $R(C)$ is dense in $X$. Later, Tanaka and Miyadera [7] generalized their results to the case of $R(C)$ not dense, and they gave a sufficient and necessary condition for a closed linear operator with dense domain to be the generator of an exponentially bounded $C$-semigroup. After defining the contraction $C$-semigroup, we are also interested in characterizing the generator by the Hille-Yosida type theorem. Here the main difficulty we meet with is that the generator may not be densely defined, we choose the Hille-Yosida space to give an additional condition on the generator.

This paper is organized as follows. §2 is devoted to some preliminaries on $C$-semigroups. In §3 we characterize the generators of contraction $C$-semigroups in general cases, and under the assumption that $C(D(A))$ is dense in $R(C)$, the characterization can be simplified. §4 deals with several special cases of $\rho(A) \neq \emptyset$ or
$R(C) = X$, we obtain both the Hille-Yosida theorem and the Lumer-Phillips theorem in such cases. This means that we partly give the answer to the question mentioned above in the affirmative.

2. Preliminaries

Throughout this paper, $X$ will be a Banach space. The space of all bounded linear operators on $X$ will be denoted by $B(X)$, and $C$ will always be an injective operator in $B(X)$. For an operator $A$, we will write $D(A)$ for its domain, $R(A)$ for its range and $\rho(A)$ for its resolvent set, and we will write $E$ for the closure of a subspace of $X$, $E$.

First, we recall the definition of $C$-semigroups.

Definition 2.1. A strongly continuous family $\{T(t)\}_{t \geq 0} \subset B(X)$ is called a $C$-semigroup if $T(t+s)C = T(t)T(s)$ for $t, s \geq 0$ and $T(0) = C$. $\{T(t)\}_{t \geq 0}$ is exponentially bounded if there exist $M < \infty$ and $\omega \in R$ such that $\|T(t)\| \leq Me^{\omega t}$.

The generator of $\{T(t)\}_{t \geq 0}$, $A$, is defined by

$$Ax = C^{-1} \lim_{t \to 0} \frac{1}{t} (T(t)x - Cx)$$

with

$$D(A) = \{x \in X : \lim_{t \to 0} \frac{1}{t} (T(t)x - Cx) \text{ exists and is in } R(C)\}.$$

The complex number $\lambda$ is in $\rho_C(A)$, the $C$-resolvent of $A$, if $(\lambda - A)$ is injective and $R(C) \subseteq R(\lambda - A)$.

Lemma 2.2 ([4, 7]). Suppose $A$ generates a $C$-semigroup $\{T(t)\}_{t \geq 0}$ satisfying $\|T(t)\| \leq Me^{\omega t}$. Then

(a) $A$ is a closed linear operator with $\overline{D(A)} \supseteq R(C)$;
(b) $\forall x \in X$, $T(t)x = Cx + A \int_0^t T(s)xdS$, which implies $T(\cdot)x$ is a mild solution for the abstract Cauchy problem

$$\frac{d}{dt} u(t) = Au(t), \quad u(0) = x;$$

(c) $\forall x \in D(A)$ and $t \geq 0$, $T(t)x \in D(A)$ with $AT(t)x = T(t)Ax$;
(d) $A = C^{-1}AC$, where $D(C^{-1}AC) = \{x \in X : Cx \in D(A) \text{ and } ACx \in R(C)\}$;
(e) $(\omega, \infty) \subseteq \rho_C(A)$. For every $r > \omega$ and $n \in \mathbb{N}$, $D((r - A)^{-n}) \supseteq R(C)$ and

$$\begin{align*}
(r - A)^{-n}C &= \frac{1}{(n - 1)!} \int_0^\infty t^{n-1}e^{-rt}T(t)dt \\
which \ implies \ \|((r - \omega)^n(r - A)^{-n}C\| &\leq M.
\end{align*}$$

Next we need to introduce the Hille-Yosida space for an operator, for the details we refer to [4].

Definition 2.3. Suppose $A$ has no eigenvalues in $(0, \infty)$. The Hille-Yosida space for $A$, $Z_0$, is the Banach space defined by

$$Z_0 = \{x \in X : \text{ The Cauchy problem (1) has a bounded uniformly continuous mild solution } u(\cdot, x)\}$$
with
\[ \|x\|_{Z_0} = \sup\{\|u(t, x)\|; t \geq 0\} \quad \text{for } x \in Z_0. \]

And the weak Hille-Yosida space for \(A, Y\), is the Banach space defined by
\[ Y = \{x \in X: x \in R((s - A)^n) \forall s > 0, n \in \mathbb{N} \text{ with} \]
\[ \|x\|_Y = \sup\{s^n\|(s - A)^{-n}x\|; s > 0, n + 1 \in \mathbb{N}\} < \infty. \]

The relation between \(Z_0\) and \(Y\) is as follows.

**Lemma 2.4.** Suppose \(A\) has no eigenvalues in \((0, \infty)\), and \(Z_0\) and \(Y\) are defined as above. Then
(a) \(Z_0 \subset Y\) and \(\|x\|_{Z_0} = \|x\|_Y\) for all \(x \in X\);
(b) \(Z_0\) is the closure, in \(Y\), of \(D(A|Y)\), where \(D(A|Y) = \{x \in Y \cap D(A): Ax \in Y\}\);
(c) \(A|Z_0\) generates a contraction \(C_0\)-semigroup on \(Z_0\).

3. Characterizations of contraction \(C\)-semigroups

A \(C\)-semigroup \(\{T(t)\}_{t \geq 0}\) is of contractions if \(\|T(t)x\| \leq \|Cx\|\) for \(t \geq 0\) and \(x \in X\). In this section, we give the characterizations of the generators of contraction \(C\)-semigroups. We start with the following

**Proposition 3.1.** Suppose \(A\) generates a contraction \(C\)-semigroup, then
(a) \((0, \infty) \subseteq \rho_C(A)\), and for \(\lambda > 0, n \in \mathbb{N}\) and \(x \in X\), \(R(C) \subseteq R((\lambda - A)^n)\) with
\[ \lambda^n\|\lambda - A\|^nCx\| \leq \|Cx\|; \]
(b) for every \(x \in D(A)\), there exists an \(x^* \in F(Cx)\), that is, \(x^* \in X^*, \|x^*\| = \|Cx\|\) and \(x^*(Cx) = \|Cx\|^2\), such that
\[ \text{Re}(CAx, x^*) \leq 0, \]
where \(\langle x, x^* \rangle\) denotes the value of \(x^*\) at \(x\).

**Proof.** (a) follows directly from Lemma 2.2(e).

Let \(x \in D(A)\) and \(x^* \in F(Cx)\). Then
\[ \text{Re} \left( \frac{T(t)x - Cx}{t}, x^* \right) \leq \text{Re} \left( \frac{T(t)x}{t}, x^* \right) - \frac{\|Cx\|^2}{t} \leq 0 \quad \text{for } t > 0, \]
hence
\[ \text{Re}(CAx, x^*) = \lim_{t \to 0} \text{Re} \left( \frac{T(t)x - Cx}{t}, x^* \right) \leq 0. \]
This is (b).

**Remark 3.2.** If an operator \(A\) with \(CA \subseteq AC\) satisfies (b), we call \(A\) \(C\)-dissipative. Similar to the proof of [5, Chapter 1, Theorem 4.2], we can prove that \(A\) is \(C\)-dissipative if and only if \(\|\lambda - A\|^n \geq \lambda \|Cx\|\) \(\forall x \in D(A)\) and \(\lambda > 0\). Note that if \(\lambda\|\lambda - A\|^{-1}Cx\| \leq C\|Cx\|\) for all \(x \in X\), then for \(x \in D(A)\),
\[ \|\lambda - A\|^n = \|C(\lambda - A)x\| \geq \lambda(\lambda - A)^{-1}C(\lambda - A)x = \lambda\|Cx\|. \]

Using the Hille-Yosida space, we can completely characterize the generators.

**Theorem 3.3.** Let \(A\) be an operator on \(X\). Then \(A\) generates a contraction \(C\)-semigroup if and only if \(A\) satisfies
(a) \( A = C^{-1}AC \);
(b) \( (0, \infty) \subseteq D(A), R(C) \subseteq R((\lambda - A)^n) \) and \( \lambda^n\|((\lambda - A)^{-n}Cx) \leq \|Cx\| \) for \( \lambda > 0, n \in \mathbb{N} \) and \( x \in X \);
(c) for some \( \lambda \geq 0 \), the Hille-Yosida space for \( A - \lambda I \), denoted by \( Z_\lambda \), contains \( R(C) \).

Proof. For the necessity, Lemma 2.2(d) and Proposition 3.1 imply (a) and (b). It remains to show (c). Let \( \lambda > 0 \) and define \( S(t) = e^{-\lambda t}T(t) \) for \( t \geq 0 \). Thus \( \{S(t)\}_{t \geq 0} \) is a bounded uniformly strongly continuous \( C \)-semigroup, generated by \( A - \lambda I \). By Lemma 2.2(b) and Definition 2.3, \( R(C) \) is contained in the Hille-Yosida space for \( A - \lambda I \), i.e., \( Z_\lambda \).

Conversely, let \( A_\lambda = A|_{Z_\lambda} \). By Lemma 2.4, \( A_\lambda - \lambda I \) generates a \( C_0 \)-semigroup of contractions, \( e^{\{A_\lambda - \lambda I\}} \), on \( (Z_\lambda, \| \cdot \|_{Z_\lambda}) \), which implies \( e^{A_\lambda} \) is also a \( C_0 \)-semigroup on \( (Z_\lambda, \| \cdot \|_{Z_\lambda}) \).

For \( t \geq 0 \), define \( W(t) : X \to X \) by \( W(t) = e^{tA_\lambda}C \); we show that \( \{W(t)\}_{t \geq 0} \) is a \( C \)-semigroup generated by \( A \).

In fact, by (a), \( CA \subseteq AC \), so that \( C \) commutes with \( e^{tA_\lambda} \) for \( t \geq 0 \). Thus
\[
W(t+s)Cx = e^{(t+s)A_\lambda C^2x}e^{tA_\lambda}C\lambda x = W(t)W(s)x,
\]
that is, \( W(t+s)C = W(t)W(s) \).

Moreover, if \( x \in D(A) \), then \( CX \in Z_\lambda \cap D(A) \) with \( ACx = CAx \in Z_\lambda \), so that \( CX \in D(A_\lambda) \), which implies that \( e^{tA_\lambda}Cx \) is differentiable and
\[
Ae^{tA_\lambda}CX = A_\lambda e^{tA_\lambda}CX = e^{tA_\lambda}A_\lambda CX = e^{tA_\lambda}CAx,
\]
hence \( W(t)x \in D(A) \) with \( AW(t)x = W(t)Ax \). So \( W(t) \) is generated by an extension of \( A \). To show \( A \) is the generator, we only need to prove that \( A \) is closed. It is exactly as in the proof of [9, Lemma 2.2].

Finally, by (b) and the exponential formulas for \( C \)-semigroups, we have
\[
\|W(t)x\| = \lim_{n \to \infty} \left\| \left(1 - \frac{t}{n} A \right)^{-n} Cx \right\| \leq \|Cx\|,
\]
so that \( \{W(t)\} \) is of contractions.

Condition (c) in Theorem 3.3 seems to be difficult to check, but in the case of \( C(D(A)) \) dense in \( R(C) \), it can be omitted.

**Theorem 3.4.** Suppose \( C(D(A)) \) is dense in \( R(C) \). Then \( A \) generates a contraction \( C \)-semigroup if and only if \( A \) satisfies (a) and (b) in Theorem 3.3.

Proof. We only need to show the sufficiency.

If \( x \in D(A) \), then \( CX \in D(A) \) with \( ACx = CAx \) by (a). By (b), \( R(C) \subseteq Y \), the weak Hille-Yosida space for \( A \), since \( Z_0 \) is the closure of \( D(A)|_Y \) in \( Y \) by Lemma 2.4, we have \( CX \in Z_0 \).

For all \( x \in X \), there exists a sequence \( \{x_n\} \subset D(A) \), such that \( Cx_n \to CX \), in \( X \). Moreover, for \( n, m \in \mathbb{N} \), by Lemma 2.4,
\[
\|Cx_n - CX_m\|_{Z_0} = \|Cx_n - CX_m\|_Y \leq \|Cx_n - CX_m\|,
\]
so that \( \{CX_n\} \) is a Cauchy sequence in \( Z_0 \), which implies \( R(C) \subseteq Z_0 \). So Theorem 3.4 follows from Theorem 3.3. 
\[\square\]
From the proof above we know that in this case we can choose $\lambda = 0$ in Theorem 3.3(c).

Note that if $D(A)$ is dense in $X$, then $C(D(A))$ is dense in $R(C)$. However, [2, Example 6.2] gave an example of a $C$-semigroup whose generator $A$ is not densely defined while $C(D(A))$ is dense in $R(C)$.

4. Special cases

In this section, we make some applications of the results from the preceding section. First we give a sufficient condition that $C(D(A))$ is dense in $R(C)$.

**Lemma 4.1.** Suppose that $A$ generates an exponentially bounded $C$-semigroup and there exists a sequence $\{\lambda_n\} \subset \rho(A)$, such that $\lambda_n \to +\infty$, then $C(D(A))$ is dense in $R(C)$.

**Proof.** Let $\lambda \in \rho(A)$, then for $\forall x \in X$, $(\lambda - A)^{-1}Cx = C(\lambda - A)^{-1}x \in C(D(A))$. An estimation using Eq. (2) yields that $\lambda(\lambda - A)^{-1}Cx \to Cx$ as $\lambda \to +\infty$. So that $\lambda_n(\lambda_n - A)^{-1}Cx \to Cx(n \to \infty)$, and since $\lambda_n(\lambda_n - A)^{-1}Cx \in C(D(A))$, our result holds.

The next lemma will be needed in the sequel.

**Lemma 4.2.** Let $A$ be a closed linear operator with $CA \subseteq AC$. Suppose $0 \neq \lambda \in \rho_C(A)$ and $\|\lambda(\lambda - A)^{-1}Cx\| \leq \|Cx\|$, $\forall x \in X$. Then $R(\lambda - A) \supseteq R(C)$, and $\|\lambda(\lambda - A)^{-1}x\| \leq \|x\|$ for all $x \in R(C)$.

**Proof.** Let $x \in R(C)$. There exists a sequence $\{x_n\} \subset X$ such that $Cx_n \to x$ as $n \to \infty$. Define $x'_n = (\lambda - A)^{-1}Cx_n$, thus

$$\|x'_n - x'_m\| = \|\lambda(\lambda - A)^{-1}C(x_n - x_m)\| \leq \frac{1}{\lambda}\|C(x_n - x_m)\|$$

for $n, m \in \mathbb{N}$, so $\{x'_n\}$ is a Cauchy sequence. Suppose $x'_n \to x_0 \in X$ as $n \to \infty$. Since $(\lambda - A)x'_n = Cx_n$ and $A$ is closed, it follows that $x_0 \in D(A)$ and $(\lambda - A)x_0 = x$. Moreover,

$$\|\lambda(\lambda - A)^{-1}x\| = \|x_0\| = \lim_{n \to \infty} \|x'_n\| = \lim_{n \to \infty} \lambda(\lambda - A)^{-1}C\|x_n\|$$

$$\leq \lim_{n \to \infty} \|Cx_n\| = \|x\|,$$

as desired.

Now we can apply Theorem 3.4 to the case of $\rho(A) \neq \emptyset$. It is remarked that since $\rho(A) \neq \emptyset$, $CA \subseteq AC$ implies $A = C^{-1}AC$.

**Theorem 4.3.** Let $A$ be an operator on $X$. Suppose that $(0, \infty) \subseteq \rho(A)$. Then $A$ generates a contraction $C$-semigroup if and only if $A$ satisfies

(a) $CA \subseteq AC$;

(b) $\lambda\|\lambda(\lambda - A)^{-1}Cx\| \leq \|Cx\|$ for $\lambda > 0$ and $x \in X$;

(c) $C(D(A))$ is dense in $R(C)$.

**Proof.** Theorem 3.3 and Lemma 4.2 imply the necessity.

Conversely, we define an operator $B$ on $X$ by

$$D(B) = \{Cx : x \in D(A)\}, \quad Bx = Ax \quad \text{for} \ x \in D(B).$$

So that $D(B) = C(D(A))$ and $R(B) \subseteq R(C)$. For $\lambda > 0$, since $(\lambda - A)^{-1}Cx = C(\lambda - A)^{-1}x$, so $R(\lambda - B) \supseteq R(C)$ and $\lambda\|\lambda(\lambda - B)^{-1}Cx\| = \lambda\|\lambda(\lambda - A)^{-1}Cx\| \leq$
\[ Cx \mid \text{From Remark 3.2, we know that } B \text{ is dissipative on } (R(C), \| \cdot \|). \text{ Since } D(B) = C(D(A)) = R(C), \text{ by [5, Chapter 1, Theorem 4.3], } B \text{ is closable in } R(C) \text{ (hence in } X), \text{ and the closure of } B \text{ in } (R(C), \|\|) \text{ (or } X), \text{ is dissipative on } (R(C), \|\|). \text{ By Lemma 4.2, } R(\lambda - B) = R(C) \text{ for } \lambda > 0. \text{ Therefore, the Lumer-Phillips theorem for } C_0\text{-semigroups implies that } \hat{B} \text{ generates a contraction } C_0\text{-semigroup, } \{S(t)\}_{t \geq 0}, \text{ on } (R(C), \|\|). \text{ Define } T(t) : X \to X \text{ by } T(t) = S(t)C. \text{ Thus } \{T(t)\}_{t \geq 0} \text{ is a } C\text{-semigroup of contractions on } X. \text{ For } x \in D(A),
\]
\[
\frac{T(t)x - Cx}{t} = \frac{S(t)Cx - Cx}{t} \to BCx = CAx,
\]
so that an extension of } A \text{ is the generator, and since } \rho(A) \neq \emptyset, \text{ it is exactly } A. \]

**Remark 4.4.** (a) The conditions (a)–(c) in Theorem 4.3 are equivalent to (a), (c) and (b)' A is } C\text{-dissipative. } \text{In fact, by Remark 3.2, (b)' implies that } \|\lambda(\lambda - A)Cx\| \geq \lambda\|Cx\| (\lambda > 0, x \in D(A)). \text{ Since } (0, \infty) \subseteq \rho(A), \text{ for } \lambda > 0,
\[
\|Cx\| = \|(\lambda - A)C(\lambda - A)^{-1}x\| \geq \lambda\|C(\lambda - A)^{-1}x\| = \lambda\|\lambda - A\|^{-1}C\|,\]
which is (b).

(b) In [2, Theorem 3.3], it is claimed that if } \rho(A) \neq \emptyset \text{ and } A \text{ generates a } C\text{-semigroup of } O(e^{\omega t}), \text{ then } (\omega, \infty) \subseteq \rho(A). \text{ However, there appears to be a gap in the argument, because it fails to prove that, if } C^{-1} \text{ and } (r - A) \text{ both have resolvents that commute, then } C^{-1}(r - A) = (r - A)C^{-1}. \text{ Here is a counterexample, suggested by deLaubenfels himself. Take } X = BC([0, \infty)), \text{ the space of all bounded continuous functions on } [0, \infty) \text{ with supremum norm. Define } (Af)(s) = -sf(s) \text{ with } D(A) = \{f \in X, Af \in X\} \text{ and } (Cf)(s) = \frac{2}{1+s}f(s) \text{ for } s \geq 0. \text{ Then } \sigma(A)(\text{the spectrum of } A)(\lambda) = (\infty, 0], \text{ and } C^{-1}(\lambda - A) = (\lambda - A)C^{-1} \text{ for all } \lambda \in \rho(A). \text{ It is obvious that the function } f(s) = \frac{2}{1+s} \text{ is in } D(C^{-1}A) \text{ but is not in } D(AC^{-1}). \text{ Thus } C^{-1}A \neq AC^{-1}. \text{ We do not know whether the claimed result remains true.}

Let us consider the case when } \rho(A) \neq \emptyset. \text{ Let } C = (r - A)^{-n}, \text{ where } r \in \rho(A) \text{ and } n \in \mathbb{N} \cup \{0\}. \text{ From [9, Lemma 6.1], we know } \rho(C)(A) = \rho(A). \text{ Since } R(C) = D(A^n) \text{ and } C(D(A)) = D(A^{n+1}), \text{ as a direct consequence of Theorem 4.3, we have}

**Corollary 4.5.** Suppose } r \in \rho(A) \neq \emptyset, \text{ let } C = (r - A)^{-n}, \text{ } n \in \mathbb{N} \cup \{0\}. \text{ Then the following statements are equivalent.}

(a) } A \text{ generates a contraction } C\text{-semigroup;
(b) } A \text{ satisfies
(i) } (0, \infty) \subseteq \rho_C(A),
(ii) \forall x \in D(A^n) \text{ and } \lambda > 0, \|\lambda(\lambda - A)^{-1}x\| \leq \|x\|,
(iii) } D(A^{n+1}) \text{ is dense in } D(A^n);
(c) } A \text{ satisfies (i), (iii) and
(ii)' } A \text{ is } C\text{-dissipative.

In the case of } R(C) = X, \text{ the generator of a contraction } C\text{-semigroup is in fact the generator of a contraction } C_0\text{-semigroup.}

**Theorem 4.6.** Suppose } R(C) = X. \text{ Then the following assertions are equivalent:
(a) } A \text{ generates a contraction } C\text{-semigroup } \{T(t)\}_{t \geq 0};
(b) } A \text{ generates a contraction } C_0\text{-semigroup } \{S(t)\}_{t \geq 0} \text{ and } CA \subseteq AC;
(c) } A \text{ satisfies
(i) } A \text{ is closed and } CA \subseteq AC,
(ii) $(0, \infty) \subseteq \rho_C(A)$ and $\lambda \| (\lambda - A)^{-1} C x \| \leq \| C x \|$ for $\lambda > 0, x \in X$,
(iii) $D(A)$ is dense;
(d) $A$ satisfies (i), (iii) and
(ii') $(0, \infty) \subseteq \rho_C(A)$ and $A$ is $C$-dissipative.

Proof. (a)$\Rightarrow$(c) and (c)$\Rightarrow$(d) are obvious.
(c)$\Rightarrow$(b). By Lemma 4.2, (ii) implies $R(\lambda - A) = X$ and $\lambda \| (\lambda - A)^{-1} x \| \leq \| x \|$ for $\lambda > 0$ and $x \in X$, applying the Hille-Yosida theorem for $C_0$-semigroups to $A$ gives (b).
(b)$\Rightarrow$(a). By defining $T(t) = S(t)C$, it is not hard to show that $\{T(t)\}_{t \geq 0}$ is a contraction $C$-semigroup generated by $A$.
(d)$\Rightarrow$(c). Since $A$ is $C$-dissipative, $\forall x \in D(A)$, we have $\| (\lambda - A) C x \| \geq \lambda \| C x \|$, so that for $x \in R(\lambda - A)$,
$$\| C x \| = \| (\lambda - A) C (\lambda - A)^{-1} x \| \geq \lambda \| C (\lambda - A)^{-1} x \| = \lambda \| (\lambda - A)^{-1} C x \|.$$ 
Since $R(\lambda - A) \supseteq R(C)$, which is dense in $X$, a similar proof as that of Lemma 4.2 will do. \hfill \Box

Consider when $B$ generates a contraction $C_0$-semigroup $\{S(t)\}_{t \geq 0}$ on $R(C)$ and $CB \subseteq BC$. Define $T(t) = S(t)C$ $(t \geq 0)$, we get a contraction $C$-semigroup $\{T(t)\}_{t \geq 0}$ on $X$. Suppose $A$ is the generator. It is not hard to verify that $B = A|_{R(C)}$. For the converse, in the case of Theorem 4.3, we know it is true.

Open Question. Suppose $A$ is the generator of a contraction $C$-semigroup on a Banach space $X$. Does there exist a restriction of $A$, $A'$, which is a generator of a contraction $C_0$-semigroup on $R(C)$?

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